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Full Length Research Paper

Results of symmetric groups S_n ($n \le 7$) acting on unordered triples and ordered quadruples

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In this paper, we examined the results of fixed point set of symmetric groups S_n ($n \le 7$) acting on $X^{(3)}$ and $X^{[4]}$. In order to find the fixed point set| fix (g) | of these permutation groups, we used the method developed by Higman (1970) to compute the number of orbits, ranks and sub degrees of these actions. The results were used to find the number of orbits as proposed by Harary (1969) in Cauchy-Frobenius Lemma and hence deduce transitivity.

Key words: Cycles, | Fix (g) |, Lemma.

INTRODUCTION

In 1970, Higman calculated the rank and the sub-degrees of the symmetric group S_n acting on 2-elements subsets from the set X= {1, 2, 3… n.}. He showed that the rank is 3 and the sub-degrees are $1, 2\ n-2\ , \binom{n-2}{2}$. Faradžev and Ivanov (1990) calculated the subdegrees of primitive permutation representations of PSL (2, q). They showed that if G = PSL (2, q) acts on the cosets of its maximal sub-group H, then the rank is at least $\frac{|G|}{|H|^2}$ and if q>100, the rank is greater than 5.

In 1992, Kamuti devised a method for constructing some of the suborbital graphs of PSL (2,q) and PGL (2,q) acting on the cosets of their maximal dihedral subgroups of orders q-1 and 2(q-I) respectively. This method gave an alternative way of constructing the Coxeter graph which was first constructed by Coxeter (1986).

Neumann (1977) gave general properties of suborbital graphs. In this paper, he gave a construction of the famous Petersen graph which was first constructed by Petersen in 1898.

In 2001, Akbas investigated the suborbital graphs for the modular group. He proved the conjecture by Jones et al. (1991) that a suborbital graph for the modular group is a forest if and only if it contains no triangles. Kamuti (2006) calculated the sub degrees of primitive permutation representations of PGL (2,q). He showed that when PGL (2,q) acts on the cosets of its maximal dihedral subgroup of order 2 (q-1) then its rank is ½ (q+3) if q is odd, and ½ (q+2) if q is even.

This shows that finding the fixed point set | fix (g) |, for the action of S_n ($n \le 7$) on $X^{(3)}$ and $X^{[4]}$ do not seem to have been published so far. Therefore, in this paper we find some formulas for fixed point set of these actions.

List of notations

S_n - Symmetric group of degree n and order n!
| G | - The order of a group G
{a, b, c} - An unordered triple
[a, b, c, d] -An ordered quadruple
$X^{(3)}$ - The set of all unordered triples from the set X = {1, 2… n}
$X^{[4]}$ - The set of all ordered quadruples from the set:

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X = {1, 2... n}

$\begin{pmatrix} r \\ s \end{pmatrix}$ - r combinations

| Fix (g) | - The number of elements in the fixed point set of g.

PRELIMINARY DEFINITIONS

In this area, we look briefly at some results and definitions on permutation groups which we are interested in.

Definition 1

Let X be a set; a group G acts on the left on X if for each $g \in G$ and each $x \in X$ there corresponds a unique element $gx \in X$ such that:

(i) $(g_1g_2) = g_1 (g_2x)$, $\forall g_1, g_2 \in G$ and $x \in X$
(ii) For any $x \in X$, 1x=x, where 1 is the identity in G.

Definition 2

Let G act on a set X. The set of elements of X fixed by $g \in G$ is called the fixed point set of g and is denoted by Fix (g). Thus Fix (g) = $\{x \in X \mid gx=x\}$.

Definition 3

If the action of a group G on a set X has only one orbit, then we say that G acts transitively on X. In other words, G acts transitively on X if for every pair of points x, $y \in X$, there exists $g \in G$ such that gx = y.

Theorem 1 (Harary, 1969:98)

Cauchy – Frobenius Lemma

Let G be a finite group acting on a set X. Then the number of orbits of G is:

$$\frac{1}{|G|}\sum_{g \in G}|Fix\ \ g\ |$$

Definition 4

If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation б of X, which can be written uniquely as a product of disjoint cycles. If σ has α_1 cycles of length 1, α_2 cycles of length 2,… α_n cycles of length n, we say that σ and hence g has cycle type $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Theorem 2 (Krishnamurthy, 1985:68)

Two permutations in S_n are conjugate if and only if they have the same cycle type; and if $g \in G$ has cycle type $\alpha_1, \alpha_2, \ldots, \alpha_n$, then the number of permutations in S_n conjugate to *g* is $\dfrac{n!}{\prod_{i=1}^{n} \alpha_i !\, i^{\alpha_i}}$.

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Lemma 1

Let the cycle type of $g \in S_n$ be $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then | Fix (g) | in $X^{(3)}$ is given by the formula:

$$\text{Fix (g) |} = \begin{pmatrix} \alpha_1 \\ 3 \end{pmatrix} + \alpha_2\ \alpha_1 + \alpha_3$$

Proof

Let $g \in S_n$ has cycle type $\alpha_1, \alpha_2, \ldots, \alpha_n$. {a, b, c}$\in X^{(3)}$ is fixed by g if each of a, b, and c come from a single cycle in g or one of a, b, or c come from single cycle in g and the other two come from a 2 – cycle in g or a, b, c come from a 3 – cycle in g. From the first case, the number of unordered triples fixed by g is $\begin{pmatrix} \alpha_1 \\ 3 \end{pmatrix}$; from the second case the number of unordered triples fixed by g is $\alpha_2\ \alpha_1$ and in the third case the number of unordered triples fixed by g is α_3 . Therefore the number of unordered triples fixed by g is:

$$\begin{pmatrix} \alpha_1 \\ 3 \end{pmatrix} + \alpha_2\ \alpha_1 + \alpha_3$$

Lemma 2

Let $g \in S_n$ be a permutation with cycle type $\alpha_1, \alpha_2, \ldots, \alpha_n$ Then the number of permutations in S_n fixing {a, b, c}$\in X^{(3)}$ and having the same cycle type as g is given by:

$$\frac{(n-3)!}{(\alpha_1-3)!1^{\alpha_1-3}\prod_{i=2}^{n}\alpha_i!i^{\alpha_i}}+\frac{3(n-3)!}{(\alpha_1-1)!1^{\alpha_1-1}(\alpha_2-1)!2^{\alpha_2-1}\prod_{i=3}^{n}\alpha_i!i^{\alpha_i}}+\frac{2(n-3)!}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}(\alpha_3-1)!3^{\alpha_3-1}\prod_{i=4}^{n}\alpha_i!i^{\alpha_i}}$$

Proof

A permutation $g \in S_n$ fixes an unordered triple say {a, b, c} $\in X^{(3)}$:

a) If g maps each element a, b and c onto itself, that is each of the elements a, b and c comes from a single cycle. To get the number of permutations in S_n that fix {a, b, c} and having the same cycle type as g, we apply Theorem 6 to a permutation of S_n-3 with cycle type $\alpha_1-3,\alpha_2,\alpha_3,\ldots,\alpha_n$ to get

$$\frac{(n-3)!}{(\alpha_1-3)!1^{\alpha_1-3}\prod_{i=2}^{n}\alpha_i!i^{\alpha_i}}$$ permutations.

b) If one of the elements a, b and c comes from a single cycle and other two comes from a 2 – cycle. In this case a, b and c may come from any of the following three permutations; (ab) (c)…, (ac) (b)…or (bc)(a)… Applying Theorem 6 to a permutation of S_n -3 with cycle type $\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4,\ldots,\alpha_n$

We get

$$\frac{(n-3)!}{(\alpha_1-1)!1^{\alpha_1-1}(\alpha_2-1)!2^{\alpha_2-1}\prod_{i=3}^{n}\alpha_i!i^{\alpha_i}}$$ permutations.

Considering the three cases pointed so far, we get

$$\frac{3(n-3)!}{(\alpha_1-1)!1^{\alpha_1-1}(\alpha_2-1)!2^{\alpha_2-1}\prod_{i=3}^{n}\alpha_i!i^{\alpha_i}}$$ permutations.

c) If the elements a, b and c come from a 3-cycle in g. In this case a, b and c may come from the permutation (abc),… or (acb)….. Applying Theorem 6 to a permutation of S_n -3 with cycle type $\alpha_1,\alpha_2,\alpha_3-1,\alpha_4,\ldots,\alpha_n$ we get

$$\frac{(n-3)!}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}(\alpha_3-1)!3^{\alpha_3-1}\prod_{i=4}^{n}\alpha_i!i^{\alpha_i}}$$ permutations.

Considering the 2 cases pointed so far, we get

$$\frac{2(n-3)!}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}(\alpha_3-1)!3^{\alpha_3-1}\prod_{i=4}^{n}\alpha_i!i^{\alpha_i}}$$

Combining cases (a), (b) and (c), we get the required result.

Lemma 3

Let g $\in S_n$ be a permutation with cycle type $\alpha_1,\alpha_2,\ldots,\alpha_n$. Then | Fix (g) | in $X^{[4]}$ is given by:

$$4!\binom{\alpha_1}{4}$$

Proof

Let [a, b, c, d] $\in X^{[4]}$ and $g \in S_n$. Then g fixes [a, b, c, d] if each of the elements a, b, c, d are mapped onto themselves, that is g [a, b, c, d] = [ga, gb, gc, gd] = [a, b, c, d] implying ga=a, gb=b, gc=c and gd=d. Each of a, b, c and d comes from single cycles. The number of unordered quadruples fixed by $g \in S_n$ is:

$$\binom{\alpha_1}{4}$$

But unordered quadruple, can be rearrange to give 24=4! distinct ordered quadruples. Thus the number of ordered quadruples fixed by $g \in S_n$ is'

$$4!\binom{\alpha_1}{4}$$

Lemma 4

Let g $\in S_n$ be a permutation with cycle type $\alpha_1,\alpha_2,\ldots,\alpha_n$ .Then the number of permutations in S_n fixing [a, b, c, d] $\in X^{[4]}$ and having the same cycle type as g is given by:

$$\frac{(n-4)!}{(\alpha_1-4)!1^{\alpha_1-4}\prod_{i=2}^{n}\alpha_i!i^{\alpha_i}}$$

Proof

Let $g \in S_n$ have cycle type $\alpha_1,\alpha_2,\ldots,\alpha_n$ and let g fix [a, b, c, d]. Then each of a, b, c and d must come from a single cycle in g. So to count the number of permutations in S_n having the same cycle type as g and fixing a, b, c and d, is the same as counting the number of permutations in S_n-4 having cycle type $\alpha_1 - 4, \alpha_2, \ldots, \alpha_n$.

By the Theorem 6, this number is:

$$\frac{(n-4)!}{(\alpha_1 - 4)!1^{\alpha_1 - 4}\prod_{i=2}^{n} \alpha_i ! i^{\alpha_i}}$$

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