African Journal of Mathematics and Computer Science Research Vol. 4(13), pp. 389-395, December 2011
Available online at http://www.academicjournals.org/AJMCSR
ISSN 2006-9731 ©2011 Academic Journals

Full Length Research Paper

Estimation of temperature distribution and thermal stresses in a thick circular plate

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Accepted 20 October, 2011

A thick circular plate with the arbitrary initial heat flux prescribed on the upper surface is considered and lower and the curved boundary surface are kept at zero temperature. The temperature distribution in the plate is determined by solving heat conduction equation with the help of variable separation technique and then stresses are determined with the help of suitable Michell's function and Goodier's thermoelastic displacement potential function. The results are obtained in series form in terms of Bessel's functions and the temperature change, displacement function and thermal stresses have been computed numerically and illustrated graphically.

Key words: Quasi-static, steady state, thermoelastic problem, thermal stresses, thick plate.

INTRODUCTION

Thermal stresses play an important role in the design of high speed flight vehicles, machine structures and also in the fields of nuclear and chemical engineering. It is essential to determine the magnitude and influence of these stresses to make a realistic design of such components. It is well known that in an infinite plate a steady heat flow with a constant temperature gradient (uniform heat flow) does not induce thermal stresses provided there is no hole in the plate and no mechanical constraints are present at the outer edges.

In this paper a thick circular plate is considered and discussed its thermoelasticity with the help of the Goodier's thermoelastic displacement potential function and the Michell's function. To obtain temperature distribution the variable separation method is applied and thermal stresses are obtained by using Goodier's thermoelastic displacement potential function and the Michell's function. The results are obtained in series form in terms of Bessel's functions and the temperature change, displacement function and thermal stresses have been computed numerically and illustrated graphically.

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LITERATURE REVIEW

The investigation of stresses in a solid body is very important for modern engineering. During the past few decades, widespread attention has been given to the thermal stress problems in a thick plate. Qian and Batra (2004) studied transient thermoelastic deformation of thick functionally graded plate. Sharma et al. (2004) studied the behavior of thermoelastic thick plate under lateral loads and obtained the results for radial and axial displacements and temperature change have been computed numerically and illustrated graphically for different theories of generalized thermoelasticity. Nasser (2004, 2005) solved two dimensional problem of thick late with heat sources in generalized thermoelasticity. The thermoelastic analysis of thick walled finite length cylinders of functionally graded materials was given (Ruhi et al., 2005) and obtained the results for stress-strain and displacement components through the thickness and along the length are presented due to uniform internal pressure and thermal loading.

Most recently, Kulkarni and Deshmukh (2007) determined quasi-static thermal stresses in a thick circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature and the fixed circular edge thermally insulated while in this paper the thick circular plate one considers and discuss the thermal stresses. The circular plate is subjected to arbitrary initial heat flux on the upper face with lower face the curved surfaces are kept at zero temperature.

ANALYSIS

Consider a thick circular plate of thickness h defined by Let the arbitrary initial heat flux $F(r)$ is prescribed on the upper surface of the plate. Let lower and the curved boundary surface are kept at zero temperature. Under these conditions the quasi-static thermal stresses are required to be determined.

Heat conduction equation

Following the Kulkarni and Deshmukh (2007), the temperature of the thick circular plate at a time t, satisfies the heat conduction equation as given in Ozisik (1968)

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (1)$$

With boundary conditions

$$T(r,z,t)=0, \qquad r=a, -\frac{h}{2}\le z<\frac{h}{2}, t\ge 0 \quad (2)$$

$$-k\frac{\partial T}{\partial z}=F(r), \qquad z=\frac{h}{2}, 0\le r\le a, t=0 \quad (3)$$

$$z=-\frac{h}{2}, 0\le r\le a, t\ge 0. \quad (4)$$

Initially,

$$T(r,z,t)=T_i \qquad t=0, -\frac{h}{2}\le z\le\frac{h}{2}, 0\le r\le a, \quad (5)$$

Where, α and k are thermal diffusivity and thermal conductivity of the material of the plate respectively.

Displacement potential and thermal stresses

The governing differential equation for the Goodier's thermoelastic displacement potential ϕ is given by Noda et al. (2003)

$$\nabla^2\phi = K\,\tau, \quad (6)$$

where, K is the resistant coefficient and τ is the temperature change given by

$$\tau = T - T_i \quad (7)$$

In cylindrical co-ordinate system the displacements are represented by the Michell's function M as

$$u_r=\frac{\partial\phi}{\partial r}-\frac{\partial^2 M}{\partial r\,\partial z} \quad (8)$$

$$u_z=\frac{\partial\phi}{\partial z}+2(1-\nu)\nabla^2 M-\frac{\partial^2 M}{\partial z^2}. \quad (9)$$

The Michell's function M satisfies the equation

$$\nabla^2\nabla^2 M=0. \quad (10)$$

The components of the stress are represented in terms of the thermoelastic displacement potential ϕ and the Michell's function M as

$$\sigma_{rr}=2G\left\{\frac{\partial^2\phi}{\partial r^2}-K\tau+\frac{\partial}{\partial z}\left[\nu\,\nabla^2 M-\frac{\partial^2 M}{\partial r^2}\right]\right\} \quad (11)$$

$$\sigma_{\theta\theta}=2G\left\{\frac{1}{r}\frac{\partial\phi}{\partial r}-K\tau+\frac{\partial}{\partial z}\left[\nu\,\nabla^2 M-\frac{1}{r}\frac{\partial M}{\partial r}\right]\right\} \quad (12)$$

$$\sigma_{zz}=2G\left\{\frac{\partial^2\phi}{\partial z^2}-K\tau+\frac{\partial}{\partial z}\left[(2-\nu)\,\nabla^2 M-\frac{\partial^2 M}{\partial z^2}\right]\right\} \quad (13)$$

$$\sigma_{rz}=2G\left\{\frac{\partial^2\phi}{\partial r\,\partial z}+\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 M-\frac{\partial^2 M}{\partial z^2}\right]\right\} \quad (14)$$

Where, is shear modulus and ν be the Poisson's ratio. The boundary conditions on traction free surfaces are

$$\sigma_{zz}=0, \sigma_{rz}=0, \qquad z=-\frac{h}{2} \quad (15)$$

Equations (1) to (15) constitute the mathematical formulation of the problem.

Temperature distribution function

Assume that the temperature distribution in the thick circular plate is

$$T(r,z,t) = \sum_{m=1}^{\infty} J_0(\lambda_m r) \left(z+\frac{h}{2}\right) f_m(t) \tag{16}$$

Where, are the positive roots of the transcendental equation

$$J_0(\lambda_m a) = 0 \tag{17}$$

Where $J_n(x)$ is Bessel's function of the first kind of order n and the function $f_m(t)$ is to be etermined. Hence, the temperature distribution obtained as

$$T(r,z,t) = \sum_{m=1}^{\infty} A_m J_0(\lambda_m r) \left(z+\frac{h}{2}\right) e^{-\alpha \lambda_m^2 t} \tag{18}$$

where,

$$A_m = \frac{-2}{k\, a^2 J_1^2(\lambda_m a)} \int_0^a r' F(r') J_0(\lambda_m r')\, dr' \tag{19}$$

Using the initial condition (5) in equation (19), one obtains

$$T_i = \sum_{m=1}^{\infty} A_m J_0(\lambda_m r) \left(z+\frac{h}{2}\right) \tag{20}$$

Thus, the temperature change τ is obtained as

$$\tau = \sum_{m=1}^{\infty} A_m J_0(\lambda_m r) \left(z+\frac{h}{2}\right) \left(e^{-\alpha \lambda_m^2 t} - 1\right) \tag{21}$$

Michell's function

Now suitable form of Michell's function M satisfying equation (20) is given by

$$M = \sum_{m=1}^{\infty} J_0(\lambda_m r) \left[\!\left[B_m \sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] + C_m \left[\lambda_m\left(z+\frac{h}{2}\right)\right] \cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] \right]\!\right] \tag{22}$$

Where, B_m and C_m are arbitrary functions to be determined.

Goodier's thermoelastic potential function

Assume that displacement function $\phi(r,z,t)$ is given by

$$\phi = \sum_{m=1}^{\infty} D_m J_0(\lambda_m r) \left[\!\left[\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] - \left[\lambda_m\left(z+\frac{h}{2}\right)\right] \right]\!\right] \left(e^{-\alpha \lambda_m^2 t} - 1\right) \tag{23}$$

Putting Equations (21) and (23) in the Equation (6) and comparing coefficients, one obtains

$$D_m = \frac{K A_m}{\lambda_m^3} \tag{24}$$

So that the Goodier's thermoelastic potential function is obtained as

$$\phi = \sum_{m=1}^{\infty} \frac{K A_m}{\lambda_m^3} J_0(\lambda_m r) \left[\!\left[\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] - \left[\lambda_m\left(z+\frac{h}{2}\right)\right] \right]\!\right] \left(e^{-\alpha \lambda_m^2 t} - 1\right) \tag{25}$$

Displacement function and thermal stresses

Using the expressions (22) and (25) in the Equations (8), (9) and (11) to (14) and using conditions for the traction free surface one obtains expressions for displacement and thermal stresses as follows.

$$u_r = \sum_{m=1}^{\infty} -\frac{K A_m}{\lambda_m^2} J_1(\lambda_m r) \left[\!\left[\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] - \lambda_m\left(z+\frac{h}{2}\right) \right]\!\right] \left(e^{-\alpha \lambda_m^2 t} - 1\right) + \sum_{m=1}^{\infty} \left\{ \lambda_m^2 J_1(\lambda_m r) \left[\!\left[2(1-\nu) \cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] + \left[\lambda_m\left(z+\frac{h}{2}\right)\right] \sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] \right]\!\right] \right\} \tag{26}$$

$$u_z = \sum_{m=1}^{\infty} \left[\frac{K\,A_m}{\lambda_m^2} J_0(\lambda_m r) \left[\!\left[\cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] - 1\right]\!\right] \times \left(e^{-\alpha\,\lambda_m^2\,t} - 1\right) + \sum_{m=1}^{\infty}\left\{[(1-2\nu)\ \lambda_m^2 J_0(\lambda_m r)\,]\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] + \left[\lambda_m\left(z+\frac{h}{2}\right)\right]\cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right]\right]\!\right]\right\} \tag{27}$$

$$\frac{\sigma_{rr}}{2\,G} = \sum_{m=1}^{\infty}\left(-\frac{K\,A_m}{\lambda_m}\right) J_0(\lambda_m r)\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] \times \left(e^{-\alpha\,\lambda_m^2\,t} - 1\right) + \sum_{m=1}^{\infty}\frac{K\,A_m}{\lambda_m^2}\,\frac{J_1(\lambda_m r)}{r} \times \left[\!\left[\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] - \left[\lambda_m\left(z+\frac{h}{2}\right)\right]\right]\!\right]\left(e^{-\alpha\,\lambda_m^2\,t} - 1\right) + \sum_{m=1}^{\infty}(1-2\nu)\,\lambda_m^2\left[\lambda_m J_0(\lambda_m r) - \frac{J_1(\lambda_m r)}{r}\right] \times \cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] + \sum_{m=1}^{\infty}\left\{2\nu\,\lambda_m^3 J_0(\lambda_m r)\cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] + \lambda_m^2\left[\lambda_m J_0(\lambda_m r) - \frac{J_1(\lambda_m r)}{r}\right] \times \left[\!\left[\left[\lambda_m\left(z+\frac{h}{2}\right)\right]\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] + \cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right]\right]\!\right]\right\} \tag{28}$$

$$\frac{\sigma_{\theta\theta}}{2\,G} = \sum_{m=1}^{\infty}\left[\left(-\frac{K\,A_m}{\lambda_m^2}\right)\frac{J_1(\lambda_m r)}{r} \times \left[\!\left[\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] - \left[\lambda_m\left(z+\frac{h}{2}\right)\right]\right]\!\right]\left(e^{-\alpha\,\lambda_m^2\,t} - 1\right)\right] - \sum_{m=1}^{\infty} K\,A_m\ J_0(\lambda_m r)\left(z+\frac{h}{2}\right)\ \left(e^{-\alpha\,\lambda_m^2\,t} - 1\right) + \sum_{m=1}^{\infty}(1-2\nu)\,\lambda_m^2\,\frac{J_1(\lambda_m r)}{r}\cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] + \sum_{m=1}^{\infty}\left\{\lambda_m^2\left\{2\nu\lambda_m J_0(\lambda_m r)\cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] + \left[\frac{J_1(\lambda_m r)}{r}\right] \times \left[\!\left[\cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] + \left[\lambda_m\left(z+\frac{h}{2}\right)\right]\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right]\right]\!\right]\right\}\right\} \tag{29}$$

$$\frac{\sigma_{zz}}{2\,G} = \sum_{m=1}^{\infty}\left\{\frac{K\,A_m}{\lambda_m} J_0(\lambda_m r) \times \left[\!\left[\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] - \left[\lambda_m\left(z+\frac{h}{2}\right)\right]\right]\!\right]\left(e^{-\alpha\,\lambda_m^2\,t} - 1\right)\right\} - \sum_{m=1}^{\infty}\left[\!\left[\left[\lambda_m\left(z+\frac{h}{2}\right)\right]\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right]\right]\!\right] \tag{30}$$

$$\frac{\sigma_{rz}}{2\,G} = \sum_{m=1}^{\infty}\left\{\left(-\frac{K\,A_m}{\lambda_m}\right) J_1(\lambda_m r) \times \left[\!\left[\cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] - 1\right]\!\right]\left(e^{-\alpha\,\lambda_m^2\,t} - 1\right)\right\} + \sum_{m=1}^{\infty}\lambda_m^3 J_1(\lambda_m r)\left[\!\left[(1-\nu)\sinh\left[\lambda_m\left(z+\frac{h}{2}\right)\right] + \left[\lambda_m\left(z+\frac{h}{2}\right)\right]\cosh\left[\lambda_m\left(z+\frac{h}{2}\right)\right]\right]\!\right] \tag{31}$$

Special case and numerical calculations

To construct the mathematical thermoelastic model of a thick circular plate, one considers the following function and parameters.

Set

$$F(r) = T_0\,\delta(r-b), b < a \tag{32}$$

where, T_0 is a constant and $\delta(x)$is Dirac delta function of the argument x and

$$A_m = \frac{-2\,b\,T_0\ J_0(\lambda_m b)}{k\ a\ J_1^2(\lambda_m a)}$$

A plate with its thickness less than 1/5 of its smallest dimension is known as a thin plate. While if the thickness of disk exceed 1/5 of its smallest dimension it is termed as thick plate. Here one considers a thick circular plate for which the dimensions are considered as follows.

Dimensions

Radius of the cylinder	$a = 1\,m$
Constant for Dirac delta function	$b = 0.5\,m$
Thickness of circular plate	$h = 0.25\,m$

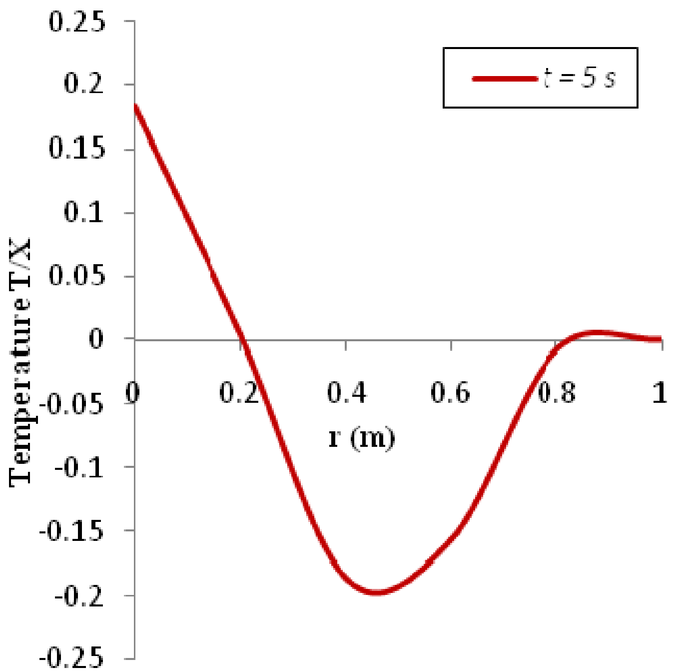

Figure 1. Temperature distribution along radial direction at upper face.

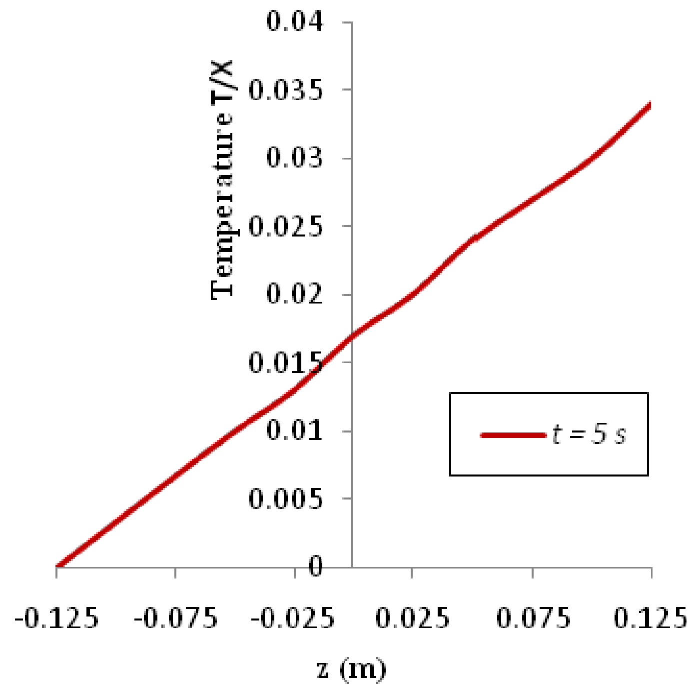

Figure 2. Temperature distribution along axial direction at $r = 0.1\ m$.

Material properties

The numerical calculations have been carried out for an Aluminum (pure) thick circular plate with the material properties as,

Thermal diffusivity $\alpha = 84.18 \times 10^{-6}\ m^2 s^{-1}$

Poisson ratio $\nu = 0.35$

Coefficient of linear thermal expansion

$a_t = 16.5 \times 10^{-6}\ \mathrm{K}^{-1}$

Lamé constant $\mu = 26.67$

Restraint coefficient $K = 1.686 \times 10^{-4}$

Thermal conductivity $k = 204.2\ W/mk$

Roots of the transcendental equation

The first ten positive roots of transcendental equation $J_0(\lambda_m\, a) = 0$ are

$\lambda_1 = 2.4048$ $\lambda_6 = 18.0711$
$\lambda_2 = 5.5201$ $\lambda_7 = 21.2116$
$\lambda_3 = 8.6537$ $\lambda_8 = 24.3525$
$\lambda_4 = 11.7915$ $\lambda_9 = 27.4935$
$\lambda_5 = 14.9309$ $\lambda_{10} = 30.6346$

For convenience one set,

$$X = \frac{2\, b\, T_0}{k\, a}, \quad A = 2G \times 10^{-6}$$

The computational mathematical software Mathcad-2000 has been used to carry out the numerical calculations and the graphs been plotted using Microsoft Office Excel 2007.

From Figure 1, it can be observed that the maximum temperature change occurs at centre of the plate and it decreases along radial direction in the region and then increases in the region $0.5 \le r \le 0.8\ m.$ Since outer curved boundary is maintained at zero temperature, the temperature distribution remains steady and negligible near outer curved surface. In Figure 2 the temperature distribution function is zero at lower face and linearly increases along axial direction. It is observed in Figure 3 that maximum radial stress developed at centre of the plate and it rapidly decreases towards radial position $r = 0.5\ m.$ In the region $0.5 \le r \le 1\ m$ it remains steady and negligible. In the Figure 4, it can be observed that the angular stresses decrease rapidly in the central part of the plate and afterward it remains steady and negligible towards outer curved surface. From Figure 5, it can be observed that the radial and angular stresses are plotted along axial direction and radial stress function increases from lower face to upper face while large stress can be observed in the region $0.025 \le z \le 0.125.$ The maximum

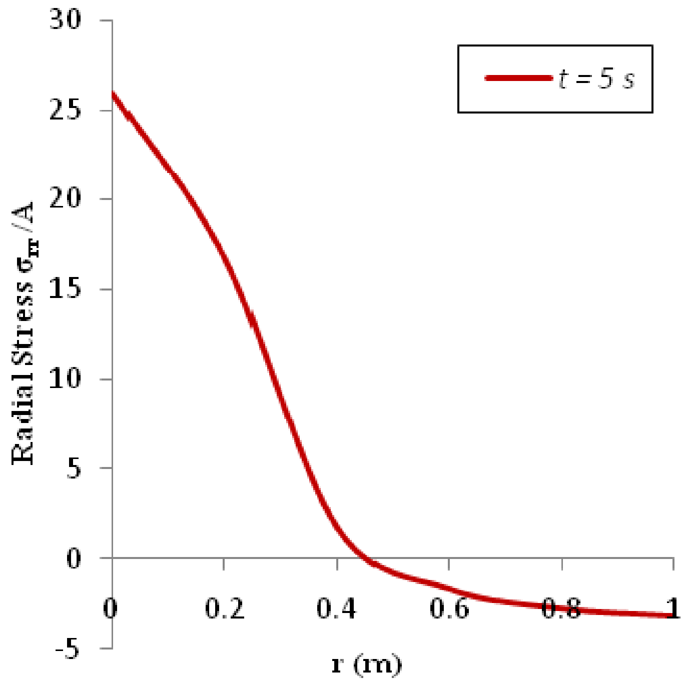

Figure 3. Radial stress function along radial direction at upper face.

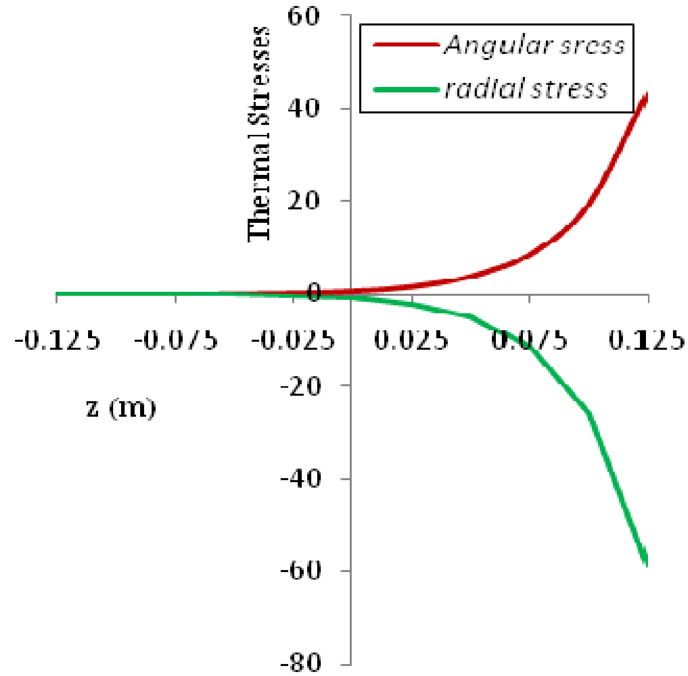

Figure 5. Radial and angular stresses along axial direction at $r = 0.1\ m$.

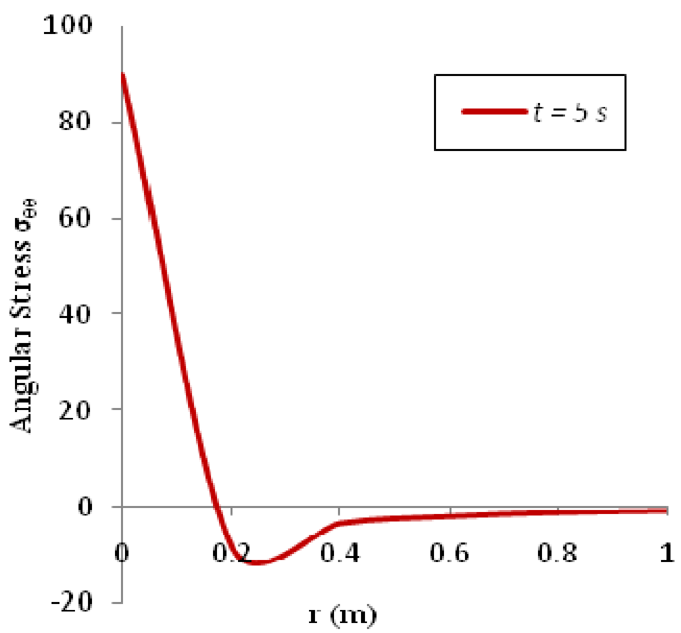

Figure 4. Angular stress function along radial direction at upper face

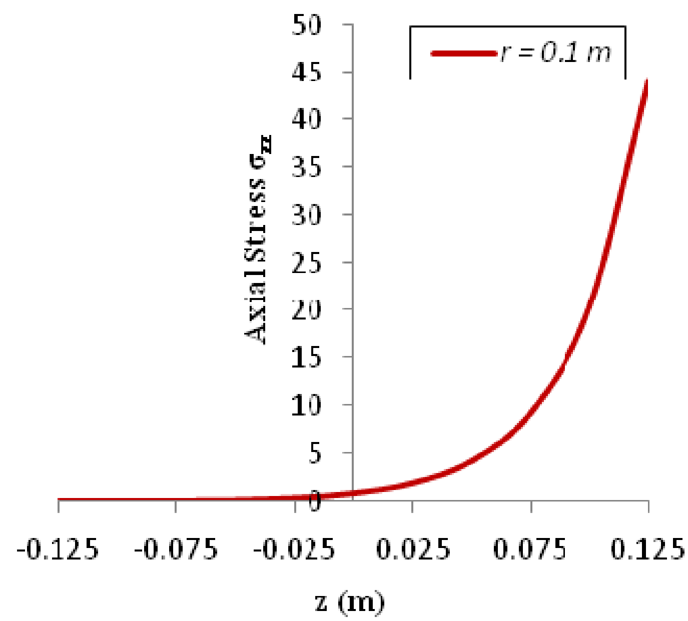

Figure 6. Axial stress along axial direction at 5=5.5 5.

increases exponentially along axial direction and large axial stress observed at upper face of the plate. Figure 7 illustrates the stress component σ_{rz} which rapidly decreases along radial direction in the central region and then monotonically increases towards outer curved surface of the plate. Figure 8 shows stress function component increases exponentially along axial direction and large stress observed at upper face of the plate.

CONCLUSION

In this problem one considered a thick circular plate and

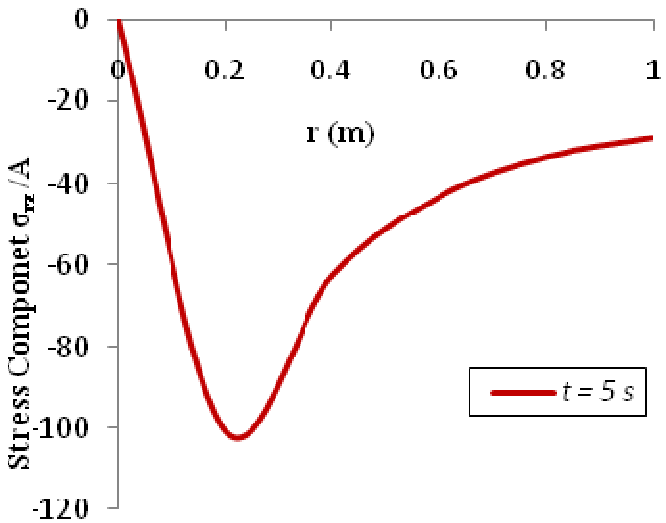

Figure 7. Stress component σ_{rz} along radial direction at upper face.

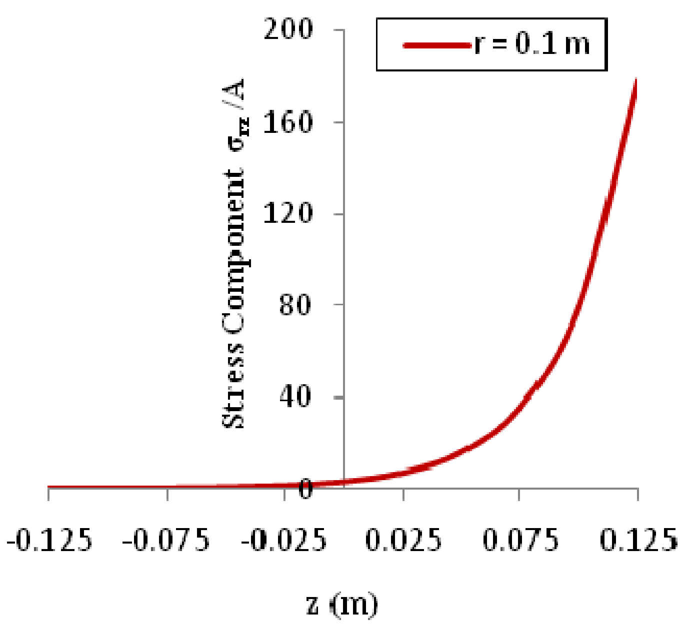

Figure 8. Stress component σ_{rz} along axial direction at 5=5.5 5.

determined the expressions of temperature, displacement and thermal stresses due to arbitrary initial heat flux on the upper face of the plate. As a special case, Mathematical model is constructed for Aluminum (pure) thick circular plate with the material properties specified in the numerical calculations.

Due to the initial heat flux at upper face large temperature change occurs at the upper surface of the plate. Temperature decreases along radial direction from centre to outer surface of the plate.

Due to large temperature changes at central part of the plate stresses are developed in this part. The radial stress develops tensile stresses in the central region $0 \leq r \leq 0.5\ m$ and afterward it is negligible towards outer surface. Similarly, the angular stress develops tensile stresses in the central region and afterward it is negligible towards outer surface. Along axial direction, radial stress develops tensile stresses while that of angular stress develops compressive stresses.

Axial stress components σ_{zz} and σ_{rz} increases along axial direction and achieves their maximum value at upper face. Both stress components develops tensile stresses in the plate along axial direction. The stress component σ_{rz} rapidly decreases along radial direction in the central region and then monotonically increases towards outer curved surface of the plate. The stress component σ_{rz} develops compressive stresses in the plate along axial direction.

One can summarize that due to initial heat flux at upper surface of the plate maximum stresses are developed at the upper face of the plate. Also, in the central part of the plate large stresses occurs.

ACKNOWLEDGEMENTS

Authors express their sincere thanks to the reviewers for their expert comments. Also the authors are thankful to university grants commission, New Delhi, India to provide the partial financial assistance under major research project and special assistance programme.

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