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Short Communication

Some inequalities of a formula of population size due to epidemics model problem

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In the present paper, we evaluate a formula of population size due to epidemics model problem for communicable diseases, in which the daily contact rate is supposed to be varied with population size N(t) that is large, so that it is considered as a continuous variable of time (t). Then, we obtain some inequalities of above analytic formula of population size that are useful to plot some graphs on population dynamics of the system.

Key words: Population size, epidemics model, hypergeometric function, inequalities, communicable diseases.

INTRODUCTION

In population dynamics, various deterministic epidemics model problems involving differential equations were studied and obtained due to Kermack and McKendriek (1927 a,b) and Bailey (1957) etc. Kapur (1998) has introduced and solved several mathematical models of epidemics through systems of ordinary differential equations of first order. Recently, Joshi (2004) has presented a solution of the deterministic epidemics model in terms of the hypergeometric functions ${}_0F_1$ (.) (Erde'lyi et al., 1953; Rainville, 1971). An epidemic usually describes as occurrence of a disease in excess to normal expectations. Since contiguousness is one of the main causes of spread of such epidemics, the term epidemiology has been applied to general study and deriving measures of controlling all communicable diseases (like, cholera, influenza, T. V., HIV etc). The general model for communicable disease in which an infected person does not recover is known as S I model. In this model at a time t, total population N (t) is divided into two disjoint classes namely infective class I (t), consisting of totality of infected individuals who can transmit disease and the susceptible class S (t) of the individuals who can incur disease by contact with the infected individuals.

The confluent hypergeometric differential equation, satisfying by the confluent hypergeometric function $u = {}_1F_1\left({a; \atop c;}z\right)$, is given by Rainville (1971) as:

$$z\frac{d^2}{dz^2} + (c-z)\ \frac{du}{dz}\ - au = 0 \qquad (1)$$

where, c ≠ 0, -1, -2, ….. and | z | < ∞

Again, the inequality $exp[\rho x]$ is given by (Zhou, 1999)

where, $\rho > 0$ and x > 0. (2)

Here, in our investigations, we evaluate a formula as a solution of Susceptible Infective (SI) model problem for communicable disease, in which the daily contact rate (C (N)) is supposed to be varied with population size N (t), that is large so that it is considered as a continuous variable of time t. Then, we obtain some inequalities of this formula of population size that are useful to analyze and plot the graphs of the population dynamics of the system.

Deterministic mathematical model problem of epidemics and its solution

Here, we construct an epidemics model problem and then convert it in terms of the differential equation satisfied by the Kummer's confluent hypergeometric function ${}_1F_1\left({a; \atop c;}z\right)$ (Rainville, 1971). In this model, the population size N (t) is so large that it can be considered as a continuous variable of time t. The population is changing on account of immigration, births, emigration and deaths (due to disease in question or other causes). Let β be the rate at which the population is receiving new individuals due to immigration and birth and μ be the rate at which individuals are being removed on account of emigration and death. Hence, all the new entrants are assumed to be susceptible. The population is assumed to be uniform or homogeneous. The daily contact rate C (N), at which number of susceptible become infected, is taken by (non dimensional):

$$\frac{\alpha IS + (\beta - \mu)(\delta S - nI)N}{N}, n \in \{0, 1, 2, ...\} \text{ and } \alpha, \beta, \gamma, \delta \text{ are real such that } \beta > \mu > 0, \delta > 0.$$

The initial value problem for the S I model can be put in the form:

$$\frac{dS}{dt} = \beta N - \mu S - \alpha IS - (\beta - \mu)(\delta S - nI), \quad (3)$$

$$\frac{dI}{dt} = \alpha IS - \mu I + (\beta - \mu)(\delta S - nI), \quad (4)$$

N = I + S; (5),

Where, n is a parameter such that $n \in \{0, 1, 2, ...)$ and $\alpha, \beta, \mu, \delta \in$ R (set of real numbers) Such that $(\beta > \mu) > 0, \delta > 0$; N (0) = N0 > 0, S (0) = S0 > 0, and I (0) = I0 > 0.

Now, adding (3) and (4) and then making an appeal to (5), we find

$$\frac{dN}{dt} = N(\beta - \mu) \quad (6)$$

So that (6) gives us

$$N = N_0 \exp[(\beta - \mu)t] \quad (7),$$

provided that all conditions given in (5) are followed and t > 0. Further, setting $\frac{du}{dt} = -\alpha$ Su in (3) and then making an application of (5), we find that

$$\frac{d^2u}{dt^2} + (\mu + \alpha N + (\beta - \mu)(\delta + n))\frac{du}{dt} + ((\beta - \mu)n + \beta)\alpha Nu = 0 \quad (8)$$

Now, making an appeal to (6), (7) and (8), we get a transformed equation in the form,

$$N^2\frac{d^2u}{dt^2} + \left(\frac{\beta}{\beta - \mu} + \delta + n + \frac{\alpha}{\beta - \mu}N\right)N\frac{du}{dN} + \left(\frac{\beta}{\beta - \mu} + n\right)\frac{\alpha}{\beta - \mu}Nu = 0 \quad (9)$$

Again, set $\frac{\alpha}{\beta - \mu}N = -M$, $M > 0$in (9), we find another transformed equation in the form of confluent hypergeometric differential equation as:

$$M\frac{d^2u}{dM^2} + \left(\frac{\beta}{\beta - \mu} + \delta + n - M\right)\frac{du}{dM} - \left(\frac{\beta}{\beta - \mu} + n\right)u = 0 \quad (10)$$

Hence, with the aid of (1), the general solution of (10) may be written in the form

$$u(\beta, \mu, \delta; M) = \sum_{n=0}^{\infty} C_n \; {}_1F_1\left(\begin{matrix}\frac{\beta}{\beta - \mu} + n; \\ \frac{\beta}{\beta - \mu} + \delta + n;\end{matrix} \; M\right) \quad (11)$$

where, $(\beta > \mu) > 0, \delta > 0, M > 0$ and all Cn, where $n \in \{0, 1, 2,....\}$ are arbitrary constants.

Inequalities of a formula of epidemics

Here, we obtain some inequalities of the function $\upsilon(.)$ through following theorem:

Theorem A: If $(\beta > \mu) > 0$, $C_0 = 1$ $C_0 = 1$ and $\langle C_n \rangle \forall n = 0,1,2,3, ...$ .is monotonically decreasing sequence, then

$$\left| eM\left(\begin{matrix}\frac{\beta}{\beta - \mu} \\ \frac{\beta}{\beta + \mu} + \delta\end{matrix}\right) - u(\beta, \mu, \delta; M)\right| \le eM \lim_{n\to\infty}\left\{\left(\sum_{k=1}^{n}\left(\frac{\frac{\beta}{\beta - \mu} + k}{\frac{\beta}{\beta + \mu} + \delta + k}\right)^2\right)^{1/2}\left(\sum_{k=1}^{n}(C_k)^2\right)^{1/2}\right\} \quad (12)$$

Proof: In right hand side of (11), use following Euler type integral formula for Kummer's confluent hypergeometric function (Srivastava and Manocha, 1984)

$${}_1F_1(a; b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b - a)}\int_0^1 e^{zt}t^{a-1}(1 - t)^{b-a-1}dt, Re(b) > Re(a) > 0. \quad (13)$$

Then, make an appeal to the inequality (2), we get

$$u(\beta, \mu, \delta; M) \ge eM \lim_{n\to\infty}\sum_{k=0}^{n}\left\{\left(\frac{\frac{\beta}{\beta - \mu} + k}{\frac{\beta}{\beta + \mu} + \delta + k}\right)\right\}C_k \quad (14)$$

Now, in right hand side of (14) make an application of Cauchy-Schwarz inequality, given by (Kapur, 1997; p.65, eqn. 90)

$$\sum_{k=1}^{n} a_k b_k \ge -\left(\sum_{k=1}^{n} a_k^2\right)^{1/2}\left(\sum_{k=1}^{n} b_k^2\right)^{1/2} \quad (15)$$

where, all $a_k's$ and $b_k's$ are any real numbers. Finally, we get the inequality (12).

Theorem B: If $(\beta > \mu) > 0, C_0 = 1$, $< C_n > = < \frac{1}{n^2} >$, Then,

$$|u(\beta,\mu,\delta;M)| \geq eM\left(\frac{\frac{\beta}{\beta-\mu}-\frac{1}{\sqrt{3}}}{\frac{\beta}{\beta+\mu}+\delta}\right) \quad (16)$$

Proof: We know that

$$\sum_{k=1}^{n}\left(\frac{1}{k^4}\right) \sim \frac{1}{n^3} \ and \ \sum_{k=1}^{n}\left(\frac{\frac{\beta}{\beta-\mu}+k}{\frac{\beta}{\beta+\mu}+\delta+k}\right)^2$$

$$< \frac{\left[\left(\frac{\beta}{\beta-\mu}\right)^2 n + \frac{n\beta(n+1)}{(\beta-\mu)} + \frac{n(n+1)(2n+1)}{6}\right]}{\left(\frac{\beta}{\beta-\mu}+\delta\right)^2} \quad (17)$$

Then, making an appeal to (12) and (17), we get the inequality (16).

CONCLUSIONS AND DISCUSSIONS

From the earlier stated theorems, we conclude that:

(i) If the ratio $\left(\frac{\beta}{\beta-\mu}\right)$ tends to infinite, then $u \geq eM$

(ii) If $\delta \to \infty$, then $u \geq 0$

(iii) If $\mu \to \infty$, β and δ are constant, then $u \geq -\frac{1}{\sqrt{3}}eM$

For, $\delta \to 0$, and n= 0, the above epidemics model becomes the epidemics model problem due to Joshi (2004).

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