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Existence of at least one solution of singular Volterra-Hammerstein integral equation and its numerical solution

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In this paper, we prove the existence of at least one solution for Volterra- Hammerstein integral equation (V-HIE) of the second kind, under certain conditions, in the space $L_p(\Omega)\times C[0,T], T<1$, Ω is the domain of integration and T is the time. The kernel of Hammerstein integral term has a singularity, while the kernel of Volterra is continuous in time. Using a quadratic numerical method with respect to time, we have a system of Hammerstein integral equations (SHIEs) in position. The existence of at least one solution for the SHIEs is considered and discussed. Moreover, using Toeplitz matrix method (TMM), the SHIEs are transformed into a nonlinear algebraic system (NAS). Many theorems related to the existence of at least one solution for this system are proved. Finally, numerical results and the estimate error of it are calculated and computed using Mable 12.

Key words: Volterra- Hammerstein integral equation, nonlinear algebraic system (NAS), singular kernel, Toeplitz matrix method, Hölder inequality.

INTRODUCTION

Linear and nonlinear singular integral equations have received considerable interest the mathematical applications in different areas of sciences. The different numerical methods play an important role in solving the nonlinear integral equations (NIE). Kummer and Sloan (2003) used a new collection type method to discuss the solution of HIE with continuous kernel. Kummer (1988) used a discrete collection-type method to discuss the solution of HIE with continuous kernel. Hacia (1993) used projection-iteration methods, and an approximate method to discuss the solution of HIE with continuous kernel. Moreover, the super convergence of some numerical methods for HIE with continuous kernels is observed and developed through the work of many authors (Zhang, 2008; Diago and Lima, 2008; Kaneko and Xu, 1996). When the kernel of HIE has a singular term, new different numerical methods were used (Lardy, 1981; Abdou et al., 2005; Abdou et al., 2009; Vainikko, 2011). More

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information for solving the NIE using different methods can be found in the work of Abdou (2003), Abdou and Al-Bigamy (2013a), Bazm and Babolian (2012) and Abdou et al. (2013b).

Consider a general formula of V-HIE of the second kind

$$\mu\varphi(x,t)=f(x,t)+\lambda\int_0^t\int_\Omega k(|x-y|)F(t,\tau)\gamma(y,\tau,\varphi(y,\tau))dyd\tau \tag{1}$$

in the space $L_P(\Omega)\times C[0,T]$; $T<1$, where f,k,F and γ are known functions. $k(|x-y|)$ is a discontinuous kernel of HI term in position. While, $F(t,\tau)$ is a continuous kernel of VI term in time. The constant μ defines the kind of IE; while λ has a physical meaning and Ω is the domain of integration with respect to position.

To discuss the existence of at least one solution of Equation (1) in $L_p(\Omega)\times C[0,T]$, $p>1$, $T<1$, we write IE (1) in the integral operator form

$$\overline{W}\phi(x,t)=\frac{1}{\mu}f(x,t)+\frac{\lambda}{\mu}W\phi(x,t), \qquad (\mu\neq 0), \tag{2}$$

$$W\phi(x,t)=\int_0^t\int_\Omega F(t,\tau)k(|x-y|)\gamma(\tau,y,\phi(y,\tau))dy\,d\tau, \quad (\forall t,\tau\in[0,T],0\le\tau\le t\le T<1). \tag{3}$$

Then, Let S_α be the set of functions ϕ in $L_p(\Omega)\times C[0,T]$ for which $\|\phi\|\le\alpha,\alpha$ is a constant, and assume the following necessary conditions:

i) The kernel of position $k(|x-y|)$, for a constant c, satisfies:

$$\left\{\int_\Omega\{\int_\Omega|k(|x-y|)|^p dx\}^{\frac{q}{p}}dy\right\}^{\frac{1}{q}}=c \quad ;(p>1;\frac{1}{p}+\frac{1}{q}=1).$$

ii) The kernel of time $F(t,\tau)\in C[0,T]$ satisfies $|F(t,\tau)|\le M$, M is a constant.

iii) The given function $f(x,t)$ with its partial derivatives with respect to position x and time t are continuous in the space $L_p(\Omega)\times C[0,T]$ and its norm is defined as:

$$\|f(x,t)\|_{L_p(\Omega)\times C[0,T]}=\max_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega f^p(x,\tau)dx\right\}^{\frac{1}{p}}d\tau\right|=G \quad (G \text{ is a constant})$$

iv) The known continuous function $\gamma(t,x,\phi(x,t))$, for the constants $Q>P_1$ and $Q>Q_1$, satisfies the following conditions

(a) $\max_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega|\gamma(\tau,x,\phi(x,\tau))|^p dx\right\}^{\frac{1}{p}}d\tau\right|\le E \quad ,\forall\phi\in S_\alpha$, ($E$ is a constant).

(b) $\max_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega|\gamma(\tau,x,\phi_1(x,\tau))-\gamma(\tau,x,\phi_2(x,\tau))|^p dx\right\}^{\frac{1}{p}}d\tau\right|<\varepsilon$,

If $\|\phi_1(x,\tau)-\phi_2(x,\tau)\|<\delta(\varepsilon);\ \ \varepsilon<<1,\ \ \forall\phi_1,\phi_2\in S_\alpha$

BASIC THEOREMS AND DEFINITIONS

We state the famous theorems used in proving the principal theorem as follows:

Theorem 1 (without proof)

Let S be a closed and convex set in a Hilbert space, and K is a continuous mapping of S into itself. Suppose that the set S is compact, and then K has at least one fixed point in S (Abdou et al., 2005).

Theorem 2 (Modified Schauder fixed point; (without proof))

Let S be a closed set, convex set, in a Hilbert space, and K is a continuous mapping of S into itself. Suppose that $K(S)$ is compact, then K has at least one fixed point in S (Abdou et al., 2011).

In the remainder part of this paper, the existence of at least one solution for V-HIE (1), under the necessary conditions, in the space $L_p(\Omega)\times C[0,T], p>1, T<1,$ will be proved. In addition, using a quadratic numerical method with respect to time, we obtain SHIEs, where the existence of at least one solution of the system can be proved.

Moreover, using TMM, we represent the SHIEs in the form of NAS. Many different theorems are derived to prove the existence of at least one solution of the NAS. Finally, some examples, when the kernel of position takes a logarithmic form, Carleman function and Cauchy kernel are calculated numerically and the error estimate, in each case, is computed.

The principal theorem of at least one solution

Here, we state several lemmas that lead to prove the following principal theorem:

Principal Theorem 3

Under the conditions (i)-(iv), V-HIE (1) has at least one solution. To prove the principal theorem, we state and prove the following lemmas:

Lemma 1 (without proof): In the complete space $L_p(\Omega)\times C[0,T]$, if we choose any functions $\phi_1(x,t),\ \phi_2(x,t)$ in the set S_p, then S_ρ is closed. Moreover, if $\{\phi_n\}$ is a sequence in S_ρ having a limit ϕ, then S_ρ is a convex set in the space $L_p(\Omega)\times C[0,T]$. ◘(Abdou, 2003).

Lemma 2: Under the conditions (i- iii) the integral operator $\overline{W}$ of Equation (2) maps the set S_α into itself.

Proof: In the light of Equations (2) and (3), we get

$$\left\|\overline{W}\phi(x,t)\right\|\leq\frac{1}{|\mu|}\left\|f(x,t)\right\|+\frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_\Omega|F(t,\tau)|\,|K(x,y)|\,|\gamma(\tau,y,\phi(y,\tau))|\,dy\,d\tau\right\|$$

Applying Cauchy-Schwarz inequality to Hammerstein integral term, and then using the conditions (i-iii), the above inequality can be adapted in the form

$$\left\|\overline{W}\varphi(x,t)\right\|\leq\frac{G}{|\mu|}+\sigma_1,\quad\left(\sigma_1=\left|\frac{\lambda}{\mu}\right|cM\,ET\right).$$

The above inequality shows that, the operator $\overline{W}$ maps the set S_α into itself.

Lemma 3: Under the conditions (i-ii), the operator $\overline{W}$ is continuous in S_α.

Proof: For the two functions $\phi_1(x,t)$ and $\phi_2(x,t)$ in S_α, after applying Cauchy-Schwarz inequality to Hammerstein integral term, then using the conditions (i) and (ii), we get

$$\left\|\overline{W}\phi_1(x,t)-\overline{W}\phi_2(x,t)\right\|\leq\frac{|\lambda|}{|\mu|}Mc^*T\max_{0\leq t\leq T}\left|\int_0^t\left\{\int_\Omega|\gamma(\tau,y,\phi_1(y,\tau))-\gamma(\tau,y,\phi_2(y,\tau))|^2dy\right\}^{\frac{1}{2}}d\tau\right|.$$

If $\left\|\phi_1(x,t)-\phi_2(x,t)\right\|<\delta(\varepsilon)$, then in the light of condition (iv) the above inequality reduces to $\left\|\overline{W}\varphi_1(x,t)-\overline{W}\varphi_2(x,t)\right\|<\frac{|\lambda|}{|\mu|}M\,cT\,\varepsilon<\varepsilon,\quad(where\ \left|\frac{\lambda}{\mu}\right|M\,cT<1).$

This implies the continuity of $\overline{W}$ in the set S_α .

Lemma 4: Let $\{k_n(|x-y|)\}$ and $\{F_n(t,\tau)\}$ be two sequences of continuous functions satisfy the conditions

$$\lim_{n\to\infty}\left\{\int_\Omega\{\int_\Omega|k_n(|x-y|)-k(|x-y|)|^P dx\}^{\frac{p}{q}}dy\right\}^{\frac{1}{q}}=0\ ,\tag{4}$$

$$\lim_{n\to\infty}\max_{0\leq\tau,t\leq T}|F_n(t,\tau)-F(t,\tau)|=0\,.\tag{5}$$

Then, after neglecting very small constants, and for positive integer n_0, we have

$$\left\{\int_\Omega\{\int_\Omega|k_n(|x-y|)|^P dx\}^{\frac{q}{p}}dy\right\}^{\frac{1}{q}}\leq c,\quad\forall n>n_0;\tag{6}$$

$$\max_{0\leq\tau,t\leq T}|F_n(t,\tau)|\leq M\,,\quad\forall n>n_0.\tag{7}$$

Now, in view of Lemma 4, we define the sequence of operator $\{\overline{W}_n\}$ as:

$$\overline{W}_n\phi(x,t)=\frac{1}{\mu}f(x,t)+\frac{\lambda}{\mu}\int_0^t\int_\Omega F_n(t,\tau)k_n(|x-y|)\gamma(\tau,y,\phi(y,\tau))\,dy\,d\tau.\tag{8}$$

Lemma 5: The integral operator (8) maps the set S_α continuously into itself. Moreover, the integral operator (8) is continuous.

The proof of Lemma (5) can be obtained directly after using Lemmas (2) and (3).

Lemma 6: Under the same conditions (i-iv) and Lemma 4, the set $\overline{W}(S_\alpha)$ of Equation (8) is compact.

Proof: From Equations (2) and (8) we get

$$\left\|\overline{W}_n\phi(x,t)-\overline{W}\phi(x,t)\right\|\leq\frac{|\lambda|}{|\mu|}\left\{\left\|\int_0^t\int_\Omega\max_{0\leq t\leq T}|F_n(t,\tau)|.|k_n(|x-y|)-k(|x-y|)||\gamma(\tau,y,\phi(y,\tau))|dy\,d\tau\right\|\right.$$
$$\left.+\left\|\int_0^t\int_\Omega\max_{0\leq t\leq T}|F_n(t,\tau)-F(t,\tau)|.|k(|x-y|)||\gamma(\tau,y,\phi(y,\tau))|dy\,d\tau\right\|\right\}.$$

After using Lemma 4 and conditions (i-iv), and then applying Hölder inequality to HI term, the above inequality yields

$$\left\|\overline{W_n}\,\phi(x,t)-\overline{W}\,\phi(x,t)\right\|\le\frac{|\lambda|}{|\mu|}M\;E\,T\;\varepsilon+\frac{|\lambda|}{|\mu|}c^{*}E\,T\;\varepsilon<\varepsilon'\;,\;\forall n>n_0(\varepsilon). \quad (9)$$

Hence, $\overline{W_n}\phi(x,t)\to\overline{W}\,\phi(x,t)$, for all $\phi(x,t)$ in S_α. Let $\{\phi_n(x,t)\}$ be any sequence in S_α. We obtain the sequence $\{\phi_{n_n}(x,t)\}$ which is subsequence of every ϕ_{n_j} except for finite number of elements and clearly $\{\overline{W_j}\,\phi_{n_n}\}$ converges for every j. Therefore, $\left\|\overline{W}\,\phi_{n_n}-\overline{W}\phi_{m_m}\right\|\le\left\|\overline{W}\,\phi_{n_n}-\overline{W_j}\,\phi_{n_n}\right\|+\left\|\overline{W_j}\,\phi_{n_n}-\overline{W_j}\phi_{m_m}\right\|+\left\|\overline{W_j}\,\phi_{m_m}-\overline{W}\,\phi_{m_m}\right\|$. Since $\left\|\overline{W_j}\,\phi_{n_n}-\overline{W_j}\phi_{m_m}\right\|\to 0$ as $m_n,n_n\to\infty$. Then from (11) for large j, we get the Cauchy sequence $\left\|\overline{W}\,\phi_{n_n}-\overline{W}\phi_{m_m}\right\|<2\varepsilon,\,(\forall m_m,n_n>n_0(\varepsilon))$. so that $\overline{W}(S_\alpha)$ is compact.

After the above discussion, the principle theorem is proved:

SYSTEM OF HAMMERSTEIN INTEGRAL EQUATIONS

Here, quadratic numerical method is used, in Equation (1) to obtain **SHIEs** in position. For this aim, we divide the interval $[0,T]$ as $0=t_0<t_1<...<t_k<...<t_N=T$ where $t=t_k$, $k=0,1,2...,N$. Hence, the integral term of Equation (1) becomes

$$\int_0^{t_k}\int_\Omega F(t_k,\tau)\,k\,(|x-y|)\,\gamma(\tau,y,\phi(y,\tau))dy\;d\tau$$

$$=\sum_{j=0}^{k}u_jF(t_k,t_j)\int_\Omega k\,(|x-y|)\,\gamma(t_j,y,\phi(y,t_j))dy+O\left(\hbar_k^{p+1}\right),\;(\hbar_k\to 0,p>0) \quad (10)$$

Where, $\hbar=\max\limits_{0\le j\le k}h_j$, $h_j=t_{j+1}-t_j$, $u_k=\frac{1}{2}h_k$, $u_j=h_j$ $(j=0,k)$.

The values of u_j and p ; $p\approx k$ are depending on the number of derivatives of $F(t,\tau)$ with respect to time, see Atkinson (2011).

Using (10) in (1), and neglecting $O\left(\hbar_k^{p+1}\right)$, we have

$$\mu\varphi_k(x)=f_k(x)+\lambda\sum_{j=0}^{k}u_jF_{kj}\int_\Omega k\,(|x-y|)\,\gamma_j\left(y,\varphi_j(y)\right)dy. \quad (11)$$

Where, we used the following notations

$$\varphi(x,t_k)=\varphi_k(x),\;f(x,t_k)=f_k(x),\;F(t_k,t_j)=F_{jk};\gamma\left(t_j,x,\phi(x,t_j)\right)=\gamma_j\left(x,\phi_j(x)\right). \quad (12)$$

The formula (11) represents SHIEs and its solution depends on the given function $f_k(x)$, the kind of the kernel $k\,(|x-y|)$, and the degree of the known function $\gamma_j\left(x,\varphi_j(x)\right)$. The formula (11) can be written in the integral operator form

$$\bar{V}\phi_k(x)=\frac{1}{\mu}f_k(x)+\frac{\lambda}{\mu}\sum_{j=0}^{k-1}u_jF_{jk}V\;\phi_j(x);\quad V\;\phi_j(x)=\int_\Omega k\,(|x-y|)\,\gamma_j\left(y,\phi_j(y)\right)dy \quad (13)$$

Following the same way of Lemmas 2, 3 and 6 of the principal theorem 3, we can directly proof the following lemmas and principal theorem of SHIEs.

Lemma 7 (without proof): Under the conditions (i) – (iv-a), the operator $\overline{V}$ maps the space $L_p(\Omega)$ into itself.

Lemma 8 (without proof): Under that, the conditions (i), (ii) and (iv-b) $\overline{V}$ is a compact operator in the space $L_p(\Omega)$.

Principal Theorem 4 of SHIEs:

According to Lemmas 7 and 8, the **SHIEs** of the second kind (11) have at least one solution.

THE TOEPLITZ MATRIX METHOD

Here, we will discuss the solution of Equation (1) numerically, using **TMM** in one dimensional, and $\Omega=[-b,b]$. For this, write the integral term Equation (11) in the form (Abdou et al., 2011)

$$\int_{-b}^{b}k\,(|x-y|)\,\gamma_j(y,\phi_j(y))dy=\sum_{n=-N}^{N-1}\int_{nh}^{(n+1)h}k\,(|x-y|)\,\gamma_j(y,\phi_j(y))dy$$

$$=\sum_{n=-N}^{N-1}\{A_n^{(j)}(x)\gamma_j(a,\phi_j(a))+B_n^{(j)}(x)\gamma_j(a+h,\phi_j(a+h))\}+R;\quad (h=\frac{2b}{N};a=nh). \quad (14)$$

where, $A_n^{(j)}(x)$ and $B_n^{(j)}(x)$ for all $0\le j\le i$ are arbitrary functions to be determined, and R is the estimate error $R=\max\limits_{j}R^j$.

In the light of TMM (Abdou et al., 2011), the integral formula (11) yields

$$\mu\,\phi_i(x)-\lambda\sum_{j=0}^{i}\sum_{n=-N}^{N}F_{ij}D_n^{(j)}(x)\gamma_j(nh,\phi_j(nh))=f_i(x) \quad (15)$$

where

$$D_n^{(j)}(x)=\begin{cases}A_{-N}^{(j)}(x) & ,\;n=-N\\ A_n^{(j)}(x)+B_{n-1}^{(j)}(x) & ,\;-N<n<N\\ B_{N-1}^{(j)}(x) & ,\;n=N\;.\end{cases} \quad (16)$$

$$A_n^{(j)}(x)=\frac{1}{h_1}\left[\gamma_j(a+h,a+h)\,I(x)-\gamma_j(a+h,1)\,J(x)\right],B_n^{(j)}(x)=\frac{1}{h_1}\left[\gamma_j(a,1)J(x)-\gamma_j(a,a)I(x)\right],$$
$$I^{(j)}(x)=\int_a^{a+h}k(|x-y|)\gamma_j(y,1)dy,\quad J^{(j)}(x)=\int_a^{a+h}k(|x-y|)\gamma_j(y,y)dy. \tag{17}$$

Putting $x=mh$ in (15), and using the following notations $\phi_i(\ell h)=\phi_{i\ell}$, $D_n^{(i)}(mh)=D_{mn}^{(i)}$, $f_i(mh)=f_{im}$, $\gamma(\tau_j,nh,\phi_j(nh))=\gamma_{jn}(\phi_{jn})$, we get the following NAS

$$\mu\phi_{im}-\lambda\sum_{j=0}^{i}\sum_{n=-N}^{N}F_{ij}D_{mn}^{(j)}\gamma_{jn}(\phi_{jn})=f_{im},\quad -N\le m\le N. \tag{18}$$

The error term $R^{(j)}$ for each value of j, is determined from the following formula

$$R^{(j)}=\left|\int_{nh}^{nh+h}\gamma_j(y,y^2)k(|x-y|)dy-A_n^{(j)}(x)\gamma_j(nh,(nh)^2)-B_n^{(j)}(x)\gamma_j(nh,(nh+h)^2)\right|=O(\gamma_j(h^3)) \tag{19}$$

The existence of at least one solution of the NAS

The existence of at least one solution of the NAS (18) in the space ℓ^∞, will be proved according to Schauder fixed point theorem. For this, we write it in the operator form:

$$\bar{T}\phi_m=T\phi_m+\frac{1}{\mu}f_m;\quad T\phi_m=\frac{\lambda}{\mu}\sum_{j=0}^{i}\sum_{n=-N}^{N}F_{ij}D_{mn}^{(j)}\gamma_n(\phi_{nj});\quad(\mu\ne 0,-N\le m\le N) \tag{20}$$

Let Λ be the set of the two families $\Phi=\{\phi_n\}$ and $\Psi=\{\psi_n\}$ in ℓ^∞ for which $\|\Phi\|_{\ell^\infty}\le\beta_1,\|\Psi\|_{\ell^\infty}\le\beta_2$, β_1,β_2 are constants, and then consider the following conditions:

$$\sup_m|f_m|\le H,\quad \sup_i\sup_N\sum_{j=0}^{i}\sum_{n=-N}^{N}|F_{ij}|\left|D_{mn}^{(j)}\right|\le E,$$ (E, H are constants). (21)

For the known set $\gamma(nh,\phi(nh))$, we have

$$\sup_n|\gamma(nh,\phi(nh))|\le Q_2,\quad(0<Q_2<1)\quad\forall\Phi\in\Lambda; \tag{22}$$

$$\sup_n|\gamma(nh,\phi(nh))-\gamma(nh,\psi(nh))|<\varepsilon_1\ if\ \sup_n|\phi(nh))-\psi(nh)|<\delta(\varepsilon_1);\ (\varepsilon_1<<1,\ \forall\Phi,\Psi\in\Lambda). \tag{23}$$

Now we state and prove some lemmas that lead directly to the principal theorem of **NAS**.

Lemma 9: Under the conditions (21)-(22), the operator $\bar{T}$ of Equation (20) maps the set Λ into itself.

Proof: In view of the formulas (20) and (21), we deduce

$$\sup_n|\bar{T}\phi_m|\le\sigma_1+\frac{H}{|\mu|}=\beta,\quad(\sigma_1=\left|\frac{\lambda}{\mu}\right|QE). \tag{24}$$

The above inequality proves that, the operator $\bar{T}$ maps the set Λ into itself. In addition, the inequality (24) define the boundedness of the operator T and $\bar{T}$.

Lemma 10: Under the conditions (21) and (23), $\bar{T}$ is continuous in the set Λ.

Proof: For the two functions Φ, Ψ in the set Λ, the operator (20), in view of Equations (21) and (23), yields

$$\sup_m|\bar{T}\phi_m-\bar{T}\psi_m|<\left|\frac{\lambda}{\mu}\right|QE\ \varepsilon_1=\tilde{\varepsilon},\quad|\lambda|QE<|\mu|.$$

The above inequality holds for every integer m under the condition $|\lambda|QE<|\mu|$.

Hence, $\|\bar{T}\Phi-\bar{T}\Psi\|_{\ell^\infty}<\tilde{\varepsilon}$ if $\|\Phi-\Psi\|_{\ell^\infty}<\delta(\tilde{\varepsilon})$. This proves the continuity of the operator $\bar{T}$ in the set Λ. ▫

Lemma 11: Let $\{\gamma_k(nh,\phi(nh))\}$ be a sequence of elements, such that

$$\lim_{k\to\infty}\sup_n|\gamma_k(nh,\phi(nh))-\gamma(nh,\phi(nh))|=0. \tag{25}$$

Then, there exists a positive integer k_0, such that

$$\sup_n|\gamma_k(nh,\phi(nh))|\le Q_2,\quad(\forall k>k_0); \tag{26}$$

$$\sup_n|\gamma_k(nh,\phi(nh))-\gamma_k(nh,\psi(nh))|<\varepsilon,\ \forall k>k_0;\ (\varepsilon<<1). \tag{27}$$

Proof: For any positive integer k, and for any two sequences $\gamma_k(nh,\phi(nh)),\ \gamma(nh,\phi(nh))$, we see that

$$\sup_n|\gamma_k(nh,\phi(nh))|\le\sup_n|\gamma_k(nh,\phi(nh))-\gamma(nh,\phi(nh))|+\sup_n|\gamma(nh,\phi(nh))|, \tag{28}$$

and

$$\sup_n|\gamma_k(nh,\phi(nh))-\gamma_k(nh,\psi(nh))|\le\sup_n|\gamma_k(nh,\phi(nh))-\gamma(nh,\phi(nh))|$$
$$+\sup_n|\gamma(nh,\phi(nh))-\gamma(nh,\psi(nh))|+\sup_n|\gamma_k(nh,\psi(nh))-\gamma(nh,\psi(nh))|. \tag{29}$$

In the light of condition (25), there exists a positive integer k_0, such that:

$$\sup_n \left|\gamma_k(nh,\phi(nh)) - \gamma(nh,\phi(nh))\right| < \varepsilon_2 , \ \forall k > k_0 ; \ \varepsilon_2 << 1. \quad (30)$$

Hence, with the aid of (25) and using (30), the inequality (27), after concluding the arbitrary small constant leads directly to (28). In addition, in view of (30) the inequality (29), leads to the same inequality (28). Hence, the lemma is proved□

Lemma 12: Under the conditions (26)- (27) of lemma 11 the sequence of operators

$$\bar{T}_k \phi_m = \frac{\lambda}{\mu} \sum_{j=0}^{i} F_{ij} \sum_{n=-N}^{N} D_{mn}^{(j)} \gamma_k(nh,\phi_j(nh)) + \frac{1}{\mu} f_m , \quad (31)$$

for each $k > k_0$, maps the set Λ continuously into itself.

Lemma 13: Under the same conditions of lemma 12, the set $\bar{T}(\Lambda)$ is compact.

Proof: From the formula (31), we have

$$\left|\bar{T}_k \phi_m - \bar{T}_k \psi_m\right| \le \left|\frac{\lambda}{\mu}\right| \sum_{j=0}^{i} \sum_{n=-N}^{N} \left|F_{ij}\right| \left|D_{mn}^{(j)}\right| \sup_n \left|\gamma_k(nh,\phi(nh)) - \gamma(nh,\phi(nh))\right|$$

Hence, with the aid of (26) and (27), there exists a positive integer k_0, such that

$$\left\|\bar{T}_k \Phi - \bar{T}_k \Psi\right\|_{\ell^\infty} < \left|\frac{\lambda}{\mu}\right| QE\,\varepsilon_2 = \tilde{\varepsilon} , \ \forall k > k_0 \ , \quad |\lambda| QE < |\mu| \quad (32)$$

Inequality (32) shows that, $\bar{T}_k \Phi \to \bar{T}\Phi$ uniformly for any $\Phi \in \Lambda$. Let $\{Y_n\}$ be any sequence in Λ, then we can select a subsequence $\{Y_{n_1}\}$ such that $\{\bar{T}_1 Y_{n_1}\}$ converges. From that subsequence, we can extract a new subsequence $\{Y_{n_2}\}$ such that $\{\bar{T}_2 Y_{n_2}\}$ converges, and so on. Thus, we obtain a chain of subsequences. Finally, we take $\{Y_{n_n}\}$ which is a subsequence of every $\{Y_{n_j}\}$ except for a finite number of elements and clearly $\{\bar{T}_j Y_{n_n}\}$ converges for every j. Therefore, we have

$$\left\|\bar{T} Y_{n_n} - \bar{T} Y_{m_m}\right\|_{\ell^\infty} \le \left\|\bar{T} Y_{n_n} - \bar{T}_j Y_{n_n}\right\|_{\ell^\infty} + \left\|\bar{T}_j Y_{n_n} - \bar{T}_j Y_{m_m}\right\|_{\ell^\infty} + \left\|\bar{T}_j Y_{m_m} - \bar{T} Y_{m_m}\right\|_{\ell^\infty} .$$

Since $\left\|\bar{T}_j Y_{n_n} - \bar{T}_j Y_{m_m}\right\|_{\ell^\infty} \to 0$ as $m_m, n_n \to \infty$, then for large j, we obtain from (32), that

$$\left\|\bar{T} Y_{n_n} - \bar{T} Y_{m_m}\right\|_{\ell^\infty} < 2\varepsilon , \ \ \forall m_m, n_n > k_0 . \quad (33)$$

Hence, the sequence $\{\bar{T} Y_{n_n}\}$ is a Cauchy sequence, so that $\bar{T}(\Lambda)$ is compact.

The previous Lemmas 9-13 show that, $\bar{T}$ is a continuous operator maps the set Λ, which is evidently closed and convex set into itself, and $\bar{T}(\Lambda)$ is a compact set. Therefore, we can state the following by theorem:

Principal Theorem 5 of the NAS

The NAS (18) has at least one solution in the set Λ under the condition $|\lambda| QE < |\mu|$. Now, it is suitable to consider the following theorem which proves the convergence of one sequence of approximate solutions to some solution of Equation (18) in the space ℓ^∞.

Theorem 6: If the conditions (19) and (20) are verified, and the sequence of functions $\{L_j\} = \{(f_m)_j\}$ converges uniformly to the function $L = \{f_m\}$ in the space ℓ^∞. Then, we have at least one sequence of the approximate solutions $\{\Phi_j\} = \{(\phi_m)_j\}$ converges uniformly to some solution $\Phi = \{\phi_m\}$ of Equation (18) in the space ℓ^∞.

Proof: By virtue of the formula (31), we get

$$\left|\phi_m - (\phi_m)_j\right| \le \left|\frac{\lambda}{\mu}\right| \sum_{j=0}^{i} \left|F_{ij}\right| \sum_{n=-N}^{N} \left|D_{mn}^{(j)}\right| \sup_n \left|\gamma(nh,\phi(nh)) - \gamma(nh,\phi_j(nh))\right| + \frac{1}{\mu} \sup_m \left|f_m - (f_m)_j\right|$$

The above inequality, after using the two conditions (30) and (31), holds for each integer m, hence

$$\left\|\Phi - \Phi_j\right\|_{\ell^\infty} < \left|\frac{\lambda}{\mu}\right| E\ \varepsilon_2 + \frac{1}{|\mu|}\left\|L - L_j\right\|_{\ell^\infty} < \varepsilon_1 + \frac{1}{|\mu|}\left\|L - L_j\right\|_{\ell^\infty} ,$$

Table 1. Principal example: logarithmic kernel $k(|x-y|)=\ln|x-y|$.

	$\ell=1$		**Table (1)**	$\ell=3$	
t	Φ	Φ_T	E	Φ_T	E
$t=0.01$	-0.0001000	-0.0000999664	0.33638E-7	-0.00009996	0.388998E-7
	-0.0000076	-0.0000007656	0.10442E-7	-0.00000076	0.108296E-7
	-0.0000300	-0.0000002994	0.53644E-7	-0.00000029	0.53917E-7
	-0.667×10^{-7}	-0.66520×10^{-7}	0.14632E-7	-0.6654×10^{-7}	0.123618E-7
$t=0.1$	-0.0100000	-0.0100178442	0.178444E-6	-0.01001732	0.17316E-6
	-0.0000766	-0.0000767245	0.57837E-5	-0.00000767	0.539567E-5
	-0.0000533	-0.0000533599	0.26569E-5	-0.00005335	0.243165E-5
	-0.0030000	-0.0030009798	0.97978E-6	-0.00300095	0.952434E-6
$t=0.4$	-0.1600000	-0.1600055961	0.26055E-5	-0.00001625	0.259703E-5
	-0.1226000	-0.1226053478	0.18676E-5	-0.12452798	0.186131E-5
	-0.0480000	-0.0487097037	0.70970E-5	-0.04870926	0.709258E-5
	-0.0106667	-0.0108199579	0.15329E-5	-0.01082365	0.156982E-5
$t=0.8$	-0.6400000	-0.6400771753	0.34377E-4	-0.64007095	0.343709E-4
	-0.4906667	-0.49065164647	0.257795E-4	-0.51646029	0.257936E-4
	-0.1920000	-0.2020090715	0.100090E-4	-0.20201104	0.100110E-4
	-0.0426667	-0.0448816191	0.221495E-4	-0.04488829	0.222162E-4

In Table 1 the linear ($\ell=1$) and nonlinear ($\ell=3$) mixed integral Equation (34) with logarithmic kernel are solved numerically, in different times, using TMM. The error, in each case, is computed. We see that as t increases the error increases. Also, the error, using Maple 12, in the linear case is less than the error in the nonlinear case.

(where $|\lambda|E<|\mu|$)

Since as $j\to\infty$, so that $\|\Phi-\Phi_j\|_{\ell^\infty}\to 0$, $\|L-L_j\|_{\ell^\infty}\to 0$,

NUMERICAL EXAMPLES AND DISCUSSION

For the integral equation

$$\phi(x,t)=f(x,t)+0.001\int_0^t\int_{-1}^1 \tau^2 k(|x-y|)\phi^\ell(y,\tau)dyd\tau; \quad (\phi(x,t)=x^2t) \tag{34}$$

We consider, in general, the nonlinear integral equation in time and position, for $\ell>1$ and the linear term for $\ell=1$. We consider the continuous function of time $F(t,\tau)=t^2$, $t\in[0,T]$, $T<1$. While the exact solution $\phi(x,t)=x^2t$. Using the exact solution the free term $f(x,t)$ is determined after assuming the kernel of position. The kernel of position is considered in three cases

Case 1: Logarithmic kernel $k(|x-y|)=\ln|x-y|$

Case 2: Carleman function $k(|x-y|)=|x-y|^{-\nu}$, $0<\nu<1$,

Case 3: Cauchy kernel $k(|x-y|)=\dfrac{1}{(x-y)}$.

1) Here, we consider a general formula of mixed integral equation in position and time with discontinuous kernel in position and continuous kernel in time.
2) Using quadratic method we have a system of Hammerstein integral equations with singular kernel. The first approximate of the system is discussed in Abdou et al. (2011) using Banach fixed point theorem.
3) We consider the first table for the linear and nonlinear case $\ell=1$, $\ell=3$ respectively, when the kernel in the logarithmic form $k(|x-y|)=\ln|x-y|$. The results are computing, using Maple 12 at, $t=0.01$, 0.1, 0.4 and $t=0.8$ and $N=30$ (Table 1).
4) In the second example, we consider the Carleman kernel $k(|x-y|)=|x-y|^{-\nu}$ and computing the results when $\nu=0.01$, 0.22 and 0.32, where ν is called Poisson's coefficient, $0<\nu<1$, $N=30$ and t=0.1 (Table 2). The importance of Carleman kernel comes from the work of Arytiunian (1959) that has shown that the plane contact problem in the nonlinear theory of plasticity, in its first approximation can be reduced to Fredholm integral equation of the first kind with Carleman kernel.
5) The third case when the kernel takes the Cauchy kernel $k(|x-y|)=\dfrac{1}{(x-y)}$. The results are computing, at, $t=0.01$, 0.1, 0.4 and $t=0.8$ and $N=30$, (Table 3).

Table 2. Principal example: Carleman function $k(|x-y|)=|x-y|^{-\nu}, 0<\nu<1,$

	$\ell=1$	Table (2)		$\ell=3$	
$\nu, t=0.1$	Φ	Φ_T	E	Φ_T	E
ν=0.01	-0.0400000	-0.0405059859	0.50599E-6	-0.04050583	0.50582E-6
	-0.0280000	-0.0283545606	0.35456E-6	-0.02835440	0.35440E-6
	-0.0160000	-0.0162027164	0.20271E=6	-0.01620256	0.20255E-6
	-0.0040000	-0.0040508069	0.25080E-6	-0.00405064	0.50644E-6
ν=0.22	-0.0400000	-0.0405133101	0.51333E-5	-0.04051315	0.51315E-5
	-0.0280000	-0.0283683819	0.36838E-5	-0.02836822	0.36821E-5
	-0.0160000	-0.0162115962	0.21159E-5	-0.01621141	0.21141E-5
	-0.0040000	-0.0040531760	0.53176E-5	-0.00405296	0.52953E-5
ν=0.32	-0.0400000	-0.0405186594	0.51865E-4	-0.40051851	0.18505E-4
	-0.0280000	-0.0283781604	0.37816E-5	-0.02837799	0.37799E-5
	-0.0160000	-0.0162177413	0.21774E-5	-0.0162176	0.21754E-5
	-0.0040000	-0.0040547969	0.54797E-5	-0.00405454	0.54538E-5

In Table 2 the linear and nonlinear mixed integral Equation (34) with Carleman function are solved numerically, in different times, using TMM. The error, in each case, is computed. As ν increases the error are increases. In addition as the time increases, the error increases

Table 3. Principal example: Cauchy kernel $k(|x-y|)=\frac{1}{(x-y)}.$

	$\ell=1$	Table (3)		$\ell=3$	
t	Φ	Φ_T	E	Φ_T	E
t=0.01	0.00000000	0.0000000000	0.00000000	0.000000000	0.00000000
	0.00000068	0.00000083683	0.14853E-7	0.000698754	0.10421E-7
	0.00206500	0.0027655451	0.70054E-7	0.002555558	0.49055E-7
	0.00275333	0.0028636356	0.10302E-7	0.002855432	0.10209E-7
t=0.1	0.00000000	0.0000000000	0.00000000	0.000000000	0.000000000
	0.06883333	0.0683543415	0.47899E-6	0.069363543	0.53021E-6
	0.27533333	0.2763424344	0.10091E-6	0.275444323	0.11099E-6
	0.34416667	0.3472882822	0.31216E-6	0.341559088	0.26075E-6
t=0.4	0.00000000	0.0000000000	0.00000000	0.000000000	0.00000000
	1.10133333	1.1123232342	0.9898E-5	1.122236658	0.20903E-5
	3.3040000	3.3549837939	0.50983E-5	3.325444459	0.21444E-5
	4.40533333	4.4827726380	0.77439E-5	4.435666577	0.30333E-5
t=0.8	0.00000000	0.0000000000	0.000000000	0.000000000	0.0000000000
	4.40533333	4.4053627627	0.26732E-4	4.405342618	0.17009E-4
	13.2160000	13.216019170	0.19170E-4	13.21662789	0.24662E-4
	17.6213333	17.621191892	0.19785E-4	17.62176526	0.19631E-4

In Table 3, the mixed integral equation in linear and nonlinear case (34) with Cauchy are solved numerically in different times, using TMM. The error increases with increase in the time and further increases the linearity of the equation.

The importance of the above kernel is found in the work of Abdou and Salama (2004).

6) The Toeplitz matrix method is considered one of the best methods for solving the singular integral equations with discontinuous kernel, where the singular part disappears and the solution is obtained directly.

7) From Tables (1) - (3), we note that as N increases the error decreases while at t increases the error increases.

8) As ν increase, $0 \le \nu < 1/2$ the error increases, for $\nu = 1/2$, we have the potential kernel; see Abdou (2002). For $1/2 < \nu < 1$, we have the generalized potential function (Abdou et al., 2013b)

Conflict of interests

The authors have not declared any conflict of interests.

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