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Full Length Research Paper

On sandwich theorems of analytic functions involving Noor integral operator

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In this paper, we introduce sufficient conditions for subordination and superordination for subclass of analytic functions containing Noor integral operator. On the other hand, we prove the subordination and superordination results by some theorem studies and proofs. There are also obtained a sandwich results by different ways.

Key words: Noor integral operator, subordination, superordination.

INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be the class of functions analytic in U and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form;

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots .$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form;

$$f(z) = z + a_2 z^2 + \dots .$$

Let F and G be analytic functions in the unit disk U. The function F is $subordinate$ to G, written $F \prec G$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. In general terms, given two functions F and G, which are analytic in U, the function F is said to be subordinate to G in U if there exists a function h, analytic in U with $h(0) = 0 \quad and \quad |h(z)| < 1 \ \ for \ all \ z \in U$ such that

$$F(z) = G(h(z)) \ \ for \ all \ z \in U.$$

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination if $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$ then p is called a solution of the differenttial superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

Denote by $D^\alpha : \mathcal{A} \to \mathcal{A}$ the operator defined by

$$D^\alpha := \frac{z}{(1-z)^{\alpha+1}} * f(z), \ \ \alpha > -1,$$

Where; $(*)$ refers to the Hadamard product or convolution. Then implies that;

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \ \ n \in N_0 = N \cup \{0\}.$$

We note that $D^0 f(z) = f(z)$ and $D' f(z) = zf'$. The operator $D^n f$ is called Ruscheweyh derivative of nth order of f. Noor (1999) defined and studied an integral operator $I_n : \mathcal{A} \to \mathcal{A}$ analogous to $D^n f$ as follows:

Let $f_n(z) = \frac{z}{(1-z)^{n+1}}, \ n \in N_0$ and let $f_n^{(-1)}$ be defined

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such that;

$$f_n(z) * f_n^{(-1)}(z) = \frac{z}{1-z}. \qquad (1)$$

Then

$$I_n f(z) = f_n^{(-1)}(z) * f(z) = [\frac{z}{(1-z)^{n+1}}]^{(-1)} * f(z). \quad (2)$$

Note that $I_0 f(z) = zf'(z)$ and $I_1(z) = f(z)$. The operator I_n is called the Noor Integral of nth order of f. Using (1), (2) and a well known identity for $D^n f$, we have

$$(n+1)I_n f(z) - nI_{n+1}f(z) = z(I_{n+1}f(z))'. \qquad (3)$$

Using hypergeometric functions ${}_2F_1$, (2) becomes

$$I_n f(z) = [z_2F_1(1,1;n+1,z)] * f(z).$$

The following definitions can be found in (Noor and Noor, 1999):

Definition 1.1. Let $f \in \mathcal{A}$. Then $f \in S^\star$ (the starlike subclass of $\mathcal{A}$) if and only if for $z \in U$

$$\Re\{\frac{z[I_n f(z)]'}{I_n f(z)}\} > 0, \ \ n \in N_0.$$

Definition 1.2. Let $f \in \mathcal{A}$. Then $f \in N^\star_{(n)}, \ n \in N_0$ if and only if $I_n f \in S^\star$ (the starlike subclass of $\mathcal{A}$) for $z \in U$.

Definition 1.3. Let $f \in \mathcal{A}$. Then $f \in M^\star_{(n)}$ for $n \in N_0$ if and only if there exists $g \in N^\star_{(n)}$ such that, for $z \in U$,

$$\Re\{\frac{z[I_n f(z)]'}{I_n g(z)}\} > 0.$$

In the present work, we apply a method based on the differential subordination in order to obtain subordination results involving Noor Integral operator for a normalized analytic function f

$$q_1(z) \prec \frac{z[I_n f(z)]'}{I_n f(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z[I_n f(z)]'}{I_n g(z)} \prec q_2(z).$$

In order to prove our subordination and superordination results, we need to the following lemmas in the sequel.

Definition 1.4. (Miller and Mocanu, 2003) Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$ where;

$$E(f) := \{\zeta \in \partial U : lim_{z \to \zeta} f(z) = \infty\}$$

and are such that

$f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Lemma 1.1. (Miller and Mocanu, 2000) Let $q(z)$ be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$

When $w \in q(U)$.
Set$Q(z) := zq'(z)\phi(q(z)), \ h(z) := \theta(q(z)) + Q(z)$.

Suppose that

1. $Q(z)$ is starlike univalent in U, and
2. $\Re \frac{zh'(z)}{Q(z)} > 0$ for $z \in U$.

If : $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$

then

$p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 1.2. (Shanmugam et al., 2006) Let $q(z)$ be convex univalent in the unit disk U and ψ and $\gamma \in \mathbb{C}$ with

$$\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0. \qquad (4)$$

If; $p(z)$ is analytic in U and

$$\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z),$$

then

$p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.3. (Bulboaca, 2002) Let $q(z)$ be convex univalent in the unit disk U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that

1. $zq'(z)\varphi(q(z))$ is starlike univalent in U,
2. $\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} > 0$ for $z \in U$.

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\vartheta(p(z)) + zp'(z)\varphi(z)$ is univalent in U and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$$

then
$q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Lemma 1.4. (Miller and Mocanu, 2003) Let q be convex univalent in the unit disk U and $\gamma \in \mathbb{C}$. Further, assume that $\Re\{\overline{\gamma}\} > 0$. If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(z) + \gamma z p'(z)$ is univalent in U then

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z)$$

Implies $q(z) \prec p(z)$ and q is the best subordinant.

SANDWICH RESULTS

By making use of Lemmas 1.1 and 1.2, we prove the following subordination results.

Theorem 2.1. Let $q(z) \neq 0$ be univalent in U such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U and

$$\Re\{1 + \frac{\alpha}{\gamma} q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\} > 0, \quad (5)$$

for $\alpha,\ \gamma \in \mathbb{C},\ \gamma \neq 0.$

If $f \in \mathcal{A}$ satisfies the subordination

$$\alpha\{\tfrac{z[I_n f(z)]'}{I_n f(z)}\} + \gamma\{1 + \tfrac{z[I_n f(z)]''}{[I_n f(z)]'} - \tfrac{z[I_n f(z)]'}{I_n f(z)}\} \prec \alpha q(z) + \tfrac{\gamma z q'(z)}{q(z)}, \quad (6)$$

then

$\frac{z[I_n f(z)]'}{I_n f(z)} \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Our aim is to apply Lemma 1.1. Setting

$$p(z) := \frac{z[I_n f(z)]'}{I_n f(z)}.$$

By a simple computation shows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z[I_n f(z)]''}{[I_n f(z)]'} - \frac{z[I_n f(z)]'}{I_n f(z)}$$

which yields the following subordination

$$\alpha p(z) + \frac{\gamma z p'(z)}{p(z)} \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)},\ \ \alpha,\ \gamma \in \mathbb{C}.$$

By setting

$$\theta(\omega) := \alpha\omega \ \ and \ \ \phi(\omega) := \frac{\gamma}{\omega},\ \ \gamma \neq 0,$$

It can be easily observed that $\theta(\omega)$ is analytic in $\mathbb{C}$ and $\phi(\omega)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(\omega) \neq 0$ when $\omega \in \mathbb{C}\backslash\{0\}$. Also, by letting

$Q(z) = zq'(z)\phi(q(z)) = \gamma z \frac{q'(z)}{q(z)}$ and

$$h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \gamma z \frac{q'(z)}{q(z)},$$

We find that Q is starlike univalent in U and that

$$\Re\{\frac{zh'(z)}{Q(z)}\} = \{1 + \frac{\alpha}{\gamma} q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\} > 0.$$

Then the relation (6) follows by an application of Lemma 1.1.

Corollary 2.1. If $f \in \mathcal{A}$ and assume that (5) holds then

$$1 + \frac{z[I_n f(z)]''}{[I_n f(z)]'} \prec \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Az)(1+Bz)}$$

implies

$$\frac{z[I_n f(z)]'}{I_n f(z)} \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. By setting $\alpha = \gamma = 1$ and $q(z) := \frac{1+Az}{1+Bz}$ where $-1 \leq B < A \leq 1.$

Corollary 2.2. If $f \in \mathcal{A}$ and assume that (5) holds then

$$1 + \frac{z[I_n f(z)]''}{[I_n f(z)]'} \prec \frac{1+z}{1-z} + \frac{2z}{1-z^2}$$

Implies

$$\frac{z[I_n f(z)]'}{I_n f(z)} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant.

Proof: By setting $\alpha = \gamma = 1$ and $q(z) := \frac{1+z}{1-z}.$

Corollary 2.3: If $f \in \mathcal{A}$ and assume that (5) holds then

$$1 + \frac{z[I_n f(z)]''}{[I_n f(z)]'} \prec e^{Az} + Az$$

implies

$$\frac{z[I_nf(z)]'}{I_nf(z)} \prec e^{Az},$$

and e^{Az} is the best dominant.

Proof. By setting $\alpha = \gamma = 1$ and $q(z) := e^{Az}, \ |A| < \pi$.

Theorem 2.2: Let $q(z)$ be convex univalent in the unit disk U and $\gamma \in \mathbb{C}$ satisfies

$$\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\gamma}\} > 0, \ \gamma \in \mathbb{C}.$$

If $f \in M^{\star}_{(n)}$ for $n \in N_0$ and exists $g \in N^{\star}_{(n)}$ such that $\frac{z[I_nf(z)]'}{I_ng(z)}$ is analytic in U and the subordination

$$\frac{z[I_nf(z)]'}{I_ng(z)}\{1 + [1 + \frac{z(I_nf(z))''}{(I_nf(z))'} - \frac{z(I_ng(z))'}{I_ng(z)}]\}$$
$$\prec q(z) + \gamma zq'(z), \ \gamma \in \mathbb{C}$$

holds, then

$$\frac{z[I_nf(z)]'}{I_ng(z)} \prec q(z) \qquad (7)$$

and q is the best dominant.

Proof: Our aim is to apply Lemma 1.2. Setting

$$p(z) := \frac{z[I_nf(z)]'}{I_ng(z)}.$$

A computation shows that

$$zp'(z) = \frac{z[I_nf(z)]'}{I_ng(z)}[1 + \frac{z(I_nf(z))''}{(I_nf(z))'} - \frac{z(I_ng(z))'}{I_ng(z)}]$$

which yields the following subordination

$$p(z) + \gamma zp'(z) \prec q(z) + \gamma zq'(z), \ \gamma \in \mathbb{C}.$$

Thus in view of Lemma 1.2, (7) is held.

Theorem 2.3: Let $q(z) \neq 0$ be convex univalent in the unit disk U. Suppose that

$$\Re\{\frac{\alpha}{\gamma} \ q(z)\} > 0, \ \alpha, \ \gamma \in \mathbb{C} \ for \ z \in U \qquad (8)$$

and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If

$\frac{z[I_nf(z)]'}{I_nf(z)} \in \mathcal{H}[q(0)] \cap Q$

Where; $f \in \mathcal{A}$,
$\alpha\{\frac{z[I_nf(z)]'}{I_nf(z)}\} + \gamma\{1 + \frac{z[I_nf(z)]''}{[I_nf(z)]'} - \frac{z[I_nf(z)]'}{I_nf(z)}\}$ is univalent in U and the subordination

$$q(z) + \frac{\gamma zq'(z)}{q(z)}$$
$$\prec \alpha\{\frac{z[I_nf(z)]'}{I_nf(z)}\} + \gamma\{1 + \frac{z[I_nf(z)]''}{[I_nf(z)]'} - \frac{z[I_nf(z)]'}{I_nf(z)}\}$$

holds, then

$$q(z) \prec \frac{z[I_nf(z)]'}{I_nf(z)} \qquad (9)$$

and q is the best subordinant.

Proof: Our aim is to apply Lemma 1.3. Setting

$$p(z) := \frac{z[I_nf(z)]'}{I_nf(z)}.$$

A computation shows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z[I_nf(z)]''}{[I_nf(z)]'} - \frac{z[I_nf(z)]'}{I_nf(z)}$$

which yields the following subordination

$$\alpha q(z) + \frac{\gamma zq'(z)}{q(z)} \prec \alpha p(z) + \frac{\gamma zp'(z)}{p(z)}, \ \alpha, \ \gamma \in \mathbb{C}.$$

By setting

$$\vartheta(\omega) := \alpha\omega \ \ and \ \varphi(\omega) := \frac{\gamma}{\omega}, \ \gamma \neq 0,$$

It can be easily observed that $\theta(\omega)$ is analytic in $\mathbb{C}$ and $\phi(\omega)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(\omega) \neq 0$ when $\omega \in \mathbb{C}\backslash\{0\}$. Also, we obtain

$$\Re\{\frac{\vartheta(q(z))}{\varphi(q(z))}\} = \Re\{\frac{\alpha}{\gamma} \ q(z)\} > 0.$$

Then (9) follows by an application of Lemma 1.3.

Theorem 2.4: Let q be convex univalent in the unit disk U and $\gamma \in \mathbb{C}$. Further, assume that $\Re\{\overline{\gamma}\} > 0$.

If $\frac{z[I_nf(z)]'}{I_ng(z)} \in \mathcal{H}[q(0), 1] \cap Q$, with

$$\frac{z[I_nf(z)]'}{I_ng(z)}\{1+[1+\frac{z(I_nf(z))''}{(I_nf(z))'}-\frac{z(I_ng(z))'}{I_ng(z)}]\}$$

is univalent in U then

$$q(z)+\gamma zq'(z) \prec \frac{z[I_nf(z)]'}{I_ng(z)}\{1+[1+\frac{z(I_nf(z))''}{(I_nf(z))'}-\frac{z(I_ng(z))'}{I_ng(z)}]\}$$

implies

$$q(z) \prec \frac{z[I_nf(z)]'}{I_ng(z)} \qquad (10)$$

and q is the best subordinant.

Proof: Our aim is to apply Lemma 1.4. Setting

$$p(z) := \frac{z[I_nf(z)]'}{I_ng(z)}.$$

A computation shows that

$$zp'(z) = \frac{z[I_nf(z)]'}{I_ng(z)}[1+\frac{z(I_nf(z))''}{(I_nf(z))'}-\frac{z(I_ng(z))'}{I_ng(z)}]$$

which yields the following subordination

$$q(z)+\gamma zq'(z) \prec p(z)+\gamma zp'(z),\ \gamma \in \mathbb{C}.$$

Thus in view of Lemma 1.4, we obtain (10).

Combining Theorems 2.1 and 2.3, and also combining Theorems 2.2 and 2.4, we get the following Sandwich theorems:

Theorem 2.5: Let $q_1(z) \neq 0,\ q_2(z) \neq 0$ be convex univalent in the unit disk U satisfy (8) and (5) respectively. Suppose that and $\frac{zq_i'(z)}{q_i(z)},\ \ i=1,2$ is starlike univalent in U. If $\frac{z[I_nf(z)]'}{I_nf(z)} \in \mathcal{H}[q(0)] \cap Q$ where $f \in \mathcal{A}$,

$$\alpha\{\frac{z[I_nf(z)]'}{I_nf(z)}\}+\gamma\{1+\frac{z[I_nf(z)]''}{[I_nf(z)]'}-\frac{z[I_nf(z)]'}{I_nf(z)}\}$$

is univalent in U and the subordination

$$q_1(z)+\frac{\gamma zq_1'(z)}{q_1(z)} \prec \alpha\{\frac{z[I_nf(z)]'}{I_nf(z)}\}+\gamma\{1+\frac{z[I_nf(z)]''}{[I_nf(z)]'}-\frac{z[I_nf(z)]'}{I_nf(z)}\} \prec \alpha q_2(z)+\frac{\gamma zq_2'(z)}{q_2(z)}$$

holds, then

$$q_1(z) \prec \frac{z[I_nf(z)]'}{I_nf(z)} \prec q_2(z)$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.

Theorem 2.6. Let $q_1(z),\ q_2(z)$ be convex univalent in the unit disk U such that

$$\Re\{1+\frac{zq_2''(z)}{q_2'(z)}+\frac{1}{\gamma}\}>0,\ \ \gamma \in \mathbb{C},\ \ \Re\{\overline{\gamma}\}>0.$$

If $f \in M^{\star}_{(n)}$ for $n \in N_0$ and exists $g \in N^{\star}_{(n)}$ such that $\frac{z[I_nf(z)]'}{I_ng(z)} \in \mathcal{H}[q_1(0),1] \cap Q$, with

$$\frac{z[I_nf(z)]'}{I_ng(z)}\{1+[1+\frac{z(I_nf(z))''}{(I_nf(z))'}-\frac{z(I_ng(z))'}{I_ng(z)}]\}$$

is univalent in U then

$$q_1(z)+\gamma zq_1'(z) \prec \frac{z[I_nf(z)]'}{I_ng(z)}\{1+[1+\frac{z(I_nf(z))''}{(I_nf(z))'}-\frac{z(I_ng(z))'}{I_ng(z)}]\} \prec q_2(z)+\gamma zq_2'(z)$$

Implies

$$q_1(z) \prec \frac{z[I_nf(z)]'}{I_ng(z)} \prec q_2(z)$$

and $q_1(z)$ is the best subordinant. and $q_2(z)$ is the best dominant.

Conclusions

There are too many other results have been obtained by embedding the Noor integral in various other new form of classes, namely can be found in (Ibrahim and Darus, 2008a, b). Many other results can be obtained by using simple technique of differential subordination introduced by Miller and Mocanu (2000), and present study have provide sufficient conditions for subordination and superordination for subclass of analytic functions containing Noor integral operator. On the other hand, we prove the subordination and superordination results by some theorem studies and proofs.

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