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Full Length Research Paper

A class of one-point zero-stable continuous hybrid methods for direct solution of second-order differential equations

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This paper considers the development of a class of one-point hybrid implicit methods for direct solution of general second order ordinary differential equations. The main predictors needed for the evaluation of the implicit methods are obtained to be of the same order with the methods at whatever hybrid point of collocation. The methods and their respective predictors are consistent and zero-stable. Numerical results are presented to show the accuracy of the methods.

Key words: One-point continuous hybrid, collocation and interpolation equations, grid point.

INTRODUCTION

In this paper, a direct numerical solution to the general second order initial value differential equations of the form

$$F(t, y, y^1) = 0,\ y^{(s)}(t_0) = y_0^{(s)}, s = 0,\ 1 \qquad (1)$$

is proposed without recourse to the conventional way of reducing it to a system of first order equations (Chan et al., 2004), which has many disadvantages (Awoyemi and Kayode, 2002).

Attempts have been made by various authors to solve Equation (1) in which the first derivative (y') is absent, (Al-Said and Noor, 2001; Golbabai and Arabshahi, 2010; Gonzalez and Thompson, 1997; Saravi et al., 2009). This limits the solution to a special class of differential equations. Efforts have also been made to develop methods for solving Equation (1) directly with little attention at solutions at off- grid points, (Awoyemi and Kayode, 2003; Tselyaev, 2004; Awoyemi and Kayode, 2005; Kayode and Awoyemi, 2005; Jator, 2007; Kayode, 2007; Kayode, 2010).

Yahaya and Badmus (2009) developed a class of hybrid methods for problem of Type (1) with low order of accuracy.

In this paper, a class of one-point numerical hybrid methods with higher order of accuracy is developed for directly approximating the solution of Equation (1).

AMS 2000 subject classification: 65L05, 65L06.

THE HYBRID METHODS

Here, interpolation and collocation procedures are used by choosing interpolation points (s) at the grid points and collocation points (t) at both grid and at one off-grid points given rise to $\zeta = s + t + 1$ equations whose coefficients are determined by using appropriate procedures.

The approximate solution to Problem (1) is taken to be a polynomial of degree $\zeta = s + t + 1$ in the form:

$$y(x) = \sum_{j=0}^{\zeta} \lambda_j x^j \qquad (2)$$

$\lambda_j, j = 0,\ 1,\ 2,\ ...$ are real numbers and y is continuous differentiable.

Obtaining the second derivative of (2) and using (1) to have the differential system as:

$$\sum_{j=2}^{\zeta} j(j-1)\lambda_j x^{j-2} = f(x, y(x), y'(x)) \quad (3)$$

To obtain a one point zero-stable hybrid method, the collocation equation is obtained by collocating (3) at all grid points $x_{n+i},\ i = 0, 1, 2, \ldots,$ and at one off-grid point $x_{n+v},\ 1 < v < 2$. Interpolation equation is obtained by interpolating (2) at all grid points except at the last end point. These result into an order m matrix equation:

$$AX = B$$

Where:

$$A=\begin{bmatrix} 0 & 0 & 2 & 6x_n & 12x_n & \ldots & \tau x_n^{\zeta-2} \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1} & \ldots & \tau x_{n+1}^{\zeta-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 2 & 6x_{n+k} & 12x_{n+k} & \ldots & \tau x_{n+k}^{\zeta-2} \\ 0 & 0 & 2 & 6x_{n+v} & 12x_{n+v} & \ldots & \tau x_{n+v}^{\zeta-2} \\ 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \ldots & x_n^{\zeta} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & \ldots & x_{n+1}^{\zeta} \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & \ldots & x_{n+2}^{\zeta} \end{bmatrix} \quad (4)$$

$$X = \left[a_0\ a_1\ a_2 \ldots a_{s+t-1}\ a_{s+t}\right]^T,$$

$$B = \left[f_{n+1}\ f_{n+3} \ldots f_{n+k}\, f_{n+v}\ y_n\ y_{n+1} \ldots y_{n+k-1}\right]^T,$$

T is the matrix transpose:

$$\tau = j(j-1),\ j = 2, 3, \ldots, \zeta,$$

$$f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i}),\ i = 0, 1, 2, \ldots,$$

$$y_{n+i} \approx y(x_{n+i}).$$

Solving the matrix Equation (6) for the unknown parameters λ_j's, $j = 0(1)\zeta$. The values of these parameters are substituted into approximate solution (2) to obtain a continuous hybrid method expressed as:

$$y_k(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \sum_{j=2}^{k} \beta_j(x) f_{n+j} + \rho(x) f_{n+v} \quad (5)$$

Using the transformations:

$$t = \frac{1}{h}(x - x_{n+k-1}),\ dt = \frac{1}{h}dx, \quad t \in (0, 1], \quad (6)$$

in the continuous Method (5), the coefficients α_j, β_j are obtained, as a function of t as:

$$\alpha_{k-1}(t) = \{1 + t\}$$

$$\alpha_{k-2}(t) = -t$$

$$\alpha_{k-3}(t) = \ldots = \alpha_0(t) = 0$$

$$\beta_k = \frac{h^2}{360(3-v)}\{(8v-13)t + 20(2-v)t^3 + 5(8-3v)t^4 + 3(5-v)t^5 + 2t^6\}$$

$$\beta_{k-1} = \frac{h^2}{120(2-v)}\{(75-43v)t + 60(2-v)t^2 + 10(4-v)t^3 + 5(2v-3)t^4 + 3(v-4)t^5 - 2t^6\}$$

$$\vdots$$

$$\beta_1 = \frac{h^2}{120(v-1)}\{11(2v-3)t + 20(2-v)t^3 + 5vt^4 + 3(v-3)t^5 - 2t^6\}$$

$$\beta_0 = \frac{h^2}{360v}\{(11-7v)t + 10(v-2)t^3 - 5t^4 + 3(2-v)t^5 + 2t^6\}$$

$$\beta_v = \frac{h^2}{60v(3-v)(2-v)(v-1)}\{11t - 20t^3 - 5t^4 + 6t^5 + 2t^6\} \quad (7)$$

and the first derivatives of α_j, β_j in (7) yields:

$$\alpha'_{k-1} = \frac{1}{h}$$

$$\alpha'_{k-2} = -\frac{1}{h}$$

$$\alpha'_{k-3}(t) = ... = \alpha'_0(t) = 0$$

$$\beta'_k = \frac{h}{360(3-v)}\{(8v-13) + 60(2-v)t^2 + 20(4-3v)t^3 + 15(5-v)t^4 + 12t^5$$

$$\beta'_{k-1} = \frac{h}{120(2-v)}\{(75-43v) + 120(2-v)t + 30(4-v)t^2 + 20(2v-3)t^3 + 15(v-4)t^4 - 12t^5\}$$

$$\beta'_1 = \frac{h}{120(v-1)}\{11(2v-3) + 60(2-v)t^2 + 20vt^3 + 15(v-3)t^4 - 12t^5\}$$

$$\beta'_0 = \frac{h}{360v}\{(11-7v) + 30(v-2)t^2 - 40t^3 + 15(2-v)t^4 + 12t^5\}$$

$$\beta'_v = \frac{h}{60v(3-v)(2-v)(v-1)}\{11 - 60t^2 - 20t^3 + 30t^4 + 12t^5\} \quad (8)$$

Using the step number k = 3 and t = 1 in (6), (7) and (8) give:

$$y_{n+3} - 2_{n+2} + y_{n+1} = \frac{h^2}{60v(3-v)(2-v)(v-1)}(\delta_3 f_{n+3} + \delta_2 f_{n+2} + \delta_v f_{n+v} + \delta_1 f_{n+1} + \delta_0 f_n) \quad (9)$$

where

$$\delta_3 = v(14-5v)(2-v)(v-1)$$

$$\delta_2 = v(103-50v)(3-v)(v-1)$$

$$\delta_v = -6$$

$$\delta_1 = v(5v-2)(3-v)(2-v)$$

$$\delta_0 = -(3-v)(2-v)(v-1)$$

and

$$y'_{n+3} = \frac{1}{h}(y_{n+2} - y_{n+1}) + \frac{h}{360v(3-v)(2-v)(v-1)}(\gamma_3 f_{n+3} + \gamma_2 f_{n+2} + \gamma_v f_{n+v} + \gamma_1 f_{n+1} + \gamma_0 f_0\} \quad (10)$$

Where:

$$\gamma_3 = v(354-127v)(2-v)(v-1)$$

$$\gamma_2 = 3v(303-138v)(3-v)(v-1)$$

$$\gamma_v = -162$$

$$\gamma_1 = 9v(10-v)(3-v)(2-v)$$

$$\gamma_0 = (8v-27)(3-v)(2-v)(v-1)$$

For the purpose of numerical experiment specific values of $v \in (1,\ 2)$ are taken at three points as $\frac{5}{4}$, $\frac{3}{2}$, $\frac{7}{4}$ to obtain the following discrete schemes:

For $v = \frac{5}{4}$:

$$y_{n+3} = 2_{n+2} - y_{n+1} + \frac{h^2}{2100}(155 f_{n+3} + 1890 f_{n+2} - 512 f_{n+\frac{5}{4}} + 595 f_{n+1} - 284 f_n) \quad (11)$$

The order (p) is 5 and error constant $C_{p+2} \approx -0.002292$

$$y'_{n+3} = \frac{1}{h}(y_{n+2} - y_{n+1}) + \frac{h}{12600}(3905 f_{n+3} + 18270 f_{n+2} - 13824 f_{n+\frac{5}{4}} + 11025 f_{n+1} - 476 f_n) \quad (12)$$

Order P = 5, $C_{p+2} = 0.006344$.

For $v = \frac{3}{2}$:

$$y_{n+3} = 2_{n+2} - y_{n+1} + \frac{h^2}{180}(13 f_{n+3} + 168 f_{n+2} - 32 f_{n+\frac{3}{2}} + 33 f_{n+1} - 2 f_n) \quad (13)$$

Order P = 5, $C_{p+2} \approx -0.002083.$

$$y'_{n+3} = \frac{1}{h}(y_{n+2} - y_{n+1}) + \frac{h}{360}(109f_{n+3} + 576f_{n+2} - 288f_{n+\frac{3}{2}} + 153f_{n+1} - 10f_n) \quad (14)$$

Order P = 5, $C_{p+2} \approx 0.005407.$

For $v = \frac{7}{4}$:

$$y_{n+3} = 2_{n+2} - y_{n+1} + \frac{h^2}{2100}(147f_{n+3} + 2170f_{n+2} - 512f_{n+\frac{7}{4}} + 315f_{n+1} - 20f_n) \quad (15)$$

Order P = 5, $C_{p+2} \approx -0.001875.$

$$y'_{n+3} = \frac{1}{h}(y_{n+2} - y_{n+1}) + \frac{h}{12600}(3689f_{n+3} + 25830f_{n+2} - 13824f_{n+\frac{7}{4}} + 3465f_{n+1} - 260f_n) \quad (16)$$

Order P = 5, $C_{p+2} \approx 0.004469.$

IMPLEMENTATION OF THE METHODS

The set of implicit discrete schemes and their respective first derivatives in equation (11) through (16) are not self-starting. Thus to be able to implement them, some starting values are developed using the same technique for the main method described above. Thus at t = 1, and $r = \frac{5}{4}, \frac{3}{2}, \frac{7}{4}$, the main starting values are:

For $v = \frac{5}{4}$:

$$y_{n+3} = -\frac{23}{4}y_{n+2} + \frac{29}{2}y_{n+1} - \frac{31}{4}y_n + \frac{h^2}{240}(495f_{n+2} - 512f_{n+\frac{5}{4}} + 1990f_{n+1} + 127f_n) \quad (17)$$

having order p = 5 and $C_{p+2} \approx 0.0122396,$ and

$$y'_{n+3} = \frac{1}{24h}\{-757y_{n+2} + 1538y_{n+1} - 781y_n\} + \frac{h}{7200}\{45585f_{n+2} - 65024f_{n+\frac{5}{4}} + 248410f_{n+1} + 16129f_n\} \quad (18)$$

p = 5, $C_{p+2} \approx -0.054671.$

For $v = \frac{3}{2}$:

$$y_{n+3} = -\frac{9}{2}y_{n+2} + 12y_{n+1} - \frac{13}{2}y_n + \frac{h^2}{24}(51f_{n+2} - 32f_{n+\frac{3}{2}} + 150f_{n+1} + 11f_n) \quad (19)$$

p = 5 and $C_{p+2} \approx 0.011458,$ and

$$y'_{n+3} = \frac{1}{4h}\{-105y_{n+2} + 214y_{n+1} - 109y_n\} + \frac{h}{720}\{4749f_{n+2} - 4064f_{n+\frac{3}{2}} + 18618f_{n+1} + 1397f_n\} \quad (20)$$

order p = 5, $C_{p+2} \approx -0.051364.$

For $v = \frac{7}{4}$:

$$y_{n+3} = -\frac{13}{4}y_{n+2} + \frac{19}{2}y_{n+1} - \frac{21}{4}y_n + \frac{h^2}{336}(847f_{n+2} - 512f_{n+\frac{7}{4}} + 1638f_{n+1} + 127f_n) \quad (21)$$

order p = 5 and $C_{p+2} \approx 0.01015625,$ and

$$y'_{n+3} = \frac{1}{24h}\{-503y_{n+2} + 1030y_{n+1} - 527y_n\} + \frac{h}{100080}\{83377f_{n+2} - 65024f_{n+\frac{7}{4}} + 201978f_{n+1} + 16129f_n\} \quad (22)$$

order p = 5 and $C_{p+2} \approx -0.0458519$.

Other starting values for y_{n+2}, y'_{n+2}, y_{n+v}, y'_{n+v}, y_{n+1}, y'_{n+1} are obtained to be:

$$y_{n+2} = 2y_{n+1} - y_n + h^2 f_{n+} \quad (23)$$

p = 2, c_{p+2} = 0.0833,

$$y'_{n+2} = \frac{1}{h}(y_{n+1} - y_n) + \frac{h}{6}(11f_{n+1} - 2f_n) \quad (24)$$

p =2, cp_{+2} = - 0.375

$$y_{n+j} = y_n + (jh)y'_n + \frac{(jh)^2}{2!} f_n + \frac{(jh)^3}{3!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n \frac{\partial f_n}{\partial y_n} + f_n \frac{\partial f_n}{\partial y'_n}\right\} + O(h^4) \quad (25)$$

$$y'_{n+j} = y'_n + (jh)f_n + \frac{(jh)^2}{2!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n \frac{\partial f_n}{\partial y_n} + f_n \frac{\partial f_n}{\partial y'_n}\right\} + O(h^3)$$

where $j = 1, \frac{5}{4}, \frac{3}{2}$, z and y_n, y'_n are the initial values from the given problem.

NUMERICAL EXPERIMENTS

The accuracy of the continuous method (11) developed for the direct solution of problem (1) is tested on linear and non-linear problems with $v = \frac{3}{2}$:

(i) $y'' = 2y^3$, $y(1) = 1$, $y'(1) = -1$;

Theoretical solution:

$$y(x) = \frac{1}{x}.$$

(ii) $y'' = y + xe^{3x}$

Theoretical solution:

$$y(x) = \frac{(4x-3)}{32\exp(-3x)}.$$

(iii) $y''(x) = \frac{(y')^2}{2y} - 2y$,

$$y(\frac{\Pi}{6}) = \frac{1}{4},\ y'(\frac{\Pi}{6}) = \frac{\sqrt{3}}{2},$$

Theoretical solution

$$y(x) = Sin^2 x.$$

(iv) $y'' - x(y')^2 = 0$, $y(0) = 1$, $y'(0) = \frac{1}{2}$;

Theoretical solution

$$y(x) = 1 + \frac{1}{2}\ln\{\frac{(2+x)}{(2-x)}\}$$

RESULTS

The absolute errors obtained from the method (13) for k = 3 are compared with those obtained from the method for k = 3 in Kayode (2007) for the problems (i) to (iv).

The results of the absolute errors are shown in Tables 1 to 4.

CONCLUSION

In this paper a class of one-point continuous hybrid method has been considered for direct solution of general second order differential equations. The discrete schemes from the continuous method (8) have the same order p = 5. The major predictors for these are also of order p = 5 with these discrete methods. All the discrete methods are consistent and zero stable, satisfying the necessary and sufficient conditions for the convergence of Linear Multistep Methods (LMM), (Chou and Ding, 2004; Parand and Hojjati, 2008).

Table 1.Comparison of errors in Kayode (2007) with new Method (11).

	Kayode (2007) for Problem (i)	New method (11) for Problem (i)
X	Errors for k = 3	Errors for k = 3
1.1	0.5263931D-08	0.2102459D-08
1.2	0.3720895D-08	0.1683367D-08
1.3	0.2704051D-08	0.1371889D-08
1.4	0.2012022D-08	0.1135265D-08
1.5	0.1527879D-08	0.9520500D-09
1.6	0.8078035D-05	0.1180986D-08
1.7	0.6925521D-05	0.9271881D-09
1.8	0.5992534D-05	0.7380502D-09
1.9	0.5228356D-05	0.5947730D-09
2.0	0.4595810D-05	0.4846371D-09

Table 2. Comparison of errors in Kayode (2007) with new Method (11).

	Kayode (2007) for Problem (ii)	New method (11) for Problem (ii)
X	Errors for k = 3	Errors for k = 3
0.1	0.2086753D-09	0.5079052D-10
0.2	0.1923770D-09	0.7948121D-10
0.3	0.1391324D-09	0.1429751D-10
0.4	0.2508468D-10	0.2779986D-10
0.5	0.1857944D-09	0.4903604D-10
0.6	0.5588885D-09	0.8051933D-10
0.7	0.1157671D-08	0.1094526D-09
0.8	0.2107025D-08	0.1865553D-09
0.9	0.3578957D-08	0.3063101D-09
1.0	0.5822924D-08	0.4891233D-09

Table 3. Comparison of errors in Kayode (2007) with new method (11).

	Kayode (2007) for Problem (iii)	New Method (11) for Problem (iii)
x	Errors for k = 3	Errors for k = 3
1.1	0.2282106D-05	0.64811445D-07
1.2	0.2893084D-05	0.80343529D-07
1.3	0.3453509D-05	0.93317005D-07
1.4	0.3954212D-05	0.10334724D-06
1.5	0.4384330D-05	0.11012633D-06
1.6	0.4731177D-05	0.11342972D-06
1.7	0.4980477D-05	0.11312237D-06
1.8	0.5116961D-05	0.10916432D-06
1.9	0.5125297D-05	0.10161543D-06
2.0	0.4991312D-05	0.90639024D-07

Four linear and non-linear the test problems are solve with the methods to confirm their accuracy. The accuracy of the method is compared with Kayode (2007) of the step number k = 3. The comparison of the absolute errors

Table 4. Comparison of errors in Kayode (2007) with new method (11).

	Kayode (2007) for Problem (iv)	New method (11) for Problem (iv)
X	Absolute errors	Errors for k =3
1.1	0.8047086D-07	0.61853700D-08
1.2	0.1625604D-06	0.31695117D-07
1.3	0.2480160D-06	0.75714456D-07
1.4	0.3387987D-06	0.14304432D-06
1.5	0.4372248D-06	0.24120724D-06
1.6	0.5490446D-06	0.38177170D-06
1.7	0.6725762D-06	0.58268768D-06
1.8	0.8153498D-06	0.87233773D-07
1.9	0.9842053D-06	0.12968951D-07
2.0	0.1188939D-05	0.19343897D-06

as shown in Tables (1) to (2) confirms a better result over Kayode (2007).

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