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Full Length Research Paper

Rank reversal problem related to wash criterion in analytic hierarchy process (AHP)

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The rank reversal problem related to wash criterion in Analytic Hierarchy Process (AHP) was studied. The purpose of this paper is threefold. Firstly, the rank reversal problem did not come off when proportional adjustment of Liberatore and Nydick (2004) and Saaty and Vargas (2006) was applied. Secondly, the conditions that insure the occurrence of rank reversal by using the approach of Finan and Hurley (2002) and Lin, Chou, Chouhuang and Hsu (2008) were found. Thirdly, the study showed that the invariant phenomenon proposed by Jung, Wou, Li and Julian (2009) implied that the combined criterion is still a wash criterion. The findings will help researchers select the synthesizing method and keep the rank under some specific range when a wash criterion is deleted.

Key words: Analytic hierarchy process, rank reversal problem, wash criterion.

INTRODUCTION

A criterion, say J_0 is defined as a wash criterion if all alternatives have the same weight corresponding to J_0 (Finan and Hurley, 2002). They mentioned that sometimes during medical research some criteria (sub-criteria) in the second bottom level are wash criteria. If researchers are allowed to delete those wash criteria without influencing the final ranking, then the deletion of those wash criteria will simplify the process of constructing comparison matrices in the upper level. Deleting wash criteria will bring us a lot of economical benefits when assessing alternatives by Analytic Hierarchy Process (AHP). Hence, they started to pay attention to rank reversal problems with wash criteria. The theorem was proved for any three-level hierarchy with a wash criterion which will not influence the final ranking of alternatives. Next, they mentioned that any hierarchy can be compressed into a three-level hierarchy. Finally, they constructed a four-level hierarchy problem with and without wash criteria to imply a rank reversal phenomenon. They then raised the question of "Whether or not a wash criterion could be deleted". It was announced that their findings were a severe challenge to the legitimacy of AHP (Finan and Hurley, 2002).

Up to now, there have been several papers discussing the wash criteria for rank reversal problems in the AHP. The upper level relative weights should be reassessed after the deletion of wash criterion so that the researcher develops another AHP problem. This way whether or not rank reversal happened is irrelevant (Liberatore and Nydick, 2004; Saaty and Vargas, 2006). They also suggested that the upper level weight should be evaluated by experts again after the deletion of wash criterion. Lin et al. (2008) tried to revise Finan and Hurley (2002) to point out that their theorem is incomplete and the entries in the comparison matrix did not satisfy the 1-9 bound criterion (Saaty, 1980). After the modification of their entries, the rank reversal phenomenon disappeared.

Recently, Jung et al. (2009) tried to settle the dispute among them by discovering an invariant subspace for the final synthesizing weight. They asserted that the invariant phenomenon may be useful to highlight the character to decide the weight for multiple objective decision making problems by AHP. Finally, they found conditions to insure that the rank reversal problem will not occur for wash criterion. They also indicated that their findings will be useful to resolve the debate for rank reversal problems in

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Table 1. AHP with wash criterion.

Criterion	Goal				
		J		J'	
		a		$1-a$	
	J_0	J_1	J_2	J_1'	J_2'
	b	c	$1-b-c$	d	$1-d$
A_1	0.5	e	f	g	h
A_2	0.5	$1-e$	$1-f$	$1-g$	$1-h$

AHP. However, there still exist some questionable results. We revised their findings and pointed out the meaning of their approach and explained the original problem proposed in Jung et al. (2009) and then prepared our solution. The unnecessary assumption of the unchanged value being 0.5 in Jung et al. (2009) can be removed.

MATERIALS AND METHODS

Analysis

Some of the theoretical concepts used in this paper are briefly introduced in this section. These include original version proposed by Saaty (1980) and modified version by various authors.

Wash criteria

There are four different approaches to deal with weights related to wash criteria: (a) with a wash criterion, (b) without a wash criterion (Finan and Hurley, 2002; Lin et al., 2008), (c) without a wash criterion (Liberatore and Nydick, 2004; Saaty and Vargas, 2006), and (d) invariant phenomenon (Jung et al. 2009). We adopted the decision problem in the paper proposed by Finan and Hurley (2002) with four-level hierarchy: (a) the top level: the goal, (b) the second top level: criteria, J and J', (c) the second bottom level: sub-criteria J_0, J_1, J_2, J_1' and J_2', and (d) the bottom level: alternatives A_1 and A_2, where J_0 is a wash criterion. The abstract entries for comparison matrix are listed in Table 1.

J_0 is a wash criterion, so the relative weight of A_j to J_0 is 0.5 for $j = 1, 2$. Next, we considered the relative weight for comparison matrix after the deletion of the wash criterion. The total weight for the original case is computed as;

$$a[b + c + (1-b-c)] + (1-a)[d + (1-d)] = 1 \tag{1}$$

and the total weight after the deletion of J_0 is computed as;

$$a[c + (1-b-c)] + (1-a)[d + (1-d)] = 1 - ab \tag{2}$$

Based on Equation (2), the relative weights without wash criterion J_0 must be adjusted to satisfy the constraint in which the total weight is one.

Two approaches without a wash criterion

There are two different approaches without a wash criterion: (a) applied by Finan and Hurley (2002) and Lin et al. (2008), the relative weight in the next upper level for J and J' did not need to be modified, and (b) applied by Liberatore and Nydick (2004) and Saaty and Vargas (2006), the relative weight in the next upper level for J and J' should be renormalized according to the remaining total weights.

For approach (a), Finan and Hurley (2002) and Lin et al. (2008) assumed that with or without a wash criteria, the weights of J and J' corresponding to the goal should be kept the same. The third row of the hierarchy did not change. They only modified the relative weights for J_1 and J_2 relative to J. If the wash criterion, J_0 is deleted, the total weight corresponding to J is left as $1-b$. So researchers revised the relative weight of J_1 and J_2 to J, from c and $1-b-c$, to $(1-b)^{-1}c$ and $(1-b)^{-1}(1-b-c)$, respectively. Since the total weight is changed from one to $1-b$, they multiplied $\frac{1}{1-b}$ to the relative weights in Table 1 so that the total weight for J_1 and J_2 becomes one. For easy comparison, we have listed the detailed results in the following Table 2.

For approach (b) proposed by Liberatore and Nydick (2004) and Saaty and Vargas (2006), not only should the relative weights of J_1 and J_2 be revised, but also J and J'. Owing to the fact that after the deletion of wash criterion J_0, the weight for J is changed from $b + c + (1-b-c) = 1$ to $c + (1-b-c) = 1-b$, the total weights corresponding to J and J' are left as $a(1-b)$ and $1-a$, respectively. Therefore, the weights in the upper level should be revised proportionally to the total weight of $a(1-b) + 1 - a = 1 - ab$. In the following, we use the existing data to get the new relative weights for J and J'. The revised weights for J and J' are renormalized as $\frac{a(1-b)}{1-ab}$ and $\frac{1-a}{1-ab}$, respectively. We have listed the detailed results in the following Table 3.

Jung et al. (2009) claimed that they have discovered conditions to preserve the consistency of the final weight for A_1 and A_2 with or without wash criterion J_0. Moreover, they predicted that their findings will cease the debate among Finan and Hurley (2002), Liberatore and Nydick (2004), Saaty and Vargas (2006) and Lin et al. (2008). We briefly quoted the results in Jung et al. (2009) for later discussion.

Weight for A_1

The final weight for A_1, under three approaches: (1) with wash

Table 2. AHP without J_0, based on Finan and Hurley (2002) and Lin et al. (2008).

	Goal			
	J		J'	
	a		$1-a$	
	J_1	J_2	J'_1	J'_2
	$\frac{c}{1-b}$	$\frac{1-b-c}{1-b}$	d	$1-d$
A_1	e	f	g	h
A_2	$1-e$	$1-f$	$1-g$	$1-h$

Table 3. AHP without J_0, based on Liberatore and Nydick (2004) and Saaty and Vargas (2006).

	Goal			
	J		J'	
	$\frac{a(1-b)}{1-ab}$		$\frac{1-a}{1-ab}$	
	J_1	J_2	J'_1	J'_2
	$\frac{c}{1-b}$	$\frac{1-b-c}{1-b}$	d	$1-d$
A_1	e	f	g	h
A_2	$1-e$	$1-f$	$1-g$	$1-h$

criterion J_0, denoted as $w(J_0, A_1)$ (2) without wash criterion J_0, derived by Finan and Hurley (2002), and Lin et al. (2008), denoted as $w(\not{J}_0, A_1)_{F,L}$, and (3) without wash criterion J_0, derived by Liberatore and Nydick (2004) and Saaty and Vargas (2006), denoted as $w(\not{J}_0, A_1)_{L,S}$, is expressed by;

$$w(J_0, A_1) = \frac{ab}{2} + \alpha + \beta, \qquad (3)$$

$$w(\not{J}_0, A_1)_{F,L} = \frac{\alpha}{1-b} + \beta, \qquad (4)$$

and

$$w(\not{J}_0, A_1)_{L,S} = \frac{(1-b)\alpha}{(1-ab)(1-b)} + \frac{\beta}{1-ab} \qquad (5)$$

Where, $\alpha = ace + a(1-b-c)f$ and $\beta = (1-a)dg + (1-a)(1-d)h$. They have created a numerical example with following data: $a = 0.8$, $b = 0.2591$, $c = 0.1396$, $d = 0.1667$, $e = 0.5$, $f = 0.5$, $g = 0.5$ and $h = 0.5$ to imply that

$$w(J_0, A_1) = 0.5 = w(\not{J}_0, A_1)_{F,L} = w(\not{J}_0, A_1)_{L,S}. \qquad (6)$$

These three different approaches have the same weight for A_1. Jung et al. (2009) believed that they have discovered some invariant phenomenon in AHP that deserves further investigation. They solved Equation (6) to find that;

$$\alpha = \frac{a(1-b)}{2} \qquad (7)$$

and;

$$\beta = \frac{2-a-b}{2}. \qquad (8)$$

Consequently, Jung et al. (2009) assumed that a and b are given, c, d, e, f, g and h satisfy the following two conditions,

$$\frac{1-b}{2} = ce + (1-b-c)f \qquad (9)$$

and

$$\frac{2-a-b}{2(1-a)} = dg + (1-d)h, \qquad (10)$$

then the final weight for A_1 will be the same for the three different approaches. It seems that they have discovered a six dimensional invariant subspace for rank reversal problem in AHP that deserves further study.

Moreover, they discovered a six dimensional invariant subspace with or without wash criteria under the condition $0 < b + c < 1$ after their mathematical derivation when they announced that there is an eight dimensional problem with parameters, a, b, c, d, e, f, g and h in (0,1) By applying three different approaches, the final weights are all the same. However, they could not provide any explanation from the operational research view point in AHP for their results in Equations (9) and (10).

Questionable results in Jung's findings and research revisions

If we assume that $g = h = 0.5$ and $a = 0.8$ and plug them into Equation (10), which implies that $b = 1$. It violates $0 < b + c < 1$ and indicates their derivation contains questionable results. We examined their results and found that they have misused Equation (5) as;

$$w(J_0, A_1)_{L,S} = \frac{(1-b)\alpha}{(1-ab)} + \frac{\beta}{1-ab}. \qquad (11)$$

All the other derivations afterwards contain questionable results. Form Equation (6), the corrected expression should be;

$$\frac{ab}{2} + \alpha + \beta = \frac{\alpha}{1-ab} + \frac{\beta}{1-ab} = \frac{\alpha}{1-b} + \beta = \frac{1}{2} \qquad (12)$$

This implies Equation (7) and the revision of Equation (8) as;

$$\beta = \frac{1-a}{2} \qquad (13)$$

Therefore, the conditions that guarantee the validity of Equation (6) are;

$$\frac{1}{2} = \frac{c}{1-b}e + \frac{1-b-c}{1-b}f \qquad (14)$$

and

$$\frac{1}{2} = dg + (1-d)h. \qquad (15)$$

RESULTS AND DISCUSSIONS

After the derivation of conditions to insure the validity of Equation (6), Jung et al. (2009) did not provide further explanation for their findings. In the following, we will provide a reasonable explanation for our findings. Let us recall Table 2, the left hand side of Equation (14) is the relative weight of A_1 corresponding to criterion J, so Equation (14) indicates that after deleting J_0, the remaining criterion, J_1 union J_2, denoted as $J = J_1 \oplus J_2$, is a wash criterion. The same phenomenon happens for Table 3. Hence, for two different approaches: by Finan and Hurley (2002) and Lin et al. (2008), in Table 2, and by Liberatore and Nydick (2004) and Saaty and Vargas (2006) in Table 3, $J_1 \oplus J_2$ is a wash criterion. Consequently, $J_0 \oplus J_1 \oplus J_2$ is also a wash criterion. Similarly, $J' = J_1' \oplus J_2'$ is a wash criterion, as well.

Further argument

Based on the aforementioned discussion, every criterion becomes a wash criterion when certain conditions are fulfilled. Therefore, the alternatives A_1 and A_2 have the same weight 0.5 which is a predictable result. Moreover, the three different approaches: (a) with a wash criterion, (b) without a wash criterion, proposed by Finan and Hurley (2002) and Lin et al. (2008) and (c) without a wash criterion, proposed by Liberatore and Nydick (2004) and Saaty and Vargas (2006), clearly implies that the final weight of A_1 and A_2 is the same weight of 0.5.

For completeness, we considered their numerical example for those strange combination of values, $a = 0.8$, $b = 0.2591$, $c = 0.1396$ and $d = 0.1667$. If we exploit some value for those parameters so that $e = 0.5$, $f = 0.5$, $g = 0.5$ and $h = 0.5$, then Equations (14) and (15) are valid for any combination of $a, b, c,$ and d in $(0,1)$ when it satisfies $1 > b + c$. It also points out that this strange combination of values, $a = 0.8$, $b = 0.2591$, $c = 0.1396$ and $d = 0.1667$ is a mirage in the distance.

Revision of Jung's approach

Jung's results

Jung et al. (2009) tried to find conditions to guarantee that the weights are the same when derived by three different approaches as mentioned above. However, they used it to solve the following problem

$$w(J_0, A_1) = w(\not{J}_0, A_1)_{F,L} = w(\not{J}_0, A_1)_{L,S} = \frac{1}{2}. \quad (16)$$

We must point out that the revised problem should be improved as;

$$w(J_0, A_1) = w(\not{J}_0, A_1)_{F,L} = w(\not{J}_0, A_1)_{L,S}. \quad (17)$$

Research revision

On the other hand, we referred to the findings in previous section to construct two new criteria: $J_1 \oplus J_2$ and $J_1' \oplus J_2'$ to represent the synthesizing result for criteria in the next upper level. Hence, we use $m = \frac{ce + (1-b-c)f}{1-b}$ and $n = dg + (1-d)h$ to simplify the expression. Based on our simplified expression, Equation (17) is amended by Equations (3) and (5), so it yields that;

$$0.5ab + \alpha + \beta = (1-ab)^{-1}(\alpha + \beta), \quad (18)$$

From which we derive;

$$\alpha + \beta = 0.5(1-ab). \quad (19)$$

Again by Equation (17), we obtain that;

$$am + (1-a)n = (1-ab)^{-1}[a(1-b)m + (1-a)n] \quad (20)$$

which implies that $ab(1-a)m = ab(1-a)n$ and then;

$$m = n. \quad (21)$$

Equations (19) and (21) yield;

$$\alpha + \beta = 0.5(1-ab) = am(1-b) + (1-a)n = m(1-ab) \quad (22)$$

which will result in

$$m = n = 0.5. \quad (23)$$

Therefore, we found that $J_1 \oplus J_2$ and $J_1' \oplus J_2'$ are wash criteria. If we plug $m = n = 0.5$ into Equation (20), then

$$w(J_0, A_1) = w(\not{J}_0, A_1)_{F,L} = w(\not{J}_0, A_1)_{L,S} = 0.5. \quad (24)$$

We will summarize our findings in the next theorem.

Theorem 1

If we try to find conditions to guarantee that the final weights are the same by three different approaches then the unchanged value for A_1 must be 0.5

The original problem in Jung et al. (2009) is the rank reversal problem. However, they tried to find conditions to insure the weight will not change in two cases: (a) with a wash criterion, and (b) without a wash criterion. Unfortunately, our revision of their results and our improvement are all directed at different problem: fixing the weight. For example, with a wash criterion, say J_0, we assumed that the final weights are $w(J_0, A_1) = 0.4$ and $w(J_0, A_2) = 0.6$. First, approach (1), the problem is to find conditions to insure that without wash criterion $\not{J}_0$, then

$$w(\not{J}_0, A_1)_{F,L} < w(\not{J}_0, A_2)_{F,L} \quad (25)$$

can still be obtained. Similarly, if we study approach (2), the problem is finding conditions to insure that without wash criterion $\not{J}_0$, then

$$w(\not{J}_0, A_1)_{L,S} < w(\not{J}_0, A_2)_{L,S} \quad (26)$$

is still preserved.

Back to the rank reversal problem

In this section, we considered the rank reversal problem, so that the proposed problem should be improved as follows. If $w(J_0, A_1) \geq w(J_0, A_2)$, then

$$w(\not{J}_0, A_1)_{F,L} \geq w(\not{J}_0, A_2)_{F,L} \quad (27)$$

and

$$w(\not{J}_0, A_1)_{L,S} \geq w(\not{J}_0, A_2)_{L,S} \quad (28)$$

are both valid. The rank reversal problem will not happen. From Equation (3), we know that $w(J_0, A_1) \geq w(J_0, A_2)$ is equivalent to;

$$\frac{ab}{2} + \alpha + \beta \geq \frac{ab}{2} + [a(1-b) - \alpha] + [(1-a) - \beta], \quad (29)$$

which we then simplify Equation (29) to obtain

$$2(\alpha + \beta) \geq 1 - ab. \quad (30)$$

On the other hand, by Equation (4),

$w(\not{J}_0, A_1)_{F,L} \geq w(\not{J}_0, A_2)_{F,L}$ is equal to

$$\frac{\alpha}{1-b} + \beta \geq \frac{a}{1-b}\left(1-b-\frac{\alpha}{a}\right) + (1-a)\left(1-\frac{\beta}{1-a}\right), \quad (31)$$

that is

$$2\alpha + 2\beta \geq 1 - b + 2b\beta . \quad (32)$$

Moreover, from Equation (5), $w(\not{J}_0, A_1)_{L,S} \geq w(\not{J}_0, A_2)_{L,S}$ is equivalent to

$$\frac{1-b}{1-ab} a \frac{\alpha}{(1-b)a} + \frac{1-a}{1-ab}\left(\frac{\beta}{1-a}\right) \geq \frac{1-b}{1-ab} a \left(\frac{1}{1-b}\right)\left(1-b-\frac{\alpha}{a}\right) + \frac{1-a}{1-ab}\left(1-\frac{\beta}{1-a}\right), \quad (33)$$

which can be simplified as:

$$2(\alpha + \beta) \geq 1 - ab . \quad (34)$$

If we compare Equation (30) with (34), they are identical which also implies that $w(J_0, A_1) \geq w(J_0, A_2)$ and $w(\not{J}_0, A_1)_{L,S} \geq w(\not{J}_0, A_2)_{L,S}$ are equivalent. It yields the same rank for alternatives if we follow the proportional adjustment to derive new relative weights of upper level suggested by approach (3) as (a) with wash criterion, J_0, and (b) without wash criterion, $\not{J}_0$. Hence, the rank reversal problem will not occur as predicted. We have summarized our results in the next theorem.

Theorem 2

The same rank will be kept if we follow the proportional adjustment of relative weights in the upper level as proposed by Liberatore and Nydick (2004), and Saaty and Vargas (2006) with a wash criterion and without a wash criterion so that the rank reversal problem will not happen.

If we compare Equations (30) with (32) to study the rank reversal problem, we will find the condition

$$N < 2(\alpha + \beta) < M , \quad (35)$$

where $N = \min\{1-ab, 1-b+2b\beta\}$ and $M = \max\{1-ab, 1-b+2b\beta\}$ for the rank reversal problem to happen.

Numerical example in Finan and Hurley (2002) and Lin et al. (2008)

Let us recall the numerical example in Finan and Hurley (2002). The following is a detailed list of the data: $a = 0.55$, $b = 0.6$, $c = 0.2$, $d = 0.5$, $e = 0.8$, $f = 0.4$, $g = 0.2$ and $h = 0.6$ which implies that $\alpha = 0.132$ and $\beta = 0.18$ to the Equation (35), which yields $N = 0.616 < 2(\alpha + \beta) = 0.624 < M = 0.67$. Hence, Finan and Hurley (2002) created a rank reversal example.

On the other hand, the condition for Equation (35) is $N = 0.66 < 2(\alpha + \beta) = 0.66 < M = 0.7$ when $a = 0.5$, $f = 1/3$, $h = 2/3$. Then, $\alpha = 34/300$ and $\beta = 13/60$ after the revision of some entries of the comparison matrix to satisfy the 1-9 scale bound proposed by Saaty (1980). The rank reversal problem will not happen as proposed by Lin et al. (2008) because the inequality in Equation (35) is invalid.

Conclusion

We show that the rank reversal phenomenon will not occur when a wash criterion is deleted if we apply the proportional adjustment to revise the upper level relative weights suggested by Liberatore and Nydick (2004), and Saaty and Vargas (2006). On the other hand, we found conditions where the rank reversal phenomenon will happen according to the approach proposed by Finan and Hurley (2002) and Lin et al. (2008). Moreover, we provided some patchwork for Jung et al. (2009) and prepared a reasonable explanation for the combination of their discoveries. Every criterion will be a wash criterion for the invariant phenomenon which was proposed by Jung et al. (2009). Based on our findings, researchers may apply our proposal to amend the relative weights after a wash criterion is deleted to avoid the rank reversal problem.

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