

Review

A focus on the Riemann's hypothesis

Jean-Max CORANSON-BEAUDU

Le Lamentin, Martinique, F.W.I, France.

Received 16 January 2020; Accepted 26 August 2020

Riemann's hypothesis, formulated in 1859, concerns the location of the zeros of Riemann's Zeta function. The history of the Riemann hypothesis is well known. In 1859, the German mathematician B. Riemann presented a paper to the Berlin Academy of Mathematic. In that paper, he proposed that this function, called Riemann-zeta function takes values 0 on the complex plane when $s=0.5+it$. This hypothesis has great significance for the world of mathematics and physics. This solutions would lead to innumerable completions of theorems that rely upon its truth. Over a billion zeros of the function have been calculated by computers and shown that all are on this line $s = 0.5+it$. In this paper, we initially show that Riemann's ζ (Zêta) function and the analytical extension of this function called \aleph (Aleph) are distinct. After extending this function in the complex plane except the point $s=1$, we will show the existence and then the uniqueness of real part zeros equal to $1/2$.

Key words: Riemann' hypothesis, Hadamard product, zeta function

INTRODUCTION

Riemann's hypothesis is expressed as following:

All non-trivial zeros of the function $\zeta(s)$ are located on the complex line $\Re(s) = \frac{1}{2}$

INTRODUCTION - ON THE ANALYTICAL EXTENSION OF THE FUNCTION \aleph

The analytical extension of the function $\zeta(s)$ on \mathbb{C} will be called $\aleph(s)$ in order to distinguish it from the function of Riemann. Riemann's Zeta function is written:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

For all complex numbers $\Re(s) > 1$

The function $\frac{1}{x^s}$ for $x \in \mathbb{R}$ and $s \in \mathbb{C}$ is differentiable p times. The p -th derivative of this function is written:

$$\left(\frac{1}{x^s}\right)^{(p)} = (-1)^p s(s+1)(s+2) \dots (s+p-1) \frac{1}{x^{s+p}}$$

Applying Euler Mac-Laurin's (Havil, 2003; Poels, 2011) formula to the function:

$$\sum_{n=1}^N \frac{1}{n^s} = \int_1^N \frac{dx}{x^s} + \frac{1}{2} \left(1 + \frac{1}{N^s}\right) - \sum_{j=1}^M \frac{b_{2j}}{(2j)!} s(s+1) \dots (s+2j-2) \left(\frac{1}{N^{s+2j-1}} - 1\right) + R_M(s) \tag{1}$$

$$R_M(s) = -\frac{s(s+1) \dots (s+2M-1)}{(2M+1)!} \int_1^N B_{2M+1}^*(x) x^{-s-2M-1} dx$$

*Corresponding author. E-mail: jmcorsan@me.com

Where $B_{2M+1}^*(x) = B_{2M+1}(x - E(x))$ is a 1-periodic function $B_n(x)$, called the p-th Bernoulli polynomial and $b_n = B_n(0)$, called the p-th number of Bernoulli.

For $N \rightarrow +\infty$ the left member of the Equation 1 leans towards $\zeta(s)$ and the development of Euler MacLaurin, a right sided part of the equation is defined by:

$$\frac{1}{s-1} + \frac{1}{2} + \sum_{j=1}^M \frac{b_{2j}}{(2j)!} s(s+1) \dots (s+2j-2) + \sigma_M(s)$$

With

$$\sigma_M(s) = -\frac{s(s+1) \dots (s+2M)}{(2M+1)!} \int_1^{+\infty} B_{2M+1}^*(x) x^{-s-2M-1} dx$$

σ_M being a convergent integral for $\Re(s) > 1 - 2M \forall M \in \mathbb{N}^*$, converges for all s of the complex plan except in $s=1$.

The other members of MacLaurin's development being polynomes, the analytical extension of the Zeta function is defined by the entire complex plan except in 1. The analytical extension (Edwards, 1974; Lachaud, 2001) of Riemann's function is expressed by the following formula:

$$\Re(s) = \left\{ \begin{array}{l} \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for} \quad \Re(s) > 1 \\ \frac{1}{s-1} + \frac{1}{2} + \sum_{j=1}^M \frac{b_{2j}}{(2j)!} s(s+1) \dots (s+2j-2) + \sigma_M(s) \quad \forall s \in \mathbb{C}/\{1\} \end{array} \right\} \quad (2)$$

It is clear that calculating the value of $\zeta(s)$ for values such as 0 or -1 with the following formula,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is impossible. So,

$$\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12}$$

is nonsense.

On the other $\Re(-1)$ does exist through the converging integral $\sigma_M(-1)$.

The function $\zeta(s)$ does not admit zeronon its domain. $\Re(s) > 1$.

On the other hand $\Re(s)$ being holomorphic on $\mathbb{C}/\{1\}$ there are zeroes for $\Re(s) \leq 1$.

ON THE ZEROS OF THE FUNCTION $\Re(s)$

According to Fourier's (Andreas, 1987) analysis, the function $x \rightarrow e^{-\pi x^2}$ that belongs to Schwartz's (Schwartz, 1966) space of fast decay functions to infinity, coincides with his transformed Fourier, that is:

$$\int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2i\pi ux} dx = e^{-\pi u^2}$$

By making the variable change of $x \rightarrow \frac{x}{\sqrt{t}}$ in this integral, the Fourier transformation of the function

$$f(x) = e^{-\pi t x^2} \text{ is } \hat{f}(u) = \frac{1}{\sqrt{t}} e^{-\frac{\pi u^2}{t}}$$

And all functions of Schwartz's space we have the following relationship:

$$\forall n \in \mathbb{Z} \quad \sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

which implies that

$$\forall t > 0 \quad \mathcal{U}(t) = \sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}} \quad (3)$$

\mathcal{U} and ψ functions meet the following functional equations

$$\forall u > 0 \quad \mathcal{U}(u) = \frac{1}{\sqrt{u}} \mathcal{U}\left(\frac{1}{u}\right) \text{ et } \psi(u) = \frac{\mathcal{U}(u) - 1}{2} = \sum_{n=1}^{\infty} e^{-\pi u n^2}$$

ψ checks:

$$\psi\left(\frac{1}{u}\right) = \frac{\mathcal{U}\left(\frac{1}{u}\right) - 1}{2} = \frac{\mathcal{U}(u)\sqrt{u} - 1}{2} = \frac{\sqrt{u}(2\psi(u) + 1) - 1}{2} = \sqrt{u}\psi(u) + \frac{\sqrt{u}}{2} - \frac{1}{2}$$

That is,

$$\forall u > 0 \quad \psi(u) = \frac{1}{\sqrt{u}} \psi\left(\frac{1}{u}\right) + \frac{1}{2\sqrt{u}} - \frac{1}{2} \quad (4)$$

$$\forall s / \Re(s) > 1, \text{ et } n \neq 0$$

To calculate the full one below by posing the variable change,

$$u = \frac{1}{\pi n^2} t$$

$$\begin{aligned}
 I_n &= \int_0^\infty u^{\frac{s}{2}-1} e^{-\pi n^2 u} du = \int_0^\infty \frac{\pi n^2}{(\pi n^2)^{\frac{s}{2}+1}} t^{\frac{s}{2}-1} e^{-t} dt \\
 &= \frac{1}{\pi^{s/2}} \frac{1}{n^s} \int_0^\infty t^{\frac{s}{2}-1} e^{-t} dt \\
 I_n &= \frac{1}{\pi^{s/2}} \frac{1}{n^s} \Gamma\left(\frac{s}{2}\right)
 \end{aligned}$$

By summing n, we obtain:

$$\sum_{n=1}^{\infty} I_n = \sum_{n=1}^{\infty} \int_0^\infty u^{\frac{s}{2}-1} e^{-\pi n^2 u} du = \sum_{n=1}^{\infty} \frac{1}{\pi^{s/2}} \frac{1}{n^s} \Gamma\left(\frac{s}{2}\right)$$

The inversion between infinite summation and integration is justified by the convergence properties of the function $e^{-\pi n^2 u}$. So we obtain:

$$\int_0^\infty u^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 u} du = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

That is,

$$\int_0^\infty u^{\frac{s}{2}-1} \psi(u) du = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \tag{5}$$

With $\zeta(s)$ is the function of Riemann for $\Re(s) > 1$

The integral 5 is developed on the intervals, $[0; 1] \cup [1; +\infty]$. We have:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty u^{\frac{s}{2}-1} \psi(u) du = \int_0^1 u^{\frac{s}{2}-1} \psi(u) du + \int_1^\infty u^{\frac{s}{2}-1} \psi(u) du$$

as

$$\psi(u) = \frac{1}{\sqrt{u}} \psi\left(\frac{1}{u}\right) + \frac{1}{2\sqrt{u}} - \frac{1}{2}$$

So on the interval $[0; 1]$ we can write:

$$\int_0^1 u^{\frac{s}{2}-1} \psi(u) du = \int_0^1 u^{\frac{s}{2}-1} \left(\frac{1}{\sqrt{u}} \psi\left(\frac{1}{u}\right) + \frac{1}{2\sqrt{u}} - \frac{1}{2} \right) du$$

By placing $u = \frac{1}{v}$ in the first part of the integral we have:

$$\int_0^1 u^{\frac{s}{2}-1} \psi(u) du = - \int_\infty^1 v^{-\frac{s}{2}+1} \left(v^{\frac{1}{2}} \psi(v) \right) \frac{1}{v^2} dv + \int_0^1 u^{\frac{s}{2}-1} \left(-\frac{1}{2} + \frac{1}{2\sqrt{u}} \right) du$$

$$\int_0^1 u^{\frac{s}{2}-1} \psi(u) du = \int_1^\infty \psi(v) v^{-\frac{s}{2}+1-2+\frac{1}{2}} dv - \frac{u^{\frac{s}{2}}}{2\left(\frac{s}{2}\right)} \Big|_0^1 + \frac{u^{\frac{s}{2}-\frac{1}{2}}}{2\left(\frac{s-1}{2}\right)} \Big|_0^1$$

$$\int_0^1 u^{\frac{s}{2}-1} \psi(u) du = \int_1^\infty \psi(u) u^{-\frac{s}{2}-\frac{1}{2}} du - \frac{1}{s} + \frac{1}{s-1}$$

Therefore

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_1^\infty \left(u^{\frac{s}{2}-1} + u^{-\frac{s-1}{2}} \right) \psi(u) du - \frac{1}{s} - \frac{1}{1-s} \tag{6}$$

This integral is converging for any complex except 0 and 1. $\zeta(s)$ function is defined by continuity on $\mathbb{C}/\{0; 1\}$

as

$$\frac{1}{s} + \frac{1}{1-s} = \frac{1}{s(1-s)}$$

by multiplying Equation 6 by $s(s-1)$ we have:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) s(s-1) = s(s-1) \int_1^\infty \left(u^{\frac{s}{2}-1} + u^{-\frac{s-1}{2}} \right) \psi(u) du + 1$$

therefore we use the term $\aleph(s)$ instead of $\zeta(s)$ and define:

$$\beth(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \aleph(s) s(s-1) = s(s-1) \int_1^\infty \left(u^{\frac{s}{2}-1} + u^{-\frac{s-1}{2}} \right) \psi(u) du + 1$$

as

$$\frac{s}{2} \Gamma\left(\frac{s}{2}\right) = \Gamma\left(\frac{s}{2} + 1\right)$$

Then,

$$\beth(s) = 2\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2} + 1\right) \aleph(s) (s-1) = s(s-1) \int_1^\infty \left(u^{\frac{s}{2}-1} + u^{-\frac{s-1}{2}} \right) \psi(u) du + 1 \tag{7}$$

This integral is defined $\forall s \in \mathbb{C}$ thanks to the rapid decay property of the ψ function to infinity. It can be said that $\beth(s)$ is holomorphic in \mathbb{C} .

So, $\beth(s) = \Phi(s) \aleph(s)$, with meromorphic

$\Phi(s) = 2\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2} + 1\right) (s-1)$ and $\beth(s)$ is holomorphic, then $\aleph(s)$ is meromorphic.

On the other hand, $\beth(s) = \beth(1-s)$ is a functional relationship between $\aleph(s)$ and $\aleph(1-s)$:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \aleph(s) s(s-1) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \aleph(1-s) s(s-1)$$

That is,

$$\Gamma\left(\frac{s}{2}\right) \aleph(s) = \pi^{s-\frac{1}{2}} \Gamma\left(\frac{1-s}{2}\right) \aleph(1-s) \tag{8}$$

The function \beth is written on \mathbb{C}

$$\beth(s) = s(s-1) \int_1^\infty \left(u^{\frac{s-1}{2}} + u^{\frac{s-2}{2}} \right) u^{-\frac{3}{4}} \psi(u) du + 1$$

That is,

$$\beth(s) = 2 \int_1^\infty s(s-1) u^{-\frac{3}{4}} \psi(u) \cosh\left[\left(s-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] du + 1 \tag{9}$$

verify that,

$$\beth(s) = \beth(1-s) \text{ and } \beth(0) = \beth(1-0) = 1$$

The trivial zeroes

$$\beth(s) = 2\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}+1\right) \aleph(s) s(s-1) \Rightarrow \aleph(s) = \frac{1}{\Gamma\left(\frac{s}{2}+1\right)} \frac{\pi^{\frac{s}{2}}}{2(s-1)} \beth(s)$$

The function $\frac{1}{\Gamma\left(\frac{s}{2}+1\right)}$ included as zeroes $\frac{s}{2} + 1 = -k, k \in \mathbb{N}; s = -2(k+1)$. On $\mathbb{C} \setminus \{1\}$

$\frac{\pi^{\frac{s}{2}}}{2(s-1)} \beth(s)$ function is holomorphic.

Therefore, the function $\aleph(s)$ included the same trivial zeroes as the zeroes of function $\frac{1}{\Gamma\left(\frac{s}{2}+1\right)}$ $s = -2(1+k), k \in \mathbb{N}$ which are whole negative pairs.

Non-trivial zeroes

If there are non-trivial zeroes in the complex plan for this function \beth ,

We expressed them as $z_k = a_k + ib_k, k \in \mathbb{N}$ and these are the same zeroes as the function $\aleph(s)$. Note $\Re(\cdot)$ the real part and $\Im(\cdot)$ the imaginary part.

These zeroes check the next relationship for the \beth function.

$$\beth(z_k) = 0 \Leftrightarrow \begin{cases} \Re(\beth(z_k)) = 0 \\ \Im(\beth(z_k)) = 0 \end{cases} \quad \forall k \in \mathbb{N} \tag{10}$$

By writing the real and imaginary part of the integral 9

for $s = z_k$ we obtain:

$$\left\{ \begin{array}{l} 2 \int_1^\infty u^{-\frac{3}{4}} \psi(u) \Re\left[z_k(z_k-1) \cosh\left[\left(z_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] \right] du + 1 = 0 \\ 2 \int_1^\infty u^{-\frac{3}{4}} \psi(u) \Im\left[z_k(z_k-1) \cosh\left[\left(z_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] \right] du = 0 \end{array} \right. \tag{11}$$

We seek to identify complex z_k values that verify the Equation 11 as,

$$z_k(z_k-1) = (a_k + ib_k)(a_k - 1 + ib_k) = a_k(a_k-1) - b_k^2 + ib_k(2a_k-1)$$

and,

$$\begin{aligned} \cosh\left[\left(z_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] &= \cosh\left[\left(a_k-\frac{1}{2}+ib_k\right) \frac{\ln(u)}{2}\right] \\ &= \cosh\left[\left(a_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] \cos\left(b_k \frac{\ln(u)}{2}\right) + i \sinh\left[\left(a_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] \sin\left(b_k \frac{\ln(u)}{2}\right) \end{aligned}$$

We note $R(a_k, b_k, u)$ the real part of the product $z_k(z_k-1) \cosh\left[\left(z_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right]$

And $I(a_k, b_k, u)$ the imaginary part of the product $z_k(z_k-1) \cosh\left[\left(z_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right]$

We have got:

$$\begin{aligned} R(a_k, b_k, u) &= (a_k(a_k-1) - b_k^2) \cosh\left[\left(a_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] \cos\left(b_k \frac{\ln(u)}{2}\right) \\ &\quad - b_k(2a_k-1) \sinh\left[\left(a_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] \sin\left(b_k \frac{\ln(u)}{2}\right) \end{aligned}$$

From this expression, we obtain, because of property of $A \cos\left[b_k \frac{\ln(u)}{2}\right] + B \sinh\left[b_k \frac{\ln(u)}{2}\right]$,

$$R(a_k, b_k, u) = \sqrt{A^2 + B^2} \sin\left[b_k \frac{\ln(u)}{2} + \frac{\pi}{2} \text{Sign}(A) - \arctan\left(\frac{B}{A}\right)\right] \tag{12}$$

With

$$A = (a_k(a_k-1) - b_k^2) \cosh\left[\left(a_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] \text{ and } B = -b_k(2a_k-1) \sinh\left[\left(a_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right]$$

We also have:

$$\begin{aligned} I(a_k, b_k, u) &= (a_k(a_k-1) - b_k^2) \sinh\left[\left(a_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] \sin\left(b_k \frac{\ln(u)}{2}\right) \\ &\quad + b_k(2a_k-1) \cosh\left[\left(a_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right] \cos\left(b_k \frac{\ln(u)}{2}\right) \\ I(a_k, b_k, u) &= \sqrt{U^2 + V^2} \sin\left[b_k \frac{\ln(u)}{2} + \frac{\pi}{2} \text{Sign}(U) - \arctan\left(\frac{V}{U}\right)\right] \end{aligned} \tag{13}$$

With $U = b_k(2a_k-1) \cosh\left[\left(a_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right]$ and $V = (a_k(a_k-1) - b_k^2) \sinh\left[\left(a_k-\frac{1}{2}\right) \frac{\ln(u)}{2}\right]$

For the imaginary part of the integral of the equation

system 11 we have:

$$2 \int_1^\infty u^{-\frac{3}{4}} \psi(u) \Im \left[z_k (z_k - 1) \cosh \left[\left(z_k - \frac{1}{2} \right) \frac{\ln(u)}{2} \right] \right] du = 0$$

That is,

$$2 \int_1^\infty u^{-\frac{3}{4}} \psi(u) I(a_k, b_k, u) du = 0 \tag{14}$$

Are there couples (a_k, b_k) such as Equation 14 equals zero?

Because of the convergence characteristics of the integral, we have the following property:

$$-\int_1^\infty u^{-\frac{3}{4}} \psi(u) |I(a_k, b_k, u)| du \leq \int_1^\infty u^{-\frac{3}{4}} \psi(u) I(a_k, b_k, u) du \leq \int_1^\infty u^{-\frac{3}{4}} \psi(u) |I(a_k, b_k, u)| du$$

That is,

$$-\int_1^\infty u^{-\frac{3}{4}} \psi(u) \sqrt{U^2 + V^2} du \leq \int_1^\infty u^{-\frac{3}{4}} \psi(u) I(a_k, b_k, u) du \leq \int_1^\infty u^{-\frac{3}{4}} \psi(u) \sqrt{U^2 + V^2} du$$

Applying the properties of the full continuous and positive function, and the squeeze theorem,

we have:

$$\int_1^\infty u^{-\frac{3}{4}} \psi(u) \sqrt{U^2 + V^2} du = 0 \Rightarrow \sqrt{U^2 + V^2} = 0 \Leftrightarrow \begin{cases} U = 0 \\ V = 0 \end{cases} \forall u \geq 1$$

The existence of the couples (a_k, b_k) such as:

$$\begin{cases} U = 0 \Leftrightarrow a_k = \frac{1}{2} \text{ ou } b_k = 0 \\ V = 0 \Leftrightarrow a_k = \frac{1}{2} \text{ ou } a_k(a_k - 1) - b_k^2 = 0 \end{cases}$$

The system is reduced to three pairs of solutions:

$$\begin{cases} a_k = \frac{1}{2} \\ b_k \in \mathbb{R} \end{cases} \text{ or } \begin{cases} a_k = 1 \\ b_k = 0 \end{cases} \text{ or } \begin{cases} a_k = 0 \\ b_k = 0 \end{cases}$$

We're checking that:

$z_0 = (0,0)$ ou $z_1 = (1,0)$ are des trivial solutions $\Im(\zeta(0)) = \Im(\zeta(1)) = 0$ because $\Re(0) = \Re(1) = 1$

And $\begin{cases} a_k = \frac{1}{2} \\ b_k \in \mathbb{R} \end{cases}$ are non-trivial zeroes. As a result, we have shown that there are non-trivial zeroes on the

critical axis $\Re(s) = \frac{1}{2}$.

The imaginary part b_k of these zeroes is identified using the first integral of the equation system 11 expressed:

$$2 \int_1^\infty u^{-\frac{3}{4}} \psi(u) R(a_k, b_k, u) du + 1 = 0 \tag{15}$$

That is,

$$2 \int_1^\infty u^{-\frac{3}{4}} \psi(u) \sqrt{A^2 + B^2} \sin \left[b_k \frac{\ln(u)}{2} + \frac{\pi}{2} \text{Sign}(A) - \arctan \left(\frac{B}{A} \right) \right] du + 1 = 0$$

Taking into consideration the result found for the imaginary part of the integral, the following couples:

$$\begin{cases} a_k = \frac{1}{2} \\ b_k \in \mathbb{R} \end{cases}$$

We've got:

$$\begin{cases} A = (a_k(a_k - 1) - b_k^2) \cosh \left[\left(a_k - \frac{1}{2} \right) \frac{\ln(u)}{2} \right] = -\frac{1}{4} - b_k^2 \\ B = b_k(2a_k - 1) \sinh \left[\left(a_k - \frac{1}{2} \right) \frac{\ln(u)}{2} \right] = 0 \end{cases}$$

Therefore the Equation 15 is written:

$$2 \int_1^\infty u^{-\frac{3}{4}} \psi(u) \left(\frac{1}{4} + b_k^2 \right) \cos \left(b_k \frac{\ln(u)}{2} \right) du = 1 \tag{16}$$

and b_k is a solution of the Equation 16. So there is an infinity of zeroes on the critical axis of the $\Re(s) = 1/2$ for $z_k = \frac{1}{2} + ib_k$

We show that these zeroes are all on the critical axis Assumptions: Suppose there are zeroes outside the critical axis and in the critical band.

These zeros are written from the existing zeros on the critical line: with $y_k = z_k + \varepsilon e^{i\delta}$

$$0 < \varepsilon < \frac{1}{2}$$

We know that all $\overline{z_k} = 1 - z_k$ because $\Re(z_k) = \frac{1}{2} \quad k \in \mathbb{N}$

$\zeta(z)$ is holomorph in \mathbb{C} , thus being an entire function. Weierstrass's factorization theorem (Patterson, 1995; Vento, 2003) states that any entire function can be represented by an infinite polynomial product with its zeroes. There is g holomorph in \mathbb{C} that does not cancel in z_k and $\overline{z_k}$ such as:

$$\zeta(z) = \prod_{k=1}^\infty \left(1 - \frac{z}{z_k} \right) \left(1 - \frac{z}{\overline{z_k}} \right) g(z(1-z)) = \prod_{k=1}^\infty \left(1 - \frac{z}{|z_k|^2} (1-z) \right) g(z(1-z)) \tag{17}$$

We check that the z_k and \bar{z}_k are zeros of, $\zeta(z)$, the function verifies

$$\zeta(1-z) = \zeta(z)$$

$$\overline{\zeta(z)} = \zeta(\bar{z})$$

Suppose that $y_k = z_k + \varepsilon_k e^{i\delta_k}$ and $\bar{y}_k = \bar{z}_k + \varepsilon_k e^{-i\delta_k}$ are also zeros of $\zeta(z)$, so we have:

$$\zeta(z) = \zeta_{z_k}(z) \prod_{k=1}^{\infty} \left(1 - \frac{z}{|y_k|^2} (y_k + \bar{y}_k - z)\right) g^*(z(1-z)) \tag{18}$$

with

$$\zeta_{z_k}(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{|z_k|^2} (1-z)\right)$$

And g^* is holomorphic and does not cancel out for y_k, z_k , and their conjugates.

$$\zeta(1-z) = \zeta_{z_k}(1-z) \prod_{k=1}^{\infty} \left(1 - \frac{(1-z)}{|y_k|^2} (y_k + \bar{y}_k - (1-z))\right) g^*(z(1-z))$$

As $\zeta_{z_k}(1-z) = \zeta_{z_k}(z)$ then:

$$\zeta(1-z) = \zeta_{z_k}(z) \prod_{k=1}^{\infty} \left(1 - \frac{y_k + \bar{y}_k - 1}{|y_k|^2} - \frac{z}{|y_k|^2} (1 - y_k - \bar{y}_k + 1 - z)\right) g^*(z(1-z))$$

And $\zeta(1-z) = \zeta(z) \Leftrightarrow y_k + \bar{y}_k - 1 = 0$

So $z_k + \varepsilon_k e^{i\delta_k} + \bar{z}_k + \varepsilon_k e^{-i\delta_k} - 1 = 0$ that is,
 $\varepsilon_k e^{i\delta_k} + \varepsilon_k e^{-i\delta_k} = 0$

Which is impossible since $\varepsilon_k \neq 0$

Therefore the hypothesis of zeros outside the critical axis leads to a contradiction in relation to the symmetries of function $\zeta(X)$ in the critical band.

There are no zeroes outside the axis $\Re(z_k) = 1/2$.

Conclusion

We have demonstrated:

(i) that the holomorphic function $\zeta(s)$ had the same zeros as the function $\aleph(s)$ which is an analytical extension of Riemann's $\zeta(s)$ function because $\aleph(s) =$

$$\frac{1}{\Gamma(\frac{s}{2}+1)} \frac{\pi^{\frac{s}{2}}}{2(s-1)} \zeta(s).$$

This result well known by the mathematical world, served us to find a holomorphic function simpler to exploit at the roots.

(ii) using the squeeze theorem on the integral form of the Riemann function, we show that there are a pairs (a_k, b_k) that are zeros of the Riemann function and these zeros are on the line $s = \frac{1}{2} + it$

(iii) as Hadamard (1896) Charles-Jean (1916) have each proved that no zero of the analytical extension of the Zeta function could be found on the line $\Re(s)= 1$, and therefore that all non-trivial zeroes must be in the interior of the critical band.

(iv) we have been hypothesis that if there were zeros, $y_k = z_k + \varepsilon_k e^{i\delta_k}$, in the critical band, with $0 < \varepsilon_k < \frac{1}{2}$, then this hypothesis leads to a contradiction. We used the Weierstrass's factorization theorem of holomorphic functions for $\zeta(s)$, and applying functional relationship of symmetry, $\zeta(1-z) = \zeta(z)$, to demonstrate contradiction. Therefore, all non-trivial zeroes of ζ are non-trivial zeroes of the analytical extension of the function ζ and have a real part $\frac{1}{2}$. These zeroes, noted $z_k = a_k + ib_k$ check the equation systems below:

$$\forall k \in \mathbb{Z} \begin{cases} a_k = \frac{1}{2} \\ b_k \in \mathbb{R} / 2\left(\frac{1}{4} + b_k^2\right) \int_1^{\infty} u^{-\frac{3}{4}} \psi(u) \cos\left[2b_k \frac{\ln(u)}{2}\right] du = 1 \end{cases} \tag{19}$$

A simple digital example

A numerical integration by Rombert's method with order precision 5 and 20 iterations, we find the results of the complete system 16 with an error of 10^{-6} .

- $b_1 = 14.13472$
- $b_2 = 21.02203$
- $b_3 = 25.01085$
- $b_4 = 30.42487$
- $b_5 = 32.93506$

CONFLICT OF INTERESTS

The author has not declared any conflict of interests.

REFERENCES

Andreas S (1987). Reimann's second proof of the analytic continuation of the Riemann zeta function, Holden-day Inc.
 Charles-Jean de La VP ((1916), On the zeros of the Riemann's zeta fuction, Comptes rendus de l'Académie des Sciences Paris 163:418-421.
 Edwards HM (1974). Reimann's Zeta Function, Academic Press, New York, London.
 Hadamard J (1896). Sur la distribution des zeros de la fonction Zetaet ses conséquences arithmétiques. Bulletin de la Societé mathématique de France 24:199-220
 Havil J (2003). Gamma: Exploring Euler's Constant. Princeton, NJ: Princeton University Press pp. 85-86,
 Lachaud G (2001). L'hypothèse de Riemann. La Recherche-Paris pp. 24-30.
 Patterson SJ (1995). An introduction to the theory of the Riemann zeta function. No. 14. Cambridge University Press.

Schwartz L (1966). *Theorie des distributions (Vol. 2)*, Paris, Hermann.
Vento S (2003). *Factorisation des Fonctions holomorphes, Espace de Hardy*. Travaux d'Etudes et de Recherches.