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Review

Reducing the number of new constraints and variables in a linearised quadratic assignment problem

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The quadratic assignment problem is a well known difficult discrete combinatorial optimization problem. The problem seeks to locate *n* facilities among *n* fixed locations in the most economical way. We propose a technique to reduce the number of new constraints and variables in a linearised quadratic assignment problem. It is computationally cheaper to solve a mathematical model with half the number of new constraints and variables than the original full model. The quadratic assignment problem is common in agricultural resource or facility location, economics, production, military operations or operations research in general.

Key words: Linearised quadratic assignment problem, discrete optimization, NP hard, heuristic, lower bound.

INTRODUCTION

The quadratic assignment problem (QAP) was introduced by Koopmans and Beckmann in 1957 as the problem of allocating a set of facilities to a set of locations, with the cost being a function of the distance and flow between the facilities and the costs associated with a facility being placed at a certain location. The objective is to assign each facility to a location in such a way that the total cost is minimized. Although, extensive research has been done for over 50 years; this remains one of the hardest optimization problems the world has ever had (Çela, 1998). We are not aware of any effective exact algorithm that can solve this NP complete model consistently in reasonable computational time.

The QAP model is common in agricultural resource or facility location, economics, production, military operations or operations research in general. An effective heuristic algorithm that can consistently and accurately approximate the quadratic algorithm is also believed not to exist. Even finding an approximate solution within some constant factor from the optimal solution is also very difficult (Adams and Johnson, 1994).

Various heuristics have been developed by Hahn and Grant (2008), Ramakrishnan et al. (2002) and Drezner (2008). For more developments in solving QAP, one can consult Nagarajan and Sviridenko (2009), Rego et al. (2010), Xia (2010), Yang et al. (2008) and many others that can be found in literature.

In addition to application in facility location, the QAP has application in computer manufacturing, scheduling, process communications, turbine balancing, backboard wiring and many others.

It is easier to solve a mathematical model with less number of constraints and variables than one with all the constraints and variables.

In this paper, we propose a technique to reduce the number of new constraints and variables in a linearised quadratic assignment problem. The reduced linearised quadratic assignment problem is easier to solve than the original problem.

QUADRATIC ASSIGNMENT PROBLEM FORMULATION

There are several mathematical formulations of QAP introduced in the past five decades by various researchers. In this paper we explore and linearise the original quadratic integer formulation introduced by Koopmans and Beckmann (1957). Suppose new buildings are to be placed on a piece of land and n sites have been identified as sites for the buildings, it is assumed that each building has a special function.

Koopmans- Beckmann formulation

Let: a_{ij} be the walking distance between sites *i* and *j*; b_{kl} be the number of people per week who circulate between buildings *k* and *l*. Then the Koopmans-Beckmann formulation of the QAP is given as:

Minimize: $Z = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{ij}b_{kl}x_{ik}x_{jl} + \sum_{i=1}^{n}\sum_{k=1}^{n} c_{ik}x_{ik}$

Such that:

$$\left.\begin{aligned} &\sum_{i=1}^{n} x_{ij} = 1, \quad 1 \le j \le n \\ &\sum_{j=1}^{n} x_{ij} = 1, \quad 1 \le i \le n \end{aligned}\right\} \quad (1)$$

$$x_{ij} \in \{0,1\}, \quad 1 \le i \le n, \quad 1 \le j \le n$$

In this formulation there are n^2 variables and 2*n* constraints.

Linearising the quadratic assignment problem

The available techniques can linearise the Koopmans-Beckmann model to the form:

Minimize: $z = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{ij}b_{kl}y_{ijkl}$

Such that:

$$\left.\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} y_{ijkl} = n^2 \\ &x_{ik} + x_{jl} \ge 2y_{ijkl}, \forall i, j, k, l \\ &y_{ijkl} \ge x_{ik} + x_{jl} - 1 \end{aligned}\right\} \quad (2)$$

Solving this linearised QAP model is very difficult due to hardware restrictions as *n* becomes large. This linearised model has $(n^4 + n^2)$ variables and $O(n^4)$ constraints.

The quadratic binary problem as the general case

In this paper we classify the Koopmans-Beckmann model as a special case of a quadratic binary problem. Let a quadratic binary problem be represented by:

Minimize: $Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^{0} x_i x_j + \sum_{k}^{n} c_k^1 x_k$

$$\left.\begin{aligned} \text{Such that: } & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ & \ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \end{aligned}\right\} \quad (3)$$

Where,

a_{ij}, b_i, c_{ij}^0 and c_k^1 are constants, $1 \le i \le m, \ 1 \le j \le n,$ $x_i, x_j, x_k \in \{0,1\}, \ 1 \le i \le n, \ 1 \le j \le n, \ 1 \le k \le n.$

The variables $x_i x_j$ where $i = j$

If $i = j$, then $x_i^2 = x_j^2$.

For binary integer variables:

$$x_i(x_i - 1) = 0$$

$$x_i^2 - x_i = 0$$

$$x_i = x_i^2 \quad (4)$$

Thus, x_i^2 can be replaced by x_i in the objective function. Similarly, x_j^2 can also be replaced by x_j in the objective function. Note that this substitution on its own does not change the number of variables in the problem.

The variables $x_i x_j$ where $i \ne j$

If $i \ne j$, then in the worst case there are $\frac{n(n-1)}{2}$ combinations of such variables in the objective function.

Proof

Suppose there are:

Two variables: $(x_1$ and $x_2)$, then in the worst case we can have $x_1 x_2$ as the only possible combination of variables.

Three variables: $(x_1, x_2$ and $x_3)$, then in the worst case we can have x_1x_2, x_1x_3 and x_2x_3 as the possible combinations of variables. These three variables give 3 possible combinations.

***n* variables:** $(x_1, x_2, ..., x_{n-1}$ and $x_n)$, then in the worst case we can have $x_1x_2, x_1x_3, ..., x_1x_n, x_2x_3, x_2x_4, ..., x_2x_n, ..., x_{n-1}x_n$ as the possible combinations. The *n* variables give $(n-1)+(n-2)+...+1=\sum_{1}^{n-1} t = \frac{n(n-1)}{2}$ possible combinations.

LINEARIZING THE QUADRATIC BINARY PROBLEM

The variable combinations x_ix_j where $i \neq j$ must be removed in order to make the objective function linear. This is done by using the following substitution:

Variable substitution

Let

$$x_ix_j = \delta_r, \tag{5}$$

Where δ_r is also a binary variable and $r = 1, 2, ..., \frac{n(n-1)}{n}$.

Such that:

$$\left.\begin{array}{l} x_i + x_j = 2\delta_r + \overline{\delta}_r \\ \delta_r + \overline{\delta}_r \leq 1 \\ \delta_r, \overline{\delta}_r \in \{0,1\} \text{ and } r = 1, 2, ..., \frac{n(n-1)}{n}. \end{array}\right\} \tag{6}$$

Proof

We have to show that the solution space $\Omega(x_ix_j) = \{0,1\}$ is also the solution space for $\Omega(\delta_r)$, every point in $\Omega(x_ix_j)$ has a corresponding point in $\Omega(\delta_r)$ and that $x_ix_j = \delta_r$ for all corresponding points.

Solution space for x_ix_j that is $\Omega(x_ix_j)$

$$\left.\begin{array}{l} x_i = 0 \text{ and } x_j = 0 \\ x_i = 1 \text{ and } x_j = 0 \\ x_i = 0 \text{ and } x_j = 1 \end{array}\right\} x_ix_j = 0$$

$$x_i = 1 \text{ and } x_j = 1 \Rightarrow x_ix_j = 1$$

$$\therefore \Omega(x_ix_j) = \{0,1\}.$$

Solution space for δ_r that is $\Omega(\delta_r)$

$$\delta_r = 1 \text{ and } \overline{\delta}_r = 0 \Rightarrow x_i + x_j = 2 \Rightarrow x_i = x_j = 1 \Rightarrow x_ix_j = 1$$
$$\delta_r = 0 \text{ and } \overline{\delta}_r = 1 \Rightarrow x_i + x_j = 1,$$

$$\Rightarrow \begin{cases} \text{Either } x_i = 1 \text{ and } x_j = 0 \Rightarrow x_ix_j = 0. \\ \text{Or } x_i = 0 \text{ and } x_j = 1 \Rightarrow x_ix_j = 0. \end{cases}$$

$$\delta_r = 0 \text{ and } \overline{\delta}_r = 0 \Rightarrow x_i + x_j = 0 \Rightarrow x_i = x_j = 0 \Rightarrow x_ix_j = 0.$$
$$\therefore \Omega(\delta_r) = \{0,1\}.$$

Corresponding points

Point in $\Omega(x_ix_j)$	Corresponding point in $\Omega(\delta_r)$
$x_i = 0$ and $x_j = 0$	$\delta_r = 0$ and $\overline{\delta}_r = 0$
$x_i = 1$ and $x_j = 0$	$\delta_r = 0$ and $\overline{\delta}_r = 1$
$x_i = 0$ and $x_j = 1$	$\delta_r = 0$ and $\overline{\delta}_r = 1$
$x_i = 1$ and $x_j = 1$	$\delta_r = 1$ and $\overline{\delta}_r = 0$

NUMBER OF NEW VARIABLES AND CONSTRAINTS IN THE LINEARIZED MODEL

Two new variables are added to every product of variables x_ix_j where $i \neq j$, that appears in the objective function. In the general case of quadratic binary problem there are $\frac{n(n-1)}{2}$ such products as shown previously. Thus there are:

$$2 \times \frac{n}{2}(n-1) = n(n-1) \text{ new variables} \tag{7}$$

This gives a total of

$$n(n-1) \text{ new variables} + n \text{ original variables} = n^2 \text{variables.}$$

Also, two new constraints are added for every product of variables $x_i x_j$ where $i \neq j$ that appears in the objective function. Similarly, the total number of new constraints is given by:

$$n(n-1) \text{ extra constraints} \tag{8}$$

The total number of constraints $(\overline{m})$ is given by:

$$\overline{m} = m \text{ original constraints} + n(n-1) \text{ original constraints}$$
$$\overline{m} = (n^2 + m - n) \text{ variables} \tag{9}$$

LINEARISED QUADRATIC BINARY PROBLEM

Then linearised model becomes as follows:

$$\left.\begin{array}{l}
\text{Minimize: } Z = \sum_{r=1}^{\frac{n}{2}(n-1)} \overline{c}_r^0 \delta_r + \sum_{i}^{n} \overline{c}_k^1 x_k \\
\text{Such that } a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\
\quad \ldots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\
x_i + x_j = 2\delta_r + \overline{\delta}_r, \forall i \neq j \\
\delta_r + \overline{\delta}_r \leq 1, \forall i \neq j \\
x_i, x_j, x_k \in \{0,1\}, 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n \\
\delta_r, \overline{\delta}_r \in \{0,1\} \text{ and } r = 1, 2, \ldots, \frac{n(n-1)}{n}.
\end{array}\right\} \tag{10}$$

Numerical illustration

$$\left.\begin{array}{l}
\text{Minimize Z} = 10x_1 + 6x_2 + 7x_3 + 4x_4 + 2x_1x_2 \\
\quad + 3x_1x_3 + 5x_1x_4 + 5x_2x_3 + 9x_2x_4 \\
\quad + 8x_3x_4 + 11x_1^2 + 13x_2^2 + 12x_3^2 + 10x_4^2 \\
\text{Such that} \\
10x_1 + 18x_2 + 16x_3 + 16x_4 \geq 47 \\
20x_1 + 14x_2 + 18x_3 + 23x_4 \geq 52 \\
17x_1 + 21x_2 + 14x_3 + 19x_4 \geq 49 \\
x_j \in \{0,1\}, j = 1, 2, 3, 4
\end{array}\right\} \tag{11}$$

Making the model linear

The first stage is to linearise by making the following substitutions:

$$x_j^2 = x_j, \; j = 1, 2, 3, 4 \tag{12}$$

$$x_i x_j = \delta_r, \; r = 1, 2, 3, 4, 5, 6 \; \forall i \neq j. \tag{13}$$

For every new variable introduced two new constraints are generated.

$$\left.\begin{array}{l}
x_i + x_j = 2\delta_r + \overline{\delta}_r, \forall i \neq j \\
\delta_r + \overline{\delta}_r \leq 1, \forall i \neq j \\
\delta_r, \overline{\delta}_r \in \{0,1\} \text{ and } r = 1, 2, \ldots, 6.
\end{array}\right\} \tag{14}$$

The linear model becomes:

$$\left.\begin{array}{l}
\text{Minimize } z = 21x_1 + 19x_2 + 19x_3 + 14x_4 + 2\delta_1 \\
\quad + 3\delta_2 + 5\delta_3 + 5\delta_4 + 9\delta_5 + 8\delta_6 \\
\text{Such that:} \\
10x_1 + 18x_2 + 16x_3 + 16x_4 \geq 47 \\
20x_1 + 14x_2 + 18x_3 + 23x_4 \geq 52 \\
17x_1 + 21x_2 + 14x_3 + 19x_4 \geq 49 \\
\left.\begin{array}{l}
x_1 + x_2 = 2\delta_1 + \overline{\delta}_1 \\
x_1 + x_3 = 2\delta_2 + \overline{\delta}_2 \\
x_1 + x_4 = 2\delta_3 + \overline{\delta}_3 \\
x_2 + x_3 = 2\delta_4 + \overline{\delta}_4 \\
x_2 + x_4 = 2\delta_5 + \overline{\delta}_5 \\
x_3 + x_4 = 2\delta_6 + \overline{\delta}_6
\end{array}\right\} \text{new equality constraints} \\
\left.\begin{array}{l}
\delta_1 + \overline{\delta}_1 \leq 1 \\
\delta_2 + \overline{\delta}_2 \leq 1 \\
\delta_3 + \overline{\delta}_3 \leq 1 \\
\delta_4 + \overline{\delta}_4 \leq 1 \\
\delta_5 + \overline{\delta}_5 \leq 1 \\
\delta_6 + \overline{\delta}_6 \leq 1
\end{array}\right\} \text{new equality constraints}
\end{array}\right\} \tag{15}$$

Thus there are $4(4-1) = 12$ new variables and $4(4-1) = 12$ new constraints.

Solution

Solving the linear binary problem, the optimal solution is obtained as:

$$x_2 = x_3 = x_4 = \overline{\delta}_1 = \overline{\delta}_2 = \overline{\delta}_3 = \delta_4 = \delta_5 = \delta_6 = 1 \quad (16)$$

$$x_1 = \delta_1 = \delta_2 = \delta_3 = \overline{\delta}_4 = \overline{\delta}_5 = \overline{\delta}_6 = 0 \quad (17)$$

The solution to the original problem becomes:

$z = 74, x_2 = x_3 = x_4 = 1$ and $x_1 = 0$ (18)

REDUCING THE NUMBER OF EXTRA CONSTRAINTS IN THE LINEAR MODEL

Solving a linear model with $n(n-1)$ new constraints and $n(n-1)$, new variables becomes very difficult for large *n*. It is possible to halve the number of new constraints and variables. The following two constraints can be combined into one:

$$x_i + x_j = 2\delta_r + \overline{\delta}_r$$

$$\delta_r + \overline{\delta}_r \le 1$$

The first constraint can be expressed as:

$$x_i + x_j = \delta_r + \delta_r + \overline{\delta}_r \quad (19)$$

$$x_i + x_j - \delta_r = \delta_r + \overline{\delta}_r \quad (20)$$

Rearranging variables in the new equality constraints:

$$\left.\begin{array}{l} x_1 + x_2 = \delta_1 + \delta_1 + \overline{\delta}_1 \\ x_1 + x_3 = \delta_2 + \delta_2 + \overline{\delta}_2 \\ x_1 + x_4 = \delta_3 + \delta_3 + \overline{\delta}_3 \\ x_2 + x_3 = \delta_4 + \delta_4 + \overline{\delta}_4 \\ x_2 + x_4 = \delta_5 + \delta_5 + \overline{\delta}_5 \\ x_3 + x_4 = \delta_6 + \delta_6 + \overline{\delta}_6 \end{array}\right\} \quad (21)$$

Transposing one of the two δ_r to the left hand side and leaving the other on the right:

$$\left.\begin{array}{l} x_1 + x_2 - \delta_1 = \delta_1 + \overline{\delta}_1 \\ x_1 + x_3 - \delta_2 = \delta_2 + \overline{\delta}_2 \\ x_1 + x_4 - \delta_3 = \delta_3 + \overline{\delta}_3 \end{array}\right\} \quad (22a)$$

$$\left.\begin{array}{l} x_2 + x_3 - \delta_4 = \delta_4 + \overline{\delta}_4 \\ x_2 + x_4 - \delta_5 = \delta_5 + \overline{\delta}_5 \\ x_3 + x_4 - \delta_6 = \delta_6 + \overline{\delta}_6 \end{array}\right\} \quad (22b)$$

Since $\delta_r + \overline{\delta}_r$ cannot exceed one, then the new equality constraints and new inequality can be combined into one:

$$x_i + x_j - \delta_r \le 1 \quad (23)$$

This reduces the number of new constraints and variables to $\frac{n(n-1)}{2}$.

The 6 new equality constraints and 6 new inequality constraints can be combined into only 6 inequality constraints as follows:

$$\left.\begin{array}{l} x_1 + x_2 - \delta_1 \le 1 \\ x_1 + x_3 - \delta_2 \le 1 \\ x_1 + x_4 - \delta_3 \le 1 \\ x_2 + x_3 - \delta_4 \le 1 \\ x_2 + x_4 - \delta_5 \le 1 \\ x_3 + x_4 - \delta_6 \le 1 \end{array}\right\} \quad (24)$$

The linear model given in the numerical illustration becomes:

$$\left.\begin{array}{l} \text{Minimize } z = 21x_1 + 19x_2 + 19x_3 + 14x_4 + 2\delta_1 \\ \qquad + 3\delta_2 + 5\delta_3 + 5\delta_4 + 9\delta_5 + 8\delta_6 \\ \text{Such that:} \\ 10x_1 + 18x_2 + 16x_3 + 16x_4 \ge 47 \\ 20x_1 + 14x_2 + 18x_3 + 23x_4 \ge 52 \\ 17x_1 + 21x_2 + 14x_3 + 19x_4 \ge 49 \\ x_1 + x_2 - \delta_1 \le 1 \\ x_1 + x_3 - \delta_2 \le 1 \\ x_1 + x_4 - \delta_3 \le 1 \\ x_2 + x_3 - \delta_4 \le 1 \\ x_2 + x_4 - \delta_5 \le 1 \\ x_3 + x_4 - \delta_6 \le 1 \end{array}\right\} \quad (25)$$

The lineraised problem has been significantly reduced in size but the optimal solution is still the same:

$z = 74, x_2 = x_3 = x_4 = 1$ and $x_1 = 0.$ (26)

The linearised QAP model with a reduced number of new constraints and variables is given by:

Minimize: $Z = \sum_{r=1}^{\frac{n}{2}(n-1)} \bar{c}_r^0 \delta_r + \sum_{i}^{n} \bar{c}_k^1 x_k$

Such that

$$\left.\begin{aligned} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ & \ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\ & x_i + x_j - \delta_r \le 1, \forall i \ne j, r. \end{aligned}\right\} \quad (27)$$

OTHER FORMULATIONS

Many combinatorial optimization problems have different but equivalent mathematical formulations. One such problem is the QAP and its formulations have different structural characteristics leading to different solution approaches. Besides the reduced linearised form presented in this paper, there are other formulations available in literature. These formulations include concave, trace and Kronecker product formulations. The concave quadratic formulation was introduced by Bazaraa and Sherali in 1982. A cutting plane procedure was derived to find the optimal solution and this exact method was found to be computationally inefficient. The trace formulation was first used by Edwards (1977). The formulation was later used by Finke et al. (1987) to introduce the eigen low bounding technique for QAPs that are symmetric. Solving for the exact solution of the trace formulated problem has been proved to be difficult. There is no consistent computationally efficient and effective general purpose solution algorithm for the trace formulated problem. The use of the Kronecker product is another way of formulating the difficult QAP.

Lawler (1963) suggested an alternative way of formulating using the Kronecker product. Though he managed to linearise the Kronecker product formulation, the feasible solution that results from his formulation cannot be solved efficiently by the available techniques. More on Kronecker product formulation in detail can be found in Graham (1981).

CONCLUSIONS

The linearised model proposed in this paper is simpler than the versions available in literature. Both the number of new constraints and variables in the linearised model increase by factor of $\frac{n(n-1)}{2}$. This is a significant improvement and attempts will be made in future to lower this factor. Quadratic assignment problems with $n \ge 30$ are huge problems and usually take days to solve or approximate. It is computationally cheaper to solve a mathematical model with half the number of new constraints and variables than the original full model. Attempts will also be made to develop a heuristic that can efficiently determine a bound from the proposed linearised QAP model. A bound is useful in the case of a fire disaster, locust invasion or military operations where very quick location and facility decisions are required and exact optimal solutions are not usually possible to obtain in the limited time.

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