African Journal of Mathematics and Computer Science Research Vol. 3(11), pp. 282-287, November 2010
Available online at http://www.academicjournals.org/AJMCSR
ISSN 2006-9731 ©2010 Academic Journals

Full Length Research Paper

A note on indirect least squares and matrix partitioning

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Accepted 26 August, 2010

At times, it is discovered that in using OLS in estimating an equation, inconsistent estimates are obtained because of correlation between the independent variable and the stochastic disturbance term. In such a circumstance, it is likely that the equation so estimated belongs to a wider family of equations related to the practical situation under consideration. Inevitably, a model describing the joint dependence of variables, called simultaneous – equation model evolves. In order to obtain consistent estimator, one may resort to indirect least squares (ILS) or two – stage least squares (2SLS). For an over-identified system, ILS through unguided coefficient technique (UCT) produces non-unique estimates for a just identified equation. Unique estimates can only be possible if ILS is approached through matrix partition techniques (MPT). The authors' objective in this paper is to prove the proposition stated above. The definitions of UCT and MPT are also given in the paper.

Key words: Least squares, matrix partitioning, unguided coefficient technique, matrix partition technique, simultaneous equation model.

INTRODUCTION

Indirect Least Squares (ILS) is an estimation method for obtaining consistent estimators of exactly identified equation in a system of simultaneous equations. This technique involves two major steps. The first is the estimation of reduced-form parameters Π using OLS. The second is the estimation of structural- form parameters B and Γ using the relationship between these parameters and the reduced-form parameters and the identifying restrictions.

Theoretical formulation and procedures of ILS techniques and relevant theorems and definitions are considered in the next section. In Section 3 an example is used to illustrate the objective of this paper. Section 4 is the conclusion.

A lot of publications abound everywhere on simultaneous – equations models. Some references that may help in the subject of this paper are Dhrymes (1970), Brundy and Jorgensn (1974), Intriligator (1978), Johnston (1984) and Essi (1991). More advanced work can be seen in the papers of Krishnakumar (1997), Poskitt and Skeels (2008), Prokhorov (2009) and Klein and Vella (2010).

THEORETICAL BACKGROUND

The simultaneous equation model (SEM) can be written in the form of,

$$\underset{1\times G}{y_i}\;\underset{G\times G}{\Gamma} + \underset{1\times k}{x_i}\;\underset{k\times G}{\mathrm{B}} = \underset{1\times G}{\varepsilon_i}\;, i = 1, 2, \ldots n. \quad (1)$$

where E(ε_i) = 0, for each period i.

Cov(ε_i) = E($\varepsilon_i'\varepsilon_i$) = $\underset{G\times G}{\sum}$ is positive definite matrix of variances and covariances such that E($\varepsilon_i'\varepsilon_j$) = 0 for all $i \neq j$. The variable y_i gives the vector of endogenous variables in period i and x_i is the vector of predetermined variables in the same period. Γ and B respectively accommodate the coefficients of endogenous and predetermined variables. In using the data matrices X and Y we can re-write Equation (1) as,

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$$\underset{nxG}{Y}\ \underset{GxG}{\Gamma} + \underset{n\,xk}{X}\ \underset{k\,xG}{B} = \underset{nxG}{E} \tag{2}$$

We shall refer to Equation (2) in due course.

Let the variables of the first equation of the system Equation (1) be renumbered with one endogenous variable being made a dependent variable by setting its coefficient equation to -1. In addition, let us impose a priori restrictions of zero coefficients on some coefficients of the equation such that only the first g_1 endogenous variables and only the first k_1 predetermined variables are included in the equation, the other $(G - g_1) + (k - k_1)$ having zero coefficients. The first equation can now be written as,

$$y_{ih} = \sum_{h=2}^{g_1} y_{ih}\gamma_{h1} + \sum_{j=2}^{k_1} x_{ij}\ B_{j1} - \varepsilon_{i1}$$

That is

$$y_{i1} = Y_{i1}\gamma_1 + X_{i1}B_1 - \varepsilon_{i1} \tag{3}$$

where $Y_{i1} = \left(y_{i2}, y_{i3}, \ldots y_{ig_1}\right)$, $X_{i1} = (x_{i1}, x_{i2}, \ldots x_{ik_i})$

$\gamma_1 = \left(\gamma_{21}\ \gamma_{31} \ldots \gamma_{g_1 1}\right)'$ and $\mathrm{B}_1 = \left(B_{11}\ B_{21} \ldots B_{k_1 1}\right)'$.

The vector Y_{i1} carries g_1 -1 explanatory endogenous variables included in the first equation; $X_{i1}$is the vector of k_1 predetermined variables included in the first equation; ε_{i1} is the stochastic term in the first equation. The vectors γ_1 and B_1 are respectively the g_1 -1 coefficients of explanatory endogenous and k_1 coefficients or predetermined variables included in the first equation.

We should recall that Equation (3) is the first equation in its i^{th} period. For all the n periods, it takes the form of,

$$\underset{nx1}{y_1} = \underset{nx(g_1-1)}{Y_1}\ \underset{(g_1-1)x1}{\gamma_1} + \underset{(n\,xk_1)}{X_1}\ \underset{(k_1 x1)}{\mathrm{B}_1} + \underset{nx1}{\in_1} \tag{4}$$

where $\in_1 = -(\varepsilon_{11}\ \varepsilon_{21} \ldots \varepsilon_{n1})$, using the data matrices X and Y with the following partitions:

$$\underset{n\times G}{Y} = \left(\underset{nx1}{y_1} \vdots \underset{n\times(g_1-1)}{Y_1} \vdots \underset{nx(G-g_1)}{Y_2} \right) \tag{5}$$

$$\underset{n\times k}{X} = \left(\underset{n\times k_1}{X_1} \vdots \underset{n\times(k-k}{X_2} \right)$$

"The matrices X and Y are as specified in Equation (2) and M indicates vertical matrix partitioning."

The matrix partition in Equation (5) is made use of in Equation (4). In Equation (5), the data matrix Y on all the endogenous variables of the system is partitioned into y_1 (column vector of data on the dependent endogenous variable), Y_1 (data matrix on the g_1-1 explanatory endogenous variables in Y_{i1}, that is in the first equation) and Y_2 is the matrix of data on the G- g_1 excluded endogenous variables. Similarly, the matrix of data on the predetermined variables X can be partitioned into X_1 (data matrix on k_1 included predetermined variables in X_{i1} that is in the first equation) and X_2 the data matrix on the k-k_1 excluded predetermined variables).

Now we can write Equation (4) in the alternative form,

$$\left(\underset{nx1}{y_1} \vdots \underset{nxg_1-1}{Y_1} \vdots \underset{nx(G-g_1)}{Y_2} \right) \begin{array}{c} \\ \left[\begin{array}{c:c} -1 & \\ \hdashline \gamma_1 & \\ \hdashline O & \end{array}\right] \\ \uparrow \\ 1^{st}\text{ Column of }\Gamma \end{array} + (X_1 \vdots X_2) \begin{array}{c} 1^{st}\text{ Column of B} \\ \downarrow \\ \left[\begin{array}{c:c} \mathrm{B}_1 & \\ \hdashline & \\ O & \end{array}\right] \end{array} = (-\in_1 \vdots) \tag{6}$$

At a glance, it is easy to see that Equation (6) is equivalent to,

$$-y_1 + Y_1\gamma_1 + X_1B_1 = -\in_1$$

The ILS techniques commences by estimating the reduced form,

$$\underset{n\times G}{Y} = \underset{n\times k}{X}\ \underset{k\times G}{\Pi} + \underset{n\times G}{U} \tag{7}$$

to obtain

$$\underset{k\times G}{\hat{\Pi}} = (X'X)^{-1}X'Y \tag{8}$$

where

$$\begin{aligned} \Pi &= -B\Gamma^{-1} \\ U &= E\Gamma^{-1} \end{aligned} \tag{9}$$

Equation (9) is obtained by using Equation (2) and Equation (7).

Let us assume that the first equation of the system under consideration is a just – identified structural equation. A judicious partitioning of Π can be carried out so that Equation (7) takes the form

$$\left(\underset{nx1}{Y_1} \vdots \underset{nx(g_1-1)}{Y_1} \vdots \underset{nx(g-g_1)}{Y_2} \right) = \left(\underset{nxk_1}{X_1} \vdots \underset{nx(k-k_1)}{X_2} \right) \begin{array}{cc} \left[\begin{array}{ccc} \Pi_{11} \vdots & \Pi_{12} \vdots & \Pi_{13} \\ \Pi_{21} \vdots & \Pi_{22} \vdots & \Pi_{23} \end{array}\right] & \begin{array}{l} k_1 \\ k-k_1 \end{array} \\ \begin{array}{ccc} 1 & (g_1-1) & (G-g_1) \end{array} & \end{array}$$

$$+\left(\underset{n\,x\,1}{U_1} \vdots \underset{n\,x\,(g_1-1)}{U_2} \vdots \underset{n\,x\,(G-g_1)}{U_3} \right) \quad (10)$$

The columns of Π have been divided to correspond to the one dependent endogenous variable, the g_1-1 explanatory endogenous variables and the G-g_1 excluded endogenous variables. Its rows have been partitioned to correspond to the k_1 included predetermined variables and k-k_1 excluded predetermined variables. The sub-matrix Π_{22} is a (k-k_1) x (g_1-1) matrix. Since the equation to be estimated, that is the first equation of the system, is exactly identified, then k-k_1 = g_1-1, so that Π_{22} is a square matrix.

The relationship between the parameters of the structural and reduced models may be given by the matrix equation.

$$\Pi\Gamma = -B \quad (11)$$

Considering the partition in Equation (2.10) and normalization and zero restrictions along with only first columns of Γ and B, we have for the first equation,

$$\begin{bmatrix} \Pi_{11} & \vdots & \Pi_{12} & \vdots & \Pi_{13} \\ \Pi_{21} & \vdots & \Pi_{22} & \vdots & \Pi_{23} \end{bmatrix} \begin{bmatrix} -1 \\ \gamma_1 \\ O \end{bmatrix} = -\begin{bmatrix} B_1 \\ \cdots \\ O \end{bmatrix} \quad (12)$$

Replacing Π, γ_1, B_1 respectively by their estimators, $\hat{\Pi}$, $\hat{\gamma}_1$, $\hat{B}_1$ and writing out the equations in Equation (2.12) we have,

$$\begin{aligned} -\hat{\Pi}_{11} + \hat{\Pi}_{12}\hat{\gamma}_1 &= \hat{B}_1 \quad (k_1 \quad equations) \\ -\hat{\Pi}_{21} + \hat{\Pi}_{22}\hat{\gamma}_1 &= O \quad (k - k_1 \quad equations) \end{aligned} \quad (13)$$

from which

$$\begin{bmatrix} \hat{\gamma}_1 \\ \cdots\cdots \\ \hat{B}_1 \end{bmatrix} = \begin{bmatrix} -\hat{\Pi}_{22}^{-1}\hat{\Pi}_{21} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \hat{\Pi}_{11} - \hat{\Pi}_{12}\hat{\Pi}_{22}^{-1}\hat{\Pi}_{12} \end{bmatrix} \quad (14)$$

We shall round-off this section by considering some definitions, a theorem and a proposition.

Definition 1

An ILS estimator using Equation (14) shall be referred to as estimator obtained by matrix partition technique (MPT).

Definition 2

An ILS estimator which uses method-of-equating coefficients without resorting to Equation (14) shall be referred to as ILS estimator by unguided coefficient technique (UCT).

Theorem

The 2SLS estimator reduces to ILS estimator for an exactly identified structural equation in a system of simultaneous equations.

Proof

The proof of this theorem can be found in Intriligator (1978: 380-389).

Proposition

For an over identified system, ILS through unguided coefficient techniques (UCT) produces non-unique estimates for a just identified equation. Unique estimates can only be possible if ILS is approached though matrix partition techniques (MPT).

Proof

The proof of this proposition is considered by giving a numerical example in the next section under econometric application.

ECONOMETRIC APPLICATION

In order to prove the proposition stated above, it will suffice to show using an over identified econometric system that there exists at least a parameter say B_2 in a just– identified structural equation that has more than one estimate when UCT is used to get ILS estimates. For this illustration we shall use a prototype macro-econometric model based on Keynesian framework for national income determination as follows:

$$\left. \begin{aligned} C &= B_1 + \gamma_1 Y + u_1 \\ I &= B_3 + B_2 Y_{-1} + \gamma_2 Y + u_2 \\ Y &= C + I + G \end{aligned} \right\} \quad (15)$$

Of course, a closed economy is assumed. C is consumption, Y national income, I investment and G is government expenditure. Y_{-1} is lagged income u_1 and u_2 are stochastic disturbance terms. The first equation in (15) is over-identified while the investment equation is just identified. Our interest is to estimate the parameters of the investment equation in this over-identified system. That is, we are to find estimates of B_3, B_2 and γ_2 in the equation,

$$I = B_3 + B_2 Y_{-1} + \gamma_2 Y + u_2 \tag{16}$$

Data used are that of National Bureau of Statistics, Abuja, Nigeria (1959-85).

Now in Equation (15) the endogenous variables are C, I and Y. the predetermined variables are Y_{-1}, G and the constant terms. For purpose of clarity, let us recall Equation (4) and call it Equation (17) as follows:

$$\underset{nx1}{y_1} = \underset{nx(g_1-1)}{Y_1\gamma_1} + \underset{(nxk_1)(k_1x1)}{X_1 B_1} + \varepsilon_1 \tag{17}$$

Where y_1 = dependent endogenous variable in the first equation. Y_1 = data matrix on g-1 explanatory endogenous variables in the first equation. γ_1 = first column of Γ (coefficient of Y_1). B_1 = first column of B (coefficient of X_1)

Though Equation (16) comes second in Equation (18), we shall name it the "first equation" of the model. Equation (16) can be written as,

$$I = (1 - Y_{-1})\begin{pmatrix} B_3 \\ B_2 \end{pmatrix} + (Y \vdots G)\begin{pmatrix} \gamma_3 \\ O \end{pmatrix} + u_2$$

and comparing it with Equation (17)

Hence, we have $B_1 = \begin{pmatrix} B_3 \\ B_2 \end{pmatrix}$ $and\ \gamma_1 = \gamma_2\ (a\ scaler)$

The reduced form of (15) is,

$$\left.\begin{aligned} I &= \frac{B_1 - B_3(\gamma_1-\gamma_2)}{1-\gamma_1-\gamma_2} + \frac{B_2\ (1-\gamma_1)}{1-\gamma_1-\gamma_2}Y_{-1} + \frac{\gamma_2}{1-\gamma_1-\gamma_2}G + V_1 \\ Y &= \frac{B_1+B_3}{1-\gamma_1-\gamma_2} + \frac{B_2 Y_{-1}}{1-\gamma_1-\gamma_2} + \frac{1}{1-\gamma_1-\gamma_2}G + V_2 \\ C &= \frac{\gamma_1 B_3(1-\gamma_2)B_1}{1-\gamma_1-\gamma_2} + \frac{\gamma_2 B_2}{1-\gamma_1-\gamma_2}Y_{-1} + \frac{\gamma_1}{1-\gamma_1-\gamma_2}G + V_3 \end{aligned}\right\} \tag{18}$$

Alternatively, we can write Equation (3.4) as,

$$\left.\begin{aligned} I &= \pi_{10} + \pi_{11}Y_{-1} + \pi_{12}G + V_1 \\ Y &= \pi_{20} + \pi_{21}Y_{-1} + \pi_{22}G + V_2 \\ C &= \pi_{30} + \pi_{31}Y_{-1} + \pi_{32}G + V_3 \end{aligned}\right\} \tag{19}$$

where the π s are scalars given by the coefficients and constants in (3.4).

Let us partition the matrix of π s in Equation (19) to conform to the specification of,

$$\begin{array}{c} k_1 \\ k-k_1 \end{array}\begin{bmatrix} \Pi_{11} \vdots \Pi_{12} \vdots \Pi_{13} \\ \ldots\ldots\ldots\ldots \\ \Pi_{21} \vdots \Pi_{22} \vdots \Pi_{23} \end{bmatrix}\begin{bmatrix} -1 \\ \ldots. \\ \gamma_1 \\ \ldots. \\ O \end{bmatrix} = -\begin{bmatrix} B_1 \\ \ldots \\ O \end{bmatrix}$$

where $k_1 = 2$ and $k-k_1 = 3-2 = 1$.

Hence, we have,

$$\begin{array}{c} k_1 \\ k-k_1 \end{array}\begin{bmatrix} \pi_{10} \vdots \pi_{20} \vdots \pi_{30} \\ \pi_{11}\ \pi_{21}\ \pi_{31} \\ \ldots\ldots\ldots\ldots \\ \pi_{12} \vdots \pi_{22} \vdots \pi_{32} \end{bmatrix}\begin{bmatrix} -1 \\ \ldots. \\ \gamma_2 \\ \ldots. \\ 0 \end{bmatrix} = -\begin{bmatrix} B_3 \\ B_2 \\ \ldots \\ 0 \end{bmatrix}$$

From which

$$\begin{aligned} -\pi_{10} + \pi_{20}\gamma_2 &= -B_3 \\ -\pi_{11} + \pi_{21}\gamma_2 &= -B_2 \\ -\pi_{12} + \pi_{22}\gamma_2 &= -0 \end{aligned}$$

and

$$\begin{bmatrix} \hat{\gamma}_2 \\ \ldots. \\ \hat{B}_3 \\ \hat{B}_2 \end{bmatrix}_{ILS} = \begin{bmatrix} \hat{\pi}_{12}/\hat{\pi}_{22} \\ \ldots\ldots\ldots \\ \hat{\pi}_{10} - \hat{\pi}_{20}\hat{\pi}_{12}/\hat{\pi}_{22} \\ \hat{\pi}_{11} - \hat{\pi}_{21}\hat{\pi}_{12}/\hat{\pi}_{22} \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} \hat{\gamma}_2 \\ \ldots. \\ \hat{B}_3 \\ \hat{B}_2 \end{bmatrix}_{ILS} = \begin{bmatrix} 0.8251340 \\ \ldots\ldots\ldots \\ -0.4442496 \\ -0.6676483 \end{bmatrix} \tag{21}$$

Results in Equation 21 are obtained using Table 1. From 2SLS, see Table 2.

$$\begin{bmatrix} \hat{\gamma}_2 \\ \ldots. \\ \hat{B}_3 \\ \hat{B}_2 \end{bmatrix}_{ILS} = \begin{bmatrix} 0.8251339 \\ \ldots\ldots\ldots \\ -0.4442496 \\ -0.6676483 \end{bmatrix} \tag{22}$$

Results in (21) and (22) are the same except for rounding-off errors. This confirms the assertion that the 2SLS estimator reduces to ILS estimator for an exactly identified structural equation in a system of simultaneous equations.

Let us now show that B2 has other estimates different from that given by (18) and (19) together within Table 1 we proceed as follows

Table 1. Estimates for Equation (19) [Standard Errors are in Parentheses].

Equation	Constant	Y_{-1}	G
I	π_{10}	π_{11}	π_{12}
	0.2339829	-0.1820153	3.5729529
	(0.4473652)	(0.0740436)	(0.6800662)
	$\bar{R}^2$ =0.884721	DW=0.939009	F=96.93277
		Residual S.S= 60.46886	
Y	π_{20}	π_{21}	π_{22}
	0.8219665	0.5885505	4.3301486
	(0.5691370)	(0.0941981)	(0.8651787)
	$\bar{R}^2$ = 0.991441	DW=1.560351	F=1448.944
		Residual S.S= 97.86809	
C	π_{30}	π_{31}	π_{32}
	0.5894319	0.7706240	-0.2436277
	(0.6043660)	(0.1000289)	(0.9187324)
	$\bar{R}^2$ =0.980847	DW=1.281535	F=641.1264
		Residual S.S= 641.1264	

Table 2. 2SLS estimates for investment and consumption equation in the structural model (15) [standard errors are in parentheses].

Equation	Parameter estimates		
Investment	**γ_2**	**B_3**	**B_2**
	0.8251339	- 0.4442496	- 0.6676483
	(0.1889510)	(0.5854478)	(0.1992052)
	$\bar{R}^2$ = 0.833140	DW=1.194607	F = 63.41326
	Residual S.S = 87.5253		
	Mean investment = 4.573508		
	Standard deviation of investment = 4.775592		
Consumption	**γ_1**	**B_1**	
	0.7027999	-0.2249105	
	(0.0198664)	(0.6165624)	
	$\bar{R}^2$ = 0.980573	DW=0.2735624	F=1262.893
	Residual S.S = 116.8001		
	Mean consumption = 15.31532		
	Standard deviation of consumption 15.82768		

$$\pi_{22} = \frac{1}{1-\gamma_1-\gamma_2}$$

$$\pi_{21} = \frac{B_2}{1-\gamma_1-\gamma_2} = B_2\pi_{22}$$

$$\Rightarrow \hat{B}_2 = \hat{\pi}_{21} / \hat{\pi}_{22} = 0.1359192$$

$$Also\ \pi_{32} = \frac{\gamma_1}{1-\gamma_1-\gamma_2}$$

$$and\ \frac{\gamma_1 B_2}{1-\gamma_1-\gamma_2} = \pi_{31}$$

$$\Rightarrow B_2\pi_{32} = \pi_{31}\ and\ \hat{B}_2 = \hat{\pi}_{31} / \hat{\pi}_{32} = -3.1631214$$

Hence we have seen that when ILS is approached through UCT for estimating parameters of a just identified equation in an over-identified system extraneous estimates result in addition to the true estimates. The extraneous estimates for B_2, are

$$\left.\begin{array}{l} \hat{B}_2 = 0.1359192 \\ \hat{B}_2 = -3.1631214 \end{array}\right\} \quad (23)$$

Conclusion

We have seen that ILS through MPT gives estimates that coincide with 2SLS estimates for an exactly identified equation. However, in using UCT to get ILS estimates, results differ from those for 2SLS. Therefore, 2SLS serves as a test for discriminating between true parameter estimates and extraneous estimates. In order to serve time used for testing extraneous estimates it is better and safer to approach ILS through MPT in estimating a just identified equation in an over identified system.

ACKNOWLEDGEMENTS

We wish to thank Centre for Econometric and Allied Research (CEAR), University of Ibadan, Ibadan for allowing us use their computing facilities and library. We are also grateful to Mr. Lekan Jimoh for his skillful typing.

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