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Full Length Research Paper

Interpolation by quartic splines

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This paper proposes two special lacunary interpolation problems using quartic splines of continuity class C^2. We call the problems (0,3) and (0,4) lacunary interpolation. In these two cases, the third and fourth derivative respectively is also prescribed in between the nodes along with the function value at the nodes. This paper is divided into two parts namely cases A and B. Case A deals with the (0,3) interpolation problem whereas Case B discusses the (0,4) interpolation problem. Special lacunary interpolation problems have been solved using spline functions of continuity class C^2.

Key words: Lacunary interpolarion, quartic splines, modulus of continuity.

INTRODUCTION

Previous researches on special lacunary interpolation problems have been solved using spline functions of continuity class C^2 (Tarazi et al., 1987; Joshi and Saxena, 1982; Saxena, 1987, 1988). The present work proceeds as follows:

Let Δ: $0 = x_0 < x_1 < \ldots\ldots < x_n = 1$ be a partition of the unit interval I = [0,1], $h = x_{i+1} - x_i$, i = 0,1, ...,n –1 and Let $f \in C^3(I)$.

We denote by $S_{n,4}^{(2)}(x)$, the class of quartic splines s(x) such that

(i) $s(x) \in C^2(I)$
(ii) $s(x) \in \pi_4$ in each interval $[x_i, x_{i+1}]$, i = 0,1,......, n-1.

CASE A - (0,3)

Theorem (A1)

Given a partition Δ of the unit interval I = [0,1] and numbers I = [0,1], real number f_i, i = 0,1,..... n; $f_{i+\lambda h}'''$, i = 0,1,............ n– 1; f_0', f_n' and $0 \le \lambda < 3/8 \cup 5/8 < \lambda \le 1$,

There exists a unique spline $s(x) \in S_{n,4}^{(2)}$ such that

$$\begin{cases} s(x_i) = f(x_i), & i = 0,1,\ldots\ldots\ldots,n, \\ s'''(x_{i+\lambda h}) = f'''(x_{i+\lambda h}); & x_i \le x_i + \lambda h \le x_{i+1} \\ s'(x_0) = f'(x_0), & s'(x_n) = f'(x_n), \end{cases} \quad \text{(A1.1)}$$

where $x_i + \lambda h$, $h = x_{i+1} - x_i$, i = 0,1,……..n-1.

Preliminaries (P1)

If P(x) is a quadric on the unit interval [0,1] then (for $\lambda \neq 1/2$)

$$P(x) = P(0)A(x) + P(1)B(x) + P'(0)C(x) + P'(1)D(x) + P'''(\lambda)E(x) \quad \text{(P1.1)}$$

where

$$A(x) = 1/(1-2\lambda)\,[x^4 - 4\lambda x^3 - (2-6\lambda)x^2 + (1-2\lambda)] \quad \text{(P1.2)}$$

$$B(x) = 1/(1-2\lambda)\,[-x^4 + 4\lambda x^3 + (2-6\lambda)x^2] \quad \text{(P1.3)}$$

$$C(x) = 1/2(1-2\lambda)\,[x^4 - 4\lambda x^3 - (3-8\lambda)x^2 + (2-4\lambda)x] \quad \text{(P1.4)}$$

$D(x) = 1/2(1-2\lambda)\,[x^4 - 4\lambda x^3 - (1-4\lambda)x^2]$ (P1.5)

$E(x) = 1/12(1-2\lambda)\,[-x^4 + 2x^3 - x^2]$ (P1.6)

For later references we have

$$\begin{cases} A''(x) = 1/(1-2\lambda)[12x^2 - 24\lambda x - 2(2-6\lambda)] \\ B''(x) = 1/(1-2\lambda)[-12x^2 + 24\lambda x + 2(2-6\lambda)] \\ C''(x) = 1/2(1-2\lambda)[12x^2 - 24\lambda x - 2(3-8\lambda)] \\ D''(x) = 1/2(1-2\lambda)[12x^2 - 24\lambda x - 2(1-4\lambda)] \\ E''(x) = 1/12(1-2\lambda)[-12x^2 + 12x - 2] \end{cases} \tag{P1.7}$$

Also

$$\begin{cases} A''(0) = (-4+12\lambda)/(1-2\lambda), & A''(1) = (8-12\lambda)/(1-2\lambda), \\ B''(0) = (4-12\lambda)/(1-2\lambda), & B''(1) = (-8+12\lambda)/(1-2\lambda), \\ C''(0) = (-3+8\lambda)/(1-2\lambda), & C''(1) = (3-4\lambda)/(1-2\lambda), \\ D''(0) = (-1+4\lambda)/(1-2\lambda), & D''(1) = (5-8\lambda)/(1-2\lambda), \\ E''(0) = -1/6(1-2\lambda), & E''(1) = -1/6(1-2\lambda), \end{cases} \tag{P1.8}$$

Proof of Theorem (A1)

To prove the theorem, we shall use the following representation of s(x).

For $x_i \le x \le x_{i+1}$, i = 0,1,........ , n –1

$$s(x) = f(x_i)\frac{A(x-x_i)}{h} + f(x_{i+1})\frac{B(x-x_i)}{h} + h s'(x_i)\frac{C(x-x_i)}{h} + h s'(x_{i+1})\frac{D(x-x_i)}{h} + h^3 f'''(x_{i+\lambda h})\frac{E(x-x_i)}{h} \tag{A1.2}$$

Using continuity condition that $s(x) \in C^2[0,1]$, that is

$s''(x_i+) = s''(x_i-)$, i = 0,1,......... , n,

We have from Equation (P1.8)

$$(3-4\lambda)\,s'(x_{i-1}) + (8-16\lambda)s'(x_i) + (1-4\lambda)\,s'(x_{i+1}) = -1/h\,[(8-12\lambda)f(x_{i-1}) - 4f(x_i) - (4-12\lambda)f(x_{i-1}) + h^3/6f'''(x_{i+\lambda n}) - h^3/6f'''(x_{i-\lambda n}) \tag{A1.3}$$

It can be verified that the above system of equations is diagonally dominant for all λ ($0\le\lambda<3/8 \cup 5/8 < \lambda \le 1$). This ensures the unique existence of s(x) for all values of λ belonging to the above mentioned range, hence the theorem.

Theorem (A2)

Let $f \in C^{\ell}(I)$, ℓ = 3, 4. Then for the unique spline s(x) of Theorem (A1) with real numbers f_i, i = 0,1...... ,n, $f'''_{i+\lambda h}$, i = 0,1,........n– 1, f'_0 and f'_n associated with the function f and λ as mentioned in the Theorem (A1), we have

$$\| s^{(r)}(x) - f^{(r)}(x) \| \le h^{5-r} C_\lambda \omega_3(h) \text{ for } f \in C^3(I) \tag{A2.1}$$

$$\| s^{(q)}(x) - f^{(q)}(x) \| \le h^{6-q} C_\lambda^* \omega_4(h) \text{ for } f \in C^4(I),\ q, r = 0, 1, 2 \tag{A2.2}$$

Where $\omega_3(h)$ and $\omega_4(h)$ denote the modulus of continuity of $f(x) \in C^3(I)$ and $f(x) \in C^4(I)$ respectively. C_λ and C_λ^* represent different constants depending on λ as mentioned in the proof of the theorem.

Auxilary lemma

To prove the above theorem we shall use the following lemma.

Lemma (L1)

Set $e'_i = s'_i - f'_i$, i = 0,1,…………. ,n.

Let j be chosen in such a manner that

$\operatorname{Max} |e'_i| = |e'_j|$.

Then for $f \in C^3(I)$, we have

(L.1.1) $|e'_j| \le K_\lambda h^2 \omega_3(h)$,

Where

$$K_\lambda = \begin{cases} 1/3, & 0 \le \lambda \le 1/4 \\ (3-4\lambda)/2(5-8\lambda), & 1/4\lambda < 3/8 \\ (1-4\lambda)/2(3-8\lambda), & 5/8 \le \lambda < 2/3 \\ (11-24\lambda)/6(3-8\lambda), & 2/3 \le \lambda < 3/4 \\ (3-8\lambda)/12(1-2\lambda), & 3/4 \le \lambda < 1 \end{cases} \tag{L1.2}$$

For $f \in C^4$ (I)

$$| e_j' | \le K_\lambda * h^3 \omega_4(h),$$

Where

$$K_\lambda^* = \begin{cases} (7-8\lambda)/72(1-2\lambda) & , \quad 0 \le \lambda \le 1/4 \\ (7-8\lambda)/12(5-8\lambda) & , \quad 1/4\lambda < 3/8 \\ 1/4\ (3-8\lambda) & , \quad 5/8 < \lambda \le 3/4 \\ (6\lambda-1)/12(3+8\lambda) & , \quad 2/3 < \lambda \le 3/4 \\ (6\lambda-1)/24(3+2\lambda) & , \quad 3/4 < \lambda \le 1 \end{cases}$$

(L1.3)

Proof

From Equation (A1.3) we have

$$(3-4\lambda)\ e_{i+1}' + 8\ (1-2\lambda)\ e_i' + (1-4\lambda)\ e_{i+1}'$$

$$= -(8-12\lambda)/h\ f_{i-1} + 4/h\ f_i + 4\ (1-3\lambda)/h\ f_{i+1} - (3-4\lambda) f_{i+1}'$$

$$- 8(1-2\lambda)\ f_{i+1}' - (1-4\lambda)\ f_{i+1}' - h^2/6\ (f_{i+\lambda h}''' - f_{i-\lambda h}''')$$

Now using Taylor expansion, we have

$$(3-4\lambda)\ e_{i+1}' + 8\ (1-2\lambda)\ e_i' + (1-4\lambda)\ e_{i+1}'$$

$$= -(8-12\lambda)/h\ [f_i - h f_i' + h^2/2\ f_i'' - h^3/6\ f_i'''(\alpha_{i-1})] + 4fi$$

$$+ 4\ (1-3\lambda)\ [f_i + h f_i' + h^2/2\ f_i'' + h^3/6\ f_i'''\ (\beta_i)]$$

$$-(3-4\lambda)\ [f_i - h f_i'' + h^2/2\ f_i'''\ (\gamma_{i-1})] - 8\ (1\text{-}2\lambda)\ f_i'$$

$$-(1-4\lambda)\ [f_i' + h f_i'' + h^2/2\ f_i'''\ (\delta_i)] - h^2/6(f_{i+\lambda h}''' - f_{i-\lambda h}''')$$

$x_{i-1} \le \alpha_{i-1}, \gamma_{i-1} \le x_i,\ x_i \le \beta_i,\ \delta_j \le x_{i+1};$

$$= h^2/6\ [4\ (2-3\lambda) - \{f'''(\alpha_{i-1}) - f'''(\beta_i)\} + 12\ (1-2\lambda)$$

$$\{f'''(\beta_i) - f'''(\gamma_{i-1})\} + 3\ (1-4\square)\ \{f'''(\gamma_{i-1}) - f'''(\delta_i)\} - \{f'''(\gamma_i + \lambda h) - f'''(x_i - (1-\lambda)h\}] .$$

Therefore

$$| (3-4\lambda) + (8-16\lambda) + (1-4\lambda)| \,.\, | e_j' | \le$$

$$h^2/6\ [4\ (2\text{-}3\lambda)\ \omega_3(h) + 12\ (1-2\lambda)\ \omega_3(h) + 3\ (1-4\lambda)\ \omega_4\ (h)\ ] + h^2/6\ \omega_3(h).$$

(L1.4)

Considering values of λ in different intervals as mentioned in the lemma above, we get equation (P1.4) for $f \in C^4$ (I). Estimates for $| e_j' |$ can be obtained in a similar way so we omit the details here.

Proof of Theorem (A2)

We first consider the case when $f \in C^3$ (I).

Writing $x = x_i + th$, $0 \le t \le 1$ we have from Equation (A1.2)

$$h^2\ [s''(x) - f''(x)] = f_i\ A''(t) + f_{i+1} B''(t) + he_i' C''(t) + he_{i+1}' D''(t) +$$

$$h^3\ f'''(x_i + \lambda h)\ E''(t) + hf_i' C''(t) + hf_{i+1}' D''(t) - h^2 f''(x).$$

Applying Taylor's expansion theorem for the function f and its derivaties and rearranging the terms, we have

$$h^2\ [s''(x) - f''(x)] = f_i\ [A''(t) + B''(t)] + hf_i'[B''(t) + C''(t) + D''(t)] +$$

$$+\ h^2 f_i''[1/2 B''(t) + D''(t) - 1] + h^3[1/6 f'''(\theta_i) B''(t) +$$

$$+\ f'''(x_i + \lambda h) E''(t) + 1/2 f'''(\phi_i) D''(t)] -$$

$$(x - x_i) h^2 f'''(\Psi_i) + he_i' C''(t) + he_{i+1}' D''(t), \quad x_i \le \theta_i,\ \phi_i, \psi_i \le x_{i+1}.$$

Using equation (P1.7) we get

$$h^2[s''(x) - f''(x)] = h^3 t^2/12(1-\lambda)[-24 f'''(\theta_i) - 12 f'''(x_i + \lambda h) + 36\ f'''(\phi_i)]$$

$$+ h^3 t/12\ (1-2\lambda)\ [48\ \lambda\ f'''(\theta_i) + 12 f'''(x_i + \lambda h) - 72\ f'''(\phi_i)] +$$

$$h^3/12\ (1-2\lambda)\ [8(1\text{-}3\lambda) f'''(\theta_i) - 2 f'''(x_i + \lambda h)$$

$$+ 6\ (4\lambda - 1)\ f'''(\phi_i)] - h^2\ (x - x_i)\ f'''(\psi_i)$$

$$+\ he_i' C''(t) + he_{i+1}' D''(t) .$$

$$= h^2 t^2/12\ (1-2\lambda)\ [\ 24\{f'''(\phi_i) - f'''(\theta_i)\} + 12\{f''(\phi_i) - f'''(x_i + \lambda h)\}] +$$

$$h^2 t/12\ (1-2\lambda)\ [48\lambda\ \{f'''(\theta_i) - f'''(x_i + \lambda h)\} + (12 + 48\lambda)\ \{f'''(x_i + \lambda h)$$

$$-\ f'''(\phi_i)\}] + h^3/12(1-2\lambda)[8(1-3\lambda)\{f'''(\theta_i) - f'''(x_i + \lambda h)\} +$$

$$6(1-4\lambda)\ \{f'''(x_i + \lambda h) - f'''(\phi_i)\}] + h^3 t f'''(\phi_i) - h^2(x - x_i) f'''(\psi_i) +$$

$$he_i' C''(t) + he_{i+1}' D''(t) .$$

Therefore

$$h^2\ |s''(x) - f''(x)| \le h^3/|\ 12\ (1-2\lambda)|\ [24\ \omega_4(h) + 12\omega_3(h)]$$

$$+ h^3/|12\ (1-2\lambda)\ [48\ |\ \lambda|\ \omega_3\ (h) + |\ 12 + 48\ \lambda\ |\omega_3\ (h)]$$

$$+ h^3\ \omega_3(h) + h\ |\ e_i' \,|\, C''(t) \,| + h\,|\, e_{i+1}' \,||\, D''(t)\,|$$

$$\le (3 + 8\lambda)/\ (1-2\lambda)\ h^3\ \omega_3(h) + h\ |\, e_i' \,||\, C''(t)| + he_{i+1}' \,||\, D''(t)\,| .$$

From Equations (L1.2) and (L1.3), we have

$$|\ C''(t)\ |\le|\ (3-4\lambda)/(1\text{-}2\lambda)\ |.$$

and

$$|\ D''(t)\ |\le|\ (5-8\lambda)/(1\text{-}2\lambda)|$$

Here we use the fact that $0 \le t \le 1$.

Using these estimates for C'' (t) and D'' (t) and the estimates for $|e_j'|$ from Lemma (L1) we get Equation (P1.1) for r = 2.

Now using interpolatory conditions (A1.1)

$$|s'(x) - f'(x)| = |\int_0^x s''(x) - f''(x)\,dx|$$

$$\le h\,|s''(x) - f''(x)|$$

$$\le C_\lambda h^4 \omega_3(h)$$

Similarly

$$|s(x) - f(x)| = |\int_{x_1}^{x} s'(x) - f'(x)\,dx|$$

$$\le h\,/\,|s'(x) - f'(x)|$$

$$\le C_\lambda h^3 \omega_3(h)$$

This proves Theorem A2 for $f \in C^3$ (I). Proof for $f \in C^4$ (I) can be carried out on similar lines so we omit the details.

CASE B - (0,4)

Theorem (B1)

Given a partition Δ of the unit interval I = [0,1] and arbitrary real numbers f_i, i = 0,1, ...n; $f_{i+\lambda h}^{(4)}$, i = 0,1............, n-1; f_0', f_1' and $0 \le \lambda \le 1$, there exists a unique spline $s(x) \in S_{n,4}^{(4)}$ such that

$$\begin{cases} s(x_i) = f(x_i) & , \ i = 0,1,......,n, \\ s^{(4)}(x_{i+\lambda h}) = f^{(4)}(x_{i+\lambda h}) & , x_i \le x_{i+\lambda h} \le x_{i+1}, \\ s'(x_0) = f'(x_0) & , \ s'(x_n) = f'(x_n) \end{cases}$$

(B1.1)

where $x_{i+\lambda h} = x_i + \lambda h$, $h = x_{i+1} - x_i$, i = 0,1,..........., n-1.

Preliminaries (P2)

If Q (x) is a quartic on the unit interval [0,1] then we have

$$Q(x) = Q(0)\,A(x) + Q(1)\,B(x) + Q'(0)\,C(x) + Q'(1)\,D(x) + Q^{(4)}(\lambda)\,E(x) \qquad \text{(P2.1)}$$

where

$$A(x) = 2x^3 - 3x^2 + 1 \qquad \text{(P2.2)}$$
$$B(x) = -2x^3 + 3x^2 \qquad \text{(P2.3)}$$
$$C(x) = x^3 - 2x^2 + x \qquad \text{(P2.4)}$$
$$D(x) = x^3 - x^2 \qquad \text{(P2.5)}$$
$$E(x) = 1/24\,(x^4 - 2x^3 + x^2) \qquad \text{(P2.6)}$$

For later reference we have

$A''(x) = 12x - 6$, $A'''(x) = 12$, $A^{(4)}(x) = 0$
$B''(x) = -12x + 6$, $B'''(x) = -12$, $B^{(4)}(x) = 0$
$C''(x) = 6x - 4$, $C'''(x) = 6$, $C^{(4)}(x) = 0$
$D''(x) = 6x - 2$, $D'''(x) = 6$, $D^{(4)}(x) = 0$
$E''(x) = 1/12(6x^2 - 6x + 1)$, $E'''(x) = x - 1/2$, $E^{(4)}(x) = 1$

Proof of Theorem (B1)

For $x_i \le x \le x_{i+1}$, i = 0,1,........ , n –1 we write s(x) as

$$s(x_i) = f(x_i)\,\frac{A(x-x_i)}{h} + f(x_{i+1})\,\frac{B(x-x_i)}{h} + h\,S'(x_i)\,\frac{C(x-x_i)}{h}$$
$$+\ hs'(x_{i+1})\frac{D(x-x_i)}{h} + h^4 f^{(4)}(x_i + \lambda h)\,\frac{E(x-x_i)}{h} \qquad \text{(B1.2)}$$

Appling the continuity requirement that $s(x) \in C^2$ (I), that is,

$s''(x_i+) = s''(x_i-)$ i = 0,1,......... , n, we have.

$$-2h\,s_{i-1}' - 8h\,s_i - 2h\,s_{i+1}' = 6f_{i-1} - 6f_{i+1} - h^4/12\,f_{i+\lambda h}^{(4)} + h^4/12\,f_{i-\lambda h}^{(4)} \qquad \text{(B1.3.)}$$

The above system of equation is clearly seen to be diagonally dominant, thereby ensuring the unique existence of the spline s(x) for all values of λ between 0 and 1.

Theorem (B2)

Let $f \in C^4$ (I), then for the unique spline s(x) of Theorem (B1) with real numbers f_i, i = 0,1...... ,n, $f_{i+\lambda n}^{(4)}$ i = 0,1,........,n – 1, f_1' and f_n' associated with the function f (x), $0 \le \lambda \le 1$, we have

$$\| s^{(q)}(x) - f^{(q)}(x) \| \le K\,h^{q-4}\,\omega_4(h), \quad q = 0,1,2,3,4 \qquad \text{(B.2.1)}$$

where $\omega_4(h)$ denotes the modulus of continuity of $f(x) \in C^4(I)$ and K are different constants occurring in the proof of the theorem.

Auxilary lemma

Lemma (L2)

Set

$e_i' = s_i' - f_i'$, i = 0,1,....... ,n,

and

max | e_i' |=| e_j' |

Then for $f \in C^4(I)$, we have

$| e_j' | \le 8 h^3/144\ \omega_3(h)$ (L2.1)

Proof

The proof can be carried out as in Lemma (L2) in a much simpler way.

Proof of Theorem (B2)

Again writing $x = x_j + th$, $0 \le t \le 1$, we have from equation (A2.1)

$$h^4 [s^{(4)}(x) - f^{(4)}(x)] = f_i A^{(4)}(t) + f_{i+1} B^{(4)}(t) + he_i' C^{(4)}(t) + he_{i+1}' D^{(4)}(t) +$$

$$h^4 \quad f^{(4)}(x + \lambda h)\ E^{(4)}(t) + hf_i'\ C^{(4)}(t) + hf_{i+1}'\ D^{(4)}(t) - h^4 f^{(4)}(x).$$

Appling Taylor's expansion we get

$h^4[s^{(4)}(x)-f^{(4)}(x)] = f_i [A^{(4)}(t) + B^{(4)}(t)] + h f_i' [B^{(4)}(t) + C^{(4)}(t) + D^{(4)}(t)]$

$+ h^2 f_1''[1/2B^{(4)}(t) + D^{(4)}(t)] + h^3 f_i''' [1/6B^{(4)}(t) +$

$1/2\ D^{(4)}(t)] + h^4[1/24\ f^{(4)}(\alpha_j)\ B^{(4)}(t) + 1/6 f^{(4)}(\beta_j)\ D^{(4)}(t)$

$+ f^{(4)}(x_j+\lambda h)\ E^{(4)}(t) - f^{(4)}(x)] + he_j'\ C^{(4)}(t) + he_{j+1}'\ D^{(4)}(t),$

$x_i \le \alpha_i, \beta_j \le x_{i+1}$.

Therefore, $|s^{(4)}(x) - f^{(4)}(x)| \le \omega_4(h)$.

Similarly

$$| s'''(x) - f'''(x) | \le 8h/3\ \omega_4(h).$$

and

$$| s''(x) - f''(x) | \le 13h^2/6\ \omega_4(h).$$

Now using interpolatory conditions (A2.1), we have

$$| s'(x) - f'(x) | \le \int_{x_1}^{x} | s''(x) - f''(x) | dx$$

$\le 13\ h^3/6\ \omega_4(h)$.

and

$$| s(x) - f(x) | \le \int_{x_1}^{x} | s'(x) - f'(x) | dx \le 13\ h^4/6\ \omega_4(h).$$

This completes the proof of Theorem (B2)

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