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Full Length Research Paper

Thermal stresses induced by a point heat source in a hollow disk by quasi-static approach

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This paper deals with the two dimensional non-homogeneous boundary value problem of heat conduction in a hollow disk defined as $a \le r \le b$; $0 \le z \le h$ and discussed the thermoelastic behavior due to internal heat generation within it. A thin hollow disk is considered having arbitrary initial temperature and subjected to time dependent heat flux at the outer circular boundary $(r = b)$ whereas inner circular boundary $(r = a)$ is at zero heat flux. Also, the upper surface $(z = h)$ of the hollow disk is insulated and the lower surface $(z = 0)$ is at zero temperature. The governing heat conduction equation has been solved by using integral transform technique. The results are obtained in series form in terms of Bessel's functions. The results for displacement and stresses have been computed numerically and are illustrated graphically

Key words: Transient, thermoelastic problem, thermal stresses, heat generation non homogenous boundary value problem.

INTRODUCTION

Deshmukh et al. (2008) has considered two dimensional non-homogeneous boundary value problem of heat conduction and studied the thermoelasticity of a thin hollow circular disk.

In this paper, the work of Deshmukh et al. (2011) has been extended for a thin hollow circular cylinder and discusses the thermoelastic behavior. This problem deals with the determination of displacement and thermal stresses due to internal heat generation within it. Consider a thin hollow disk of thickness h occupying space D defined by $a \le r \le b$, $0 \le z \le h$. Initially, the disk is kept at arbitrary temperature $F(r, z)$. The inner circular boundary $(r = a)$ is at zero heat flux whereas the time dependent heat flux $Q(z, t)$ is applied on the outer circular boundary $(r = b)$. Also the upper surface $(z = h)$ of the hollow disk is insulated and the lower surface $(z = 0)$ of the disk is at zero temperature. For time $t > 0$, heat is generated within the thin hollow disk at the rate $g(r, z, t)$. The governing heat conduction equation has been solved by using integral transform technique. The results are obtained in series form in terms of Bessel's functions. The results for displacement and stresses have been computed numerically and are illustrated graphically. To our knowledge no one has studied thermal stresses due to heat generation in a thin hollow disk so far. This is new and novel contribution to the field.

The results presented here will be useful in engineering problems particularly in aerospace engineering for stations of a missile body not influenced by nose tapering. The missile skill material is assumed to have physical properties independent of temperature, so that the temperature $T(r, z, t)$ is a function of radius, thickness and time only. Under these conditions, the displacement and thermal stresses in a thin hollow disk

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due to heat generation are required to be determined.

THEORY ANALYSIS

Following Deshmukh et al. (2011), we assume that a hollow disk of small thickness *h* is in a plane state of stress. In fact, "the smaller the thickness of the hollow disk compared to its diameter, the nearer to a plane state of stress is the actual state". The displacement equations of thermoelasticity have the form as:

$$U_{i,kk} + \left(\frac{1+\nu}{1-\nu}\right)e,_i = 2\left(\frac{1+\nu}{1-\nu}\right)a_t T,_i \tag{1}$$

$$e = U_{k,k}; \quad k,i = 1,2 \tag{2}$$

where $U_i -$ Displacement component, $e -$ Dilatation, $T -$ Temperature, and ν and a_t are respectively, the Poisson's ratio and the linear coefficient of thermal expansion of the hollow disk material. Introducing

$$U_i = \psi_{,i} \quad , \quad i = 1,2$$

we have

$$\nabla_1^{\,2}\psi = (1+\nu)a_t T \qquad \nabla_1^{\,2} = \frac{\partial^2}{\partial r_1^{\,2}} + \frac{\partial^2}{\partial r_2^{\,2}} \tag{3}$$

$$\sigma_{ij} = 2\mu\left(\psi_{,ij} - \delta_{ij}\psi_{,kk}\right), \qquad i,j,k = 1,2, \tag{4}$$

where μ is the Lamé constant and δ_{ij} is the Kronecker symbol. In the axially-symmetric case

$$\psi = \psi(r,z,t)\,, \quad T = T(r,z,t)$$

and the differential equation governing the displacement potential function $\psi(r,z,t)$ is

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = (1+\nu)a_t T \tag{5}$$

with

$$\frac{\partial\psi}{\partial r} = 0 \quad \text{at } r = a, r = b \tag{6}$$

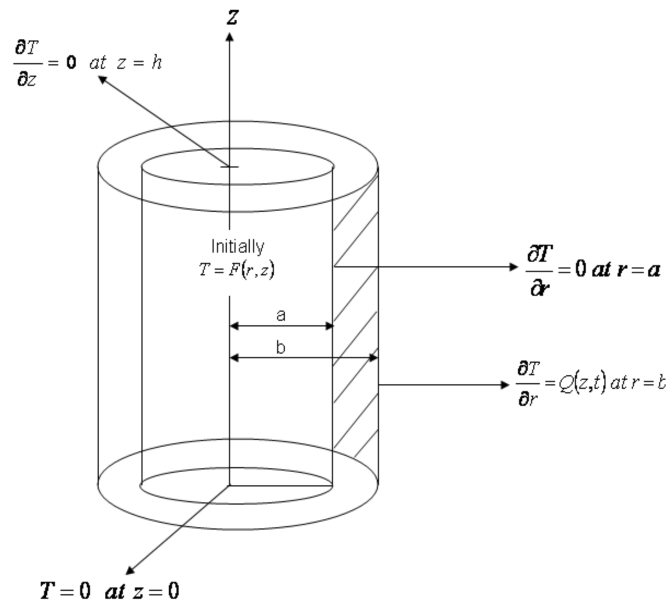

Figure 1. Shows the geometry of the Heat conduction problem.

for all time t, where ν and a_t are Poisson's ratio and linear coefficient of the thermal expansion of the disk respectively. The stress function σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = \frac{-2\mu}{r}\frac{\partial\psi}{\partial r} \tag{7}$$

$$\sigma_{\theta\theta} = -2\mu\frac{\partial^2\psi}{\partial r^2} \tag{8}$$

The surface of the thin hollow disk at $r = b$ is assumed to be traction free. The boundary condition can be taken as

$$\sigma_{rr} = 0 \quad \text{at } r = a,\ r = b \tag{9}$$

Also in the plane state of stress within the hollow disk

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0 \tag{10}$$

Initially

$$T = \psi = \sigma_{rr} = \sigma_{\theta\theta} = F(r,z) \quad \text{at } t = 0 \tag{11}$$

The temperature of the hollow disk $T(r,z,t)$ at time t satisfying the differential equation given in Ozisik (1968), (Figure 1),

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}+\frac{g(r,z,t)}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{12}$$

with the boundary conditions,

$$\frac{\partial T}{\partial r}=0 \quad \text{at } r=a, \quad t>0 \tag{13}$$

$$\frac{\partial T}{\partial r}=Q(z,t), \quad \text{at } r=b, \quad t>0 \tag{14}$$

$$T=0 \text{ at } z=0, \quad t>0 \tag{15}$$

$$\frac{\partial T}{\partial z}=0 \quad \text{at } z=h, \quad t>0 \tag{16}$$

and initial condition

$$T(r,z,t)=F(r,z) \text{ in } a\le r\le b,\ 0\le z\le h \text{ for } t=0. \tag{17}$$

where k and α are thermal conductivity and thermal diffusivity of the material of hollow disk respectively. Equations 1 to 17 constitute mathematical formulation of the problem. To obtain the expression for temperature function $T(r,z,t)$, we develop the finite Fourier transform, the henkal transform and their inversion and operate on the heat conduction Equations 12 to 17 one obtain

$$T(r,z,t)=\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)K_0(\beta_m,r)\,e^{-\alpha(\beta_m^2+\eta_p^2)\,t}$$
$$\times\left\{\int_{r'=a}^{b}\int_{z'=0}^{h} r'K_0(\beta_m,r')K(\eta_p,z')F(r',z')\,dr'\,dz'\right.$$
$$+\int_{t'=0}^{t}e^{\alpha(\beta_m^2+\eta_p^2)t'}\left(\frac{\alpha}{k}\int_{r'=a}^{b}\int_{z'=0}^{h}r'K_0(\beta_m,r')K(\eta_p,z')g(r',z',t')\,dr'\,dz'\right.$$
$$\left.\left.+\alpha bK_0(\beta_m,b)\int_{z'=0}^{h}K(\eta_p,z')Q(z',t')\,dz'\right)dt'\right\} \tag{18}$$

Where $K(\eta_p,z)=\sqrt{\frac{2}{h}}\sin(\eta_p z)$

and $\eta_1,\eta_2,\ldots..$ are the positive roots of the transcendental equation

$$\cos\eta_p h=0,\quad p=1,2,\ldots\ . \tag{19}$$

$$K_0(\beta_m r)=\frac{\pi}{\sqrt{2}}\frac{\beta_m J_0'(\beta_m b)Y_0'(\beta_m b)}{\left[1-\frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}\right]^{\frac{1}{2}}}\left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)}-\frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)}\right]$$

and $\beta_1,\beta_2,\ldots..$ are the positive roots of the transcendental equation

$$\frac{J_0'(\beta a)}{J_0'(\beta b)}-\frac{Y_0'(\beta a)}{Y_0'(\beta b)}=0 \tag{20}$$

Using Equation (18) in (5), and using the well known result

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)J_0(\beta_m r)=-\beta_m^2 J_0(\beta_m r)$$
$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)Y_0(\beta_m r)=-\beta_m^2 Y_0(\beta_m r)$$

one obtains the displacement function as

$$\psi=-(1+\nu)a_t\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)\frac{1}{\beta_m^2}K_0(\beta_m,r)\,e^{-\alpha(\beta_m^2+\eta_p^2)\,t}$$
$$\times\left\{\int_{r'=a}^{b}\int_{z'=0}^{h}r'K_0(\beta_m,r')K(\eta_p,z')F(r',z')\,dr'\,dz'\right.$$
$$+\int_{t'=0}^{t}e^{\alpha(\beta_m^2+\eta_p^2)t'}\left(\frac{\alpha}{k}\int_{r'=a}^{b}\int_{z'=0}^{h}r'K_0(\beta_m,r')K(\eta_p,z')g(r',z',t')\,dr'\,dz'\right.$$
$$\left.\left.+\alpha bK_0(\beta_m,b)\int_{z'=0}^{h}K(\eta_p,z')Q(z',t)\,dz'\right)dt'\right\} \tag{21}$$

Using Equation (21) in Equations (7) and (8), one obtains the expression of radial stress function and angular stress function as:

$$\sigma_{rr}=2\mu(1+\nu)a_t\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{r\beta_m^2}K(\eta_p,z)K_1(\beta_m,r)e^{-\alpha(\beta_m^2+\eta_p^2)t}$$

$$\times\left\{\int_{r'=a}^{b}\int_{z'=0}^{h} r' K_0(\beta_m r')\, K(\eta_p, z')\, F(r', z')\, dr'\, dz'\right. +$$

$$\int_{t'=0}^{t} e^{\alpha(\beta_m^2+\eta_p^2)t'}\left(\frac{\alpha}{k}\int_{r'=a}^{b}\int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t')\, dr'\, dz'\right.$$

$$\left.\left.+\alpha b\, K_0(\beta_m, b)\int_{z'=0}^{h} K(\eta_p, z')\, Q(z', t')\, dz'\right) dt'\right\} \tag{22}$$

$$\sigma_{\theta\theta} = 2\mu(1+\nu)a_t \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\beta_m^2} K(\eta_p, z).K_2(\beta_m, r).e^{-\alpha(\beta_m^2+\eta_p^2)t}$$

$$\times\left\{\int_{r'=a}^{b}\int_{z'=0}^{h} r' K_0(\beta_m, r')\, K(\eta_p, z')\, F(r', z')\, dr'\, dz'\right.$$

$$+\int_{t'=0}^{t} e^{\alpha(\beta_m^2+\eta_p^2)t'}\left(\frac{\alpha}{k}\int_{r'=a}^{b}\int_{z'=0}^{h} r' K_0(\beta_m, r') K(\eta_p, z') g(r', z', t')\, dr'\, dz'\right.$$

$$\left.\left.+\alpha b K_0(\beta_m, b)\int_{z'=0}^{h} K(\eta_p, z') Q(z', t')\, dz'\right) dt'\right\} \tag{23}$$

where

$$K_1(\beta_m, r) = -\frac{\pi}{\sqrt{2}}\,\frac{\beta_m^2.J_0'(\beta_m b)Y_0'(\beta_m b)}{\left[1-\frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}\right]^{\frac{1}{2}}}\times\left[\frac{J_1(\beta_m r)}{J_0'(\beta_m b)}-\frac{Y_1(\beta_m r)}{Y_0'(\beta_m b)}\right]$$

$$K_2(\beta_m, r) = -\frac{\pi}{\sqrt{2}}\,\frac{\beta_m^3.J_0'(\beta_m b).Y_0'(\beta_m b)}{\left[1-\frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}\right]^{\frac{1}{2}}}$$

$$\times\left\{\frac{1}{J_0'(\beta_m b)}\times\left[J_0(\beta_m r)-\frac{J_1(\beta_m r)}{\beta_m r}\right]-\frac{1}{Y_0'(\beta_m b)}\times\left[Y_0(\beta_m r)-\frac{Y_1(\beta_m r)}{\beta_m r}\right]\right\}$$

Setting

$$F(r, z) = (r^2 - a^2)^2 \times r^2 \times (z^2 - h^2)^2 \times z^2$$

$$Q(z, t) = z^2 \times (z^2 - h^2)^2 \times e^{-\omega t}$$

$$g(r, z, t) = g_{pi}\delta(r - r_1)\delta(z - z_1)\delta(t - \tau)$$

with

$\omega = 10$, $t \to \tau = 5$, $g_{pi} = 50$

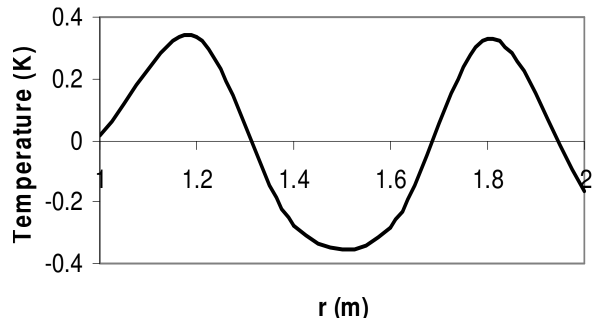

Figure 2. Temperature distribution $\frac{T}{X}$.

Dimension

Inner radius of a thin hollow circular disk, $a = 1\ m$

Outer radius of a thin hollow circular disk, $b = 2\ m$

Thickness of hollow circular disk, $z = 0.2\ m$

Central circular path of circular disk in radial and axial directions, $r_1 = 1.5\ m$ and $z_1 = 0.1\ m$

The numerical calculation has been carried out for a Copper (Pure) thin hollow disk with the material, properties as:

Thermal diffusivity, $\alpha = 11234 \times 10^{-6} m^2 s^2$

Thermal conductivity, $k = 386(W/mk)$

Density, $\rho = 8954 kg/m^3$

Specific heat, $c_p = 383\ J/kgK$,

Poisson ratio, $\nu = 0.35$,

Coefficient of linear thermal expansion, $a_t = 16.5 \times 10^{-6}\frac{1}{K}$,

Lamé constant, $\mu = 26.67$

$\beta_1 = 3.1965,\quad \beta_2 = 6.3123,\quad \beta_3 = 9.4445,\quad \beta_4 = 12.5812,\quad \beta_5 = 15.7199$ are the positive roots of transcendental Equation (20). For convenience setting

$$X = \frac{\pi^2 10^5}{h}, Y = \frac{-(1+\nu)a_t\pi^2 10^8}{h},\ Z = \frac{-2\mu(1+\nu)a_t\pi^2 10^8}{h}$$

The numerical calculation has been carried out with the help of computational mathematical software Mathcad-2000 and the graphs are plotted with the help of Excel (MS office-2000). From Figure 2, it can be observed that, the temperature distribution undergoes the form of wave due to the point heat source. From Figure 3, it can be observed that displacement variate non-uniformly in the radial direction due to the point heat source. From the Figure 4 it can be observed that the radial stresses

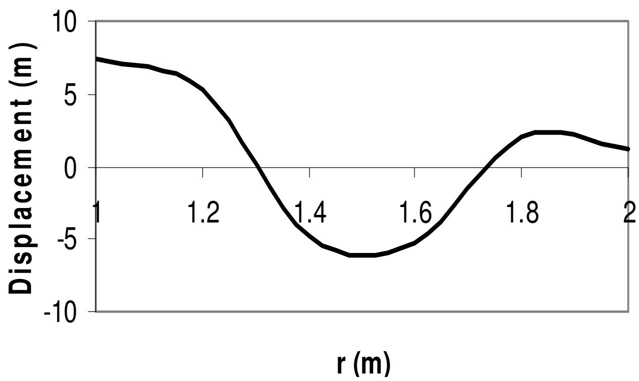

Figure 3. Displacement function: $\frac{\psi}{Y}$.

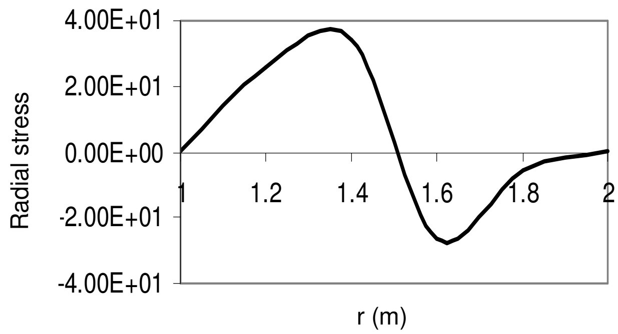

Figure 4. Radial stress function: $\frac{\sigma_{rr}}{Z}$.

develop due to the point heat source in the radial direction. From Figure 5 it can be observed that the angular stresses develop the compressive stresses in the axial direction.

RESULTS AND DISCUSSION

In this paper, we extended the work of Deshmukh et al. (2011) in two dimensional non-homogeneous boundary value problem of heat conduction in a thin hollow disk and determined the expressions for temperature, displacement and stresses due to internal heat generation within it. As a special case mathematical model is constructed for Copper (Pure) thin hollow disk with the material properties specified as above. The heat source is an instantaneous point heat source of strength g_{pi} situated at the center of the hollow circular disk along radial and axial direction and releases its heat spontaneously at the time $t = \tau$.

The results obtained here are useful in engineering

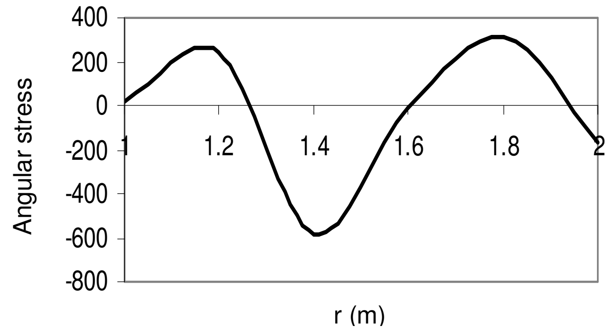

Figure 5. Angular stress function: $\frac{\sigma_{\theta\theta}}{Z}$.

problems particularly in the determination of state of stress in thin hollow disk. Also any particular case of special interest can be derived by assigning suitable values to the parameter and function in the expressions (18), (21), (22) and (23).

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