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Full Length Research Paper

A common fixed point theorem and some fixed point theorems in D*- Metric spaces

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In this paper we establish some common fixed point theorems for contraction and some generalized contraction mappings in D* - metric space which is introduced by Shaban et al. (2007). In what follows, (X, D*) will denote D* - metric space, N is the set of all natural number and R^+ is the set of all positive real number.

Key words: D*- metric, contraction mapping, complete D*- metric space, common fixed point theorem.

INTRODUCTION

There have been a number of generalization in generalized metric space (or D-Metric space) initated by Dhage (1992). He proved the existence of unique fixed point theorems of a self map satisfying contractive conditions in complete and bounded D- Metric space. Dealing with D- metric space, (Ahmad et al., 2001; Dhage, 1992, 1999; Dhange et al., 2000; Rhoades, 1996; Singh and Sharma, 2002) and others made a significant contribution in fixed point theory of D- metric space. Unfortunately almost all theorems in D- metric space are not valid (Naidu et al., 2004, 2005a, b). Here our aim is to prove some common fixed point theorems using some generalized contractive conditions in D*- metric space as a probable modification of the definition of D- metric spaces introduced by Dhage (1992).

Definition 1

Let X be a non empty set. A generalized metric (or D* - metric) on X is a function
$D^*: X^3 \to [0, \infty)$ that satisfies the following conditions for each x, y, z, a $\in$ X.
(1) D* (x, y, z) $\geq$ 0
(2) D* (x, y, z) = 0 if and only if x = y = z
(3) D* (x, y, z) = D* (ρ\{x, y, z\}) where ρ is permutation (function.

(4) D* (x, y, z) $\leq$ D* (x, y, a) + D* (a, z, z).

The pair (X, D*) is called generalized metric (or D* - metric) space.

Examples 1

(a) D*(x, y, z) = max {d(x, y), d(y, z), d(z, x)},

(b) D*(x, y, z) = d(x, y) + d(y, z) + d(z, x).

Here, d is the ordinary metric on X.

(c) If X = R^n then we define

$D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{1/p}$ for every $p \in R^+$

(d) If X = R then we define

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise,} \end{cases}$$

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Remark 1

In D* - metric space D* (x, y, y) = D* (x, x, y)

Definition 2

A open ball in a D* - metric space X with centre x and radius r is denoted by B_{D^*} (x, r) and is defined by B_{D^*} (x, r) = {y ∈X: D* (x, y, y) < r}

Example 2

Let X = R Denote D* (x, y, z) = |x-y| + |y – z| + |z – x| for all x, y, z ∈ R.

Thus, B_{D^*} (0, 1) = {y ∈ R / D* (0, y, y) < 1}
= {y ∈ R / |0 – y| + |y – y| + |y – 0| < 1}
= {y ∈ R / |y| + |y| < 1}
= {y ∈ R / |y| < ½}
= {y ∈ R / - ½ < y < ½}
= (-½, ½).

Definition 3

Let (X, D*) be a D* - metric space and A ⊆ X

(1) If for every x ∈A, there exist r > o such that B_{D^*} (x, r) ⊆ A, then subset A is called open subset of X.
(2) Subset A of X is said to be D* - bounded if there exist r > o such that
D* (x, y, y) < r for all x, y ∈ A.
(3) A sequence $\{x_n\}$ in X converges to x if and only if
D* (x_n, x_n, x) = D* (x, x, x_n) → 0 as n → ∞.
That is, for each ε > 0 there exist n_0 ∈ N such that for all n ≥ n_0 implies
D* (x, x, x_n) < ε. This is equivalent for each ε > 0, there exist n_0 ∈ N such that for all n, m ≥ n_0 implies D* (x, x_n, x_m) < ε.
It is also noted that D* (x_n, x_n, x) = D* (x, x, x_n) < ε for all n ≥ n_0, for some n_0 ∈ N.
(4) A square $\{x_n\}$ in X is called a Cauchy sequence if for each ε > 0, there exist n_0 ∈ N such that D* (x_n, x_n, x_m) < ε for each n, m ≥ n_0 The D* - metric space (X, D*) is said to be complete if every Cauchy sequence is convergent.

Remark 2

(1) D* is continuous function on X^3
(2) If sequence $\{x_n\}$ in X converges to x, then x is unique.
(3) Any convergent sequence in (X, D*) is a Cauchy sequence.

Definition 4

A point x in X is a common fixed point of two maps T_1, T_2: X → X if T_1 (x) = T_2 (x) = x.

MAIN RESULTS

Common fixed point theorems for Banach contraction mappings type in D*- metric space.

Theorem 1

Let (X, D*) be a complete D* - metric space and T_1, T_2, T_3: X→ X be three maps such that D*(T_1x, T_2y, T_3z) ≤ a D* (x, y, z) for all x, y, z, ∈ X and 0 ≤ a < 1. Then T_1, T_2, T_3 have a unique fixed point in X.

Proof

Let x_0 ∈ X be a fixed arbitrary element.
Define a sequence $\{x_n\}$ in X as
$x_{3n+1} = T_1 x_{3n}$ for n = 0, 1, 2. . .
$x_{3n+2} = T_2 x_{3n+1}$ for n = 0, 1, 2. . .
$x_{3n+3} = T_3 x_{3n+2}$ for n = 0, 1, 2. . .
For all n > 0 we define

d_n = D* (x_n , x_{n+1}, x_{n+2})
d_{3n+1} = D* (x_{3n+1}, x_{3n+2}, x_{3n+3})
= D* ($T_1 x_{3n}$, $T_2 x_{3n+1}$, $T_3 x_{3n+2}$)
≤ a D* (x_{3n} , x_{3n+1}, x_{3n+2})
< d_{3n}

Simillarly,

d_{3n+2} = D* (x_{3n+2}, x_{3n+3}, x_{3n+4})
= D* ($T_2 x_{3n+1}$, $T_3 x_{3n+2}$, $T_1 x_{3n+3}$)
≤ a D* (x_{3n+1}, x_{3n+2}, x_{3n+3})
< d_{3n+1}

Hence

$d_{3n+2} < d_{3n+1} < d_{3n}$ for all n.

Thus $\{d_n\}$ is strictly monotonically decreasing sequence of positive real numbers and bounded below, it is convergent to a positive real number .Let it be l . suppose that l ≠ 0. Then l › 0.

l = $\lim_{n\to\infty} d_n$
= $\lim_{n\to\infty} d_{3n+1}$
= $\lim_{n\to\infty}$ D* (x_{3n+1}, x_{3n+2}, x_{3n+3})
= $\lim_{n\to\infty}$ D* ($T_1 x_{3n}$, $T_2 x_{3n+1}$, $T_3 x_{3n+2}$)
≤ $\lim_{n\to\infty}$ a D* (x_{3n}, x_{3n+1}, x_{3n+2})

$\leq \lim_{n\to\infty}$ a d_{3n}

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$\leq \lim_{n\to\infty}$ a $3^n d_0 \to 0$ as $n \to \infty$.

Thus l = 0.

Hence $\lim_{n\to\infty} d_n = 0$.

we shall prove that $\{x_n\}$ is a D* - cauchy sequence in X.
For $m > n \geq n_0$, we have

$D^* (x_n, x_n, x_m) \leq D^* (x_n, x_n, x_{n+1}) + D^* (x_{n+1}, x_{n+1}, x_m)$
$\leq D^* (x_n, x_n, x_{n+1}) + D^* (x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^* (x_{m-1}, x_{m-1}, x_m)$
$\to 0$ as $m, n \to \infty$.

Thus $D^* (x_n, x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$ for some $n_0 \in N$
Thus $\{x_n\}$ is D* - Cauchy sequence in X.
Since X is D* - complete $x_n \to x$ in X as $n \to \infty$.
Now we shall prove that $T_1x = x$.
We have $D^* (T_1x, x, x) = \lim_{n\to\infty} D^* (T_1x, T_2 x_{3n+1}, T_3 x_{3n+2})$

$\leq a \lim_{n\to\infty} D^* (x, x_{3n+1}, x_{3n+2}) = 0$

Thus $D^* (T_1x, x, x) = 0$. Hence $T_1x = x$

Similarly, we prove that $T_2x = x$ and $T_3x = x$

Uniqueness:
Suppose $x \neq y$ such that $T_1y = y$, $T_2y = y$ and $T_3 y = y$
Now $D^* (x, y, y) = D^* (T_1x, T_2y, T_3 y)$
$\leq a D^* (x, y, y)$.
This implies $(1- a) D^* (x, y, y) \leq 0$.
Since $x \neq y$, $D^* (x, y, y) > 0$ we have $1- a < 0$
This implies $a > 1$ which is contradiction to $a < 1$.
Hence T_1, T_2 and T_3 have a unique common fixed point.

Theorem 2

Let (X, D*) be a D*- complete metric space and $T: X \to X$ be a map such that

$$D^*(Tx, Ty, Tz) \leq a \left\{D^*(x, y, z) + D^*(x, Tx, Ty) + D^* (y, Ty, Tz)\right\} \text{ for all } x, y, z \in X$$

and $0 \leq a < 1/4$. Then T has a unique fired point.

Proof

Let $x_0 \in X$ a fixed arbitrary element.
Define the sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for n = 0, 1, 2 . . .
For $n \geq 0$ we have,

$D^*(x_n, x_n, x_{n+1}) = D^*(Tx_{n-1}, Tx_{n-1}, Tx_n)$
$\leq a \{D^* (x_{n-1}, x_{n-1}, x_n) + D^*(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + D^*(x_{n-1}, Tx_{n-1}, Tx_n)\}$
$= a \{D^*(x_{n-1}, x_{n-1}, x_n) + D^*(x_{n-1}, x_n, x_n) + D^*(x_{n-1}, x_n, x_{n+1})\}$
$\leq a \{D^*(x_{n-1}, x_{n-1},x_n) + D^*(x_{n-1}, x_{n-1}, x_n) + D^*(x_{n-1}, x_{n-1},x_n) + D^*(x_n, x_n, x_{n+1})\}$
$(1-a) D^*(x_n, x_n, x_{n+1}) \leq 3a D^*(x_{n-1}, x_{n-1}, x_n)$
$D^*(x_n, x_n, x_{n+1}) \leq \frac{3a}{1-a} D^*(x_{n-1}, x_{n-1}, x_n)$.
$\leq b D^*(x_{n-1}, x_{n-1}, x_n)$.

where $b = \frac{3a}{1-a} < 1$.

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$\leq b^n D^* (x_0, x_0, x_1) \to 0$, as $n \to \infty$.
For $m > n \geq n_0$, we have
$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$
$\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m)$
$\to 0$ as $m, n \to \infty$.

Thus $\{x_n\}$ is D*- Cauchy sequence in X.
Since X is D*- complete $x_n \to x$ in X
Now we prove that Tx = x. Suppose $x \neq Tx$

$D^*(Tx, x, x) = \lim_{n\to\infty} D^*(Tx, x_{n+1}, x_{n+1})$

$= \lim_{n\to\infty} D^*(Tx, Tx_n, Tx_n)$

$\leq a \lim_{n\to\infty} \{D^*(x, x_n, x_n) + D^*(x, Tx, Tx_n) + D^*(x_n, Tx_n, Tx_n)\}$

$= a \lim_{n\to\infty} \{D^*(x, x_n, x_n) + D^*(x, Tx, x_{n+1}) + D^*(x_n, x_{n+1}, x_{n+1})\}$

$= a D^*(x, Tx, x)$
$< D^*(Tx, x, x)$. which is contradiction.
Thus x = Tx.

Now we prove the uniqueness. Suppose $x \neq y$ such that Ty = y

$D^*(x, y, y) = D^*(Tx, Ty, Ty)$
$\leq a \{D(x, y, y) + D^*(x, x, y) + D^*(y, y, y) \}$
$\leq 2a D^*(x, y, y)$
$< D^* (x, y, y)$, which is contradiction.

Hence T has a unique fixed point.

Theorem 3

Let (X, D*) be a D*- complete metric space and $T: X\to X$ be a map such that

$$D^*(Tx, Ty, Tz) \leq a_1 \left\{ D^*(x, y, z) + \frac{a_2}{2} \{D^*(x, Tx, Ty) + D^*(y, Ty, Tz)\} + \frac{a_3}{2} \{D^*(x, y, Ty) + D^*(y, z, Tz)\} \text{ for all } x, y, z \in X \right\}$$

and $0 \le a_1 + \frac{3}{2} a_2 + \frac{3}{2} a_3 < 1$. Then T has a unique fixed point.

Proof

Let $x_0 \in X$ be a fixed arbitrary element.
Define the sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for n = 0, 1, 2 . . .
For $n \ge 0$ we have

$$D^*(x_n, x_n, x_{n+1}) = D^*(Tx_{n-1}, Tx_{n-1}, Tx_n)$$
$$\le \Big\{ a_1 D^*(x_{n-1}, x_{n-1}, x_n) + \frac{a_2}{2} \{D^*(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + D^*(x_{n-1}, Tx_{n-1}, Tx_n)\}$$
$$+ \frac{a_3}{2} \{D^*(x_{n-1}, x_{n-1}, Tx_{n-1}) + D^*(x_{n-1}, x_n, Tx_n)\} \Big\}$$
$$= \Big\{ a_1 D^*(x_{n-1}, x_{n-1}, x_n) + \frac{a_2}{2} \{D^*(x_{n-1}, x_n, x_n) + D^*(x_{n-1}, x_n, x_{n+1})\}$$
$$+ \frac{a_3}{2} \{(x_{n-1}, x_{n-1}, x_n) + D^*(x_{n-1}, x_n, x_{n+1})\} \Big\}$$
$$\le \Big\{ a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 (D^*(x_{n-1}, x_{n-1}, x_n) + a_3 D^*(x_{n-1}, x_{n-1}, x_n)$$
$$+ \frac{a_2}{2} D^*(x_n, x_n, x_{n+1}) + \frac{a_3}{2} D^*(x_n, x_n, x_{n+1}) \Big\}$$

$$(1 - \frac{a_2}{2} - \frac{a_3}{2}) D^*(x_n, x_n, x_{n+1}) \le (a_1+a_2+a_3) D^*(x_{n-1}, x_{n-1}, x_n)$$

$$D^*(x_n, x_n, x_{n+1}) \le \frac{a_1 + a_2 + a_3}{\left(1 - \frac{a_2}{2} - \frac{a_3}{2}\right)} D^*(x_{n-1}, x_{n-1}, x_n)$$

$D^*(x_n, x_n, x_{n+1}) \le a\, D^*(x_{n-1}, x_{n-1}, x_n)$, where

$$a = \frac{a_1 + a_2 + a_3}{\left(1 - \frac{a_2}{2} - \frac{a_3}{2}\right)} < 1.$$

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$\le a^n D^*(x_0, x_0, x_1) \to 0$ as $n \to \infty$
Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X.
For m > n, we have
$D^*(x_n, x_n, x_m) \le D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$
$\le D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \dots + D^*(x_{m-1}, x_{m-1}, x_m)$
$\to 0$ as $n, m \to \infty$.
Thus $\{x_n\}$ is D* - Cauchy sequence in X
Since X is D* - Complete $x_n \to x$ in X
Now we prove that Tx = x. Suppose $x \ne Tx$.

$$D^*(Tx, x, x) = \lim_{n\to\infty} D^*(Tx, x_{n+1}, x_{n+1})$$
$$= \lim_{n\to\infty} D^*(Tx, Tx_n, Tx_n)$$
$$\le \lim_{n\to\infty} \Big\{ a_1 D^*(x, x_n, x_n) + \frac{a_2}{2} \{D^*(x, Tx, Tx_n) + D^*(x_n, Tx_n, Tx_n)\}$$
$$+ \frac{a_3}{2} \{D^*(x, x_n, Tx_n) + D^*(x_n, x_n, Tx_n)\} \Big\}$$
$$= \lim_{n\to\infty} \Big\{ a_1 D^*(x, x_n, x_n) + \frac{a_2}{2} \{D^*(x, Tx, x_{x+1}) + D^*(x_n, x_{n+1}, x_{n+1})\}$$
$$+ \frac{a_3}{2} \{D^*(x, x_n, x_{n+1}) + D^*(x_n, x_n, x_{n+1})\} \Big\}$$
$$= \frac{a_2}{2} D^*(x, x, Tx)$$

$< D^*(x, x, Tx)$. Which is contradiction Hence x = Tx

Uniqueness:
Suppose $x \ne y$ such that Ty = y

$$D^*(x, y, y) = D^*(Tx, Ty, Ty)$$
$$\le a_1 D^*(x, y, y) + \frac{a_2}{2} \{D^*(x, Tx, Ty) + D^*(y, Ty, Ty)\}$$
$$+ \frac{a_3}{2} \{D^*(x, y, Ty) + D^*(y, y, Ty)\}$$
$$= a_1 D^*(x, y, y) + \frac{a_2}{2} \{D^*(x, x, y) + D^*(y, y, y)\}$$
$$+ \frac{a_3}{2} \{D^*(x, y, y) + D^*(y, y, y)\}$$
$$= (a_1 + \frac{a_2}{2} + \frac{a_3}{2}) D^*(x, y, y)$$

$< D^*(x, y, y)$, which is contradiction.

Thus x = y. T has a unique fixed point.

Theorem 4

Let (X, D*) be a complete D* - metric space and $T: X \to X$ be a map such that
$D^*(Tx, T^2x, T^3x) \le a\, D^*(x, Tx, T^2x)$ for all $x \in X$ and $0 \le a < 1$. Then T has a unique fixed point.

Proof

Let $x_0 \in X$ be a fixed arbitrary element.
Define the sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for n = 0, 1, 2 . . .
For $n \ge 0$, we have

$D^*(x_n, x_n, x_{n+1}) = D^*(Tx_{n-1}, Tx_{n-1}, Tx_n)$
$\le a\, D^*(x_{n-1}, x_{n-1}, x_n)$
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$\le a^n D^*(x_0, x_0, x_1) \to 0$ as $n \to \infty$.
For m > n we have

$D^*(x_n, x_n, x_m) \le D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \dots + D^*(x_{m-1}, x_{m-1}, x_m)$
$\to 0$ as $m, n \to \infty$
Hence $\{x_n\}$ is a Cauchy sequence in D* - complete metric space.

Thus, $x_n \to x$ in X.
Now we prove Tx = x. Supper $x \neq Tx$

$D^*(x, x, Tx) = \lim_{n\to\infty} D^*(x_{n+3}, x_{n+2}, Tx)$

$= \lim_{n\to\infty} D^*(T^3x_n, T^2x_n, Tx)$

$\leq a \lim_{n\to\infty} D^*(T^2x_n, Tx_n, x)$

$= a \lim_{n\to\infty} D^*(x_{n+2}, x_{n+1}, x) = 0$

Thus, x = Tx

Uniqueness:
Supper $x \neq y$ such that Ty = y.
Then $D^*(x, y, y) = D^*(T^3x, T^2y, Ty)$
$\leq a D^*(T^2x, Ty, y)$
$= a D^*(x, y, y)$
This implies
$(1- a) D^* (x, y, y) \leq 0$
Hence 1 - a < 0 (since $D^*(x, y, y) > 0$)
There fore a > 1. This is contradiction to a < 1.
Thus T has a unique fixed point.

Theorem 5

Let (X, D*) be a complete D* - metric space and $T:X \to X$ be a map such that

$D^*(Tx, Ty, Tz) \leq a \max \{D^*(x, y, z), D^*(x, Tx, Ty), D^*(y, Ty, Tz), D(x, y, Ty), D(y, z, Tz)\}$

for all x, y, z $\in$ X and $0 \leq a < \frac{1}{2}$. Then T has a unique fixed point.

Proof

Let $x_0 \in X$ be a fixed arbitrary element.
Define the sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for n = 0, 1, 2 . . .
For $n \geq 0$, we have
$D^*(x_n, x_n, x_{n+1}) = D^*(Tx_{n-1}, Tx_{n-1}, Tx_n)$
$\leq a \max \{D^*(x_{n-1}, x_{n-1}, x_n), D^*(x_{n-1}, Tx_{n-1}, Tx_{n-1}), D^*(x_{n-1}, Tx_{n-1}, Tx_n),$
$D^*(x_{n-1}, x_{n-1}, Tx_{n-1}), D^*(x_{n-1}, x_n, Tx_n)\}$
$= a \max \{D^*(x_{n-1}, x_{n-1}, x_n), D^*(x_{n-1}, x_n, x_n), D^*(x_{n-1}, x_n, x_{n+1}),$
$D^*(x_{n-1}, x_{n-1}, x_n) D^*(x_{n-1}, x_n, x_{n+1})\}$
$= a \max \{D^*(x_{n-1}, x_{n-1}, x_n), D^*(x_{n-1}, x_n, x_{n+1})\}$
$\leq a \max \{D^*(x_{n-1}, x_{n-1}, x_n), D^*(x_{n-1}, x_{n-1}, x_n) + D^*(x_n, x_n, x_{n+1})\}$
$\leq a D^*(x_{n-1}, x_{n-1}, x_n) + a D^*(x_n, x_n, x_{n+1})$
$(1-a) D^*(x_n, x_n, x_{n+1}) \leq a D^*(x_{n-1}, x_{n-1}, x_n)$

$D^*(x_n, x_n, x_{n+1}) \leq \frac{a}{1-a} D^*(x_{n-1}, x_{n-1}, x_n)$

$\leq b D^*(x_{n-1}, x_{n-1}, x_n)$ where $b = \frac{a}{1-a} < 1$ for all n

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$\leq b^n D^*(x_0, x_0, x_1) \to 0$ as $n \to \infty$
Now we prove that $\{x_n\}$ is D* - cauchy sequence in X.
For m > n we have,
$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + . . . + D^*(x_{m-1}, x_{m-1}, x_m)$
$\to 0$ as m. $n \to \infty$

Thus $\{x_n\}$ is a D* cauchy sequence in X and X is D* - complete $x_n \to x$ in X.
Now we shall prove that Tx = x

$D^*(Tx, x, x) = \lim_{n\to\infty} D^*(Tx, x_{n+1}, x_{n+1})$

$= \lim_{n\to\infty} D^*(Tx, Tx_n, Tx_n)$

$\leq a \lim_{n\to\infty} \max \{D^*(x, x_n, x_n), D^*(x, Tx, Tx_n), D^*(x_n, Tx_n, Tx_n)$
$D^*(x, x_n, Tx_n), D^*(x_n, x_n, Tx_n)\}$

$\leq a \lim_{n\to\infty} \max \{D^*(x, x_n, x_n), D^*(x, Tx, x_{n+1}), D^*(x_n, x_{n+1}, x_{n+1}),$
$D^*(x, x_n, x_{n+1}), D^*(x_n, x_n, x_{n+1})\}$
$\to a \{D^*(x, Tx, x)\}$
$< D^*(Tx, x, x)$, Which is a contradiction. Thus x = Tx

Uniqueness:
Suppose $x \neq y$ such that Ty = y
$D^*(x, y, y) = D^*(Tx, Ty, Ty)$
$\leq a \max \{D^*(x, y, y), D^*(x, Tx, Ty), D^*(y, Ty, Ty), D^*(x, y, Ty), D^*(y, y, Ty)\}$
$= a \max \{D^*(x, y, y), D^*(x, x, y), D^*(y, y, y), D^*(x, y, y), D^*(y, y, y)\}$
$= a D^*(x, y, y)$
$< D^* (x, y, y)$, which in contradiction. Thus x = y
Therefore T has a unique fixed point.

Theorem 6

Let (X, D*) be a complete D* - metric space and $T: X \to X$ be a map such that
$D^*(Tx, Ty, Tz) \leq a_1 D^* (x, y, z) + a_2 \max \{D^*(x, Tx, Ty), D^*(y, Ty, Tz)\}$ for all x, y, z, $\in$ X and $0 \leq a_1 + 2a_2 < 1$. Then T has a unique fixed point.

Proof

Let $x_0 \in X$ be any arbitrary fixed element and define a sequence element and define a sequence $\{x_n\}$ in X as $x_{n+1} = x_n$ for n = 0, 1, 2 . . .

For $n \geq 0$, we have.

$D^*(x_n, x_n, x_{n+1}) = D^*(Tx_{n-1}, Tx_{n-1}, Tx_n)$
$\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max\{D^*(x_{n-1}, Tx_{n-1}, Tx_{n-1}), D^*(x_{n-1}, Tx_{n-1}, Tx_n)\}$
$= a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max\{D^*(x_{n-1}, x_n, x_n), D^*(x_{n-1}, x_n, x_{n+1})\}$
$\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \{D^*(x_{n-1}, x_{n-1}, x_n) + D^*(x_n, x_n, x_{n+1})\}$
$(1- a_2) D^*(x_n, x_n, x_{n+1}) \leq (a_1 + a_2) D^*(x_{n-1}, x_{n-1}, x_{n+1})$
$D^*(x_n, x_n, x_{n+1}) \leq \frac{a_1 + a_2}{1 - a_2} D^*(x_{n-1}, x_{n-1}, x_n)$
Thus $D^*(x_n, x_n, x_{n+1}) \leq a D^*(x_{n-1}, x_{n-1}, x_n)$, where

$a = \frac{a_1 + a_2}{1 - a_2} < 1.$

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$\leq a^n D^*(x_0, x_0, x_1)$
$\to 0$ as $n \to \infty$.
Now for $m > n \geq 0$ we have
$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$
$\leq \sum_{k=n}^{m-1} D^*(x_k, x_k, x_{k+1}) \to 0$ as $m, n \to \infty$
Thus $\{x_n\}$ is a Cauchy sequence in complete D^* - metric space.
Hence there exist a point $x \in X$ such that $x_n \to x$ in X.

Now we shall prove that x is a fixed point of T.
$D^*(x, x, Tx) = \lim_{n\to\infty} D^*(x_{n+1}, x_{n+1}, Tx)$
$= \lim_{n\to\infty} D^*(Tx_n, Tx_n, Tx)$
$\leq \lim_{n\to\infty} \{a_1 D^*(x_n, x_n, x) + a_2 \max\{D^*(x_n, Tx_n, Tx_n), D^*(x_n, Tx_n, Tx)\}$
$\leq \lim_{n\to\infty} \{a_1 D^*(x_n, x_n, x) + a_2 \max\{D^*(x_n, x_{n+1}, x_{n+1}), D^*(x_n, x_{n+1}, Tx)\}\}$
$\to a_1 (0) + a_2 D^*(x, x, Tx)$ as $n \to \infty$

Thus, $D^*(x, x, Tx) < D^*(x, x, Tx)$, which is contradiction.
Thus implies $x = Tx$.

Now we shall prove uniqueness. Suppose $x \neq y$ such that $Ty = y$
Then $D^*(x, y, y) = D^*(Tx, Ty, Ty)$

$\leq a_1 D^*(x, y, y) + a_2 \max\{D^*(x, Tx, Ty), D^*(y, Ty, Ty)\}$
$= a_1 D^*(x, y, y) + a_2 \max\{D^*(x, x, y), D^*(y, y, y)\}$
$= (a_1 + a_2) D^*(x, y, y)$
$< D^*(x, y, y)$, which is contradiction. There fore T has a unique fixed point.

Remark 3

If we put $a_2 = 0$ and $a_1 = a$ in the above theorem we get the following Theorem as corollary.

Corollary 1

Let (X, D^*) be a complete D^* - metric space and $T: X \to X$ be a map such that
$D^*(Tx, Ty, Tz) \leq a D^*(x, y, z)$ for all $x, y, z \in X$ and $0 \leq a < 1$. Then T has a unique fixed point.
The aforeseen theorem is known as Banach contraction type theorem in D^*- metric space.

Remark 4

If we put $a_1 = 0$ and $a_2 = a$ in the previous theorem 6. We get the following theorem as corollary 2.

Corollary 2

Let (X, D^*) be a complete D^* - metric space and $T: X \to X$ be a map such that
$D^*(Tx, Ty, Tz) \leq \frac{a}{2} \max\{D^*(x, Tx, Ty), D^*(y, Ty, Tz)\}$ for all $x, y, z \in X$ and
$0 \leq a < \frac{1}{2}$. Then T has a unique fixed point.

Theorem 7

Let (X, D^*) be a complete D^* - metric space and $T: X \to X$. be a map such that

$$D^*(Tx, Ty, Tz) \leq \left\{ \begin{array}{l} a_1 D^*(x, y, z) + a_2 \max\{D^*(x, Tx, Ty), D^*(y, Ty, Tz)\} \\ + a_3 \max\{D^*(x, y, Ty), D^*(y, z, Tz)\} \end{array} \right\}$$

for all $x, y, z \in X$ and $0 \leq a_1 + 2a_2 + 2a_3 < 1$. Then T has a unique fixed point.

Proof

Let $x_0 \in X$ be a fixed arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for $n = 0, 1, 2 \ldots$
Now for $n \geq 0$ we have

$D^*(x_n, x_n, x_{n+1}) = D^*(Tx_{n-1}, Tx_{n-1}, Tx_n)$

$$= \left\{ \begin{array}{l} a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max\{D^*(x_{n-1}, Tx_{n-1}, Tx_{n-1}), D^*(x_{n-1}, Tx_{n-1}, Tx_n)\} \\ + a_3 \max\{D^*\{x_{n-1}, x_{n-1}, Tx_{n-1}), D^*(x_{n-1}, x_n, Tx_n)\} \end{array} \right\}$$

$= \{ a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{D^*(x_{n-1}, x_n, x_n), D^*(x_{n-1}, x_n, x_{n+1}) + a_3 \max \{D^*(x_{n-1}, x_{n-1}, x_n), D^*(x_{n-1}, x_n, x_{n+1})\} \}$

$\leq \{ a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2\{ D^*(x_{n-1}, x_{n-1}, x_n) + D^*(x_n, x_n, x_{n+1}) \} + a_3 \{D^*(x_{n-1}, x_{n-1}, x_n) + D^*(x_n, x_n, x_{n+1}) \} \}$

$(1- a_2- a_3) D^*(x_n, x_n, x_{n+1}) \leq (a_1+ a_2 + a_3) D^*(x_{n-1}, x_{n-1}, x_n)$ for all $n \geq 0$.

$$D^*(x_n, x_n, x_{n+1}) \leq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} D^*(x_{n-1}, x_{n+1}, x_n)$$

$D^*(x_n, x_n, x_{n+1}) \leq a D^* (x_{n-1}, x_{n-1}, x_n)$ for all $n \geq 0$, where $a = \dfrac{a_1 + a_2 + a_3}{1 - a_2 - a_3} < 1$

$\leq a^n D^*(x_0, x_0, x_1) \to 0$ as $n \to \infty$

Now we shall prove that $\{x_n\}$ is a D* - Cauchy sequence in X.
For $m > n \geq 0$, we have
$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$

$\leq \sum_{k=n}^{m-1} D^*(x_k, x_k, x_{k+1}) \to 0$ as $m, n \to \infty$

Thus $\{x_n\}$ is a Cauchy sequence in D* - complete metric space X.
Hence there is a point x in X such that $x_n \to x$ in X.
Now we shall prove that x is fixed point of T.
Now $D^*(x, x, Tx) = \lim_{n\to\infty} D^*(x_{n+1}, x_{n+1}, Tx)$

$= \lim_{n\to\infty} D^*(Tx_n, Tx_n, Tx)$

$\leq \lim_{n\to\infty} \{ a_1 D^*(x_n, x_n, x) + a_2 \max \{D^*(x_n, Tx_n, Tx_n), D^*(x_n, Tx_n, Tx)\} + a_3 \max \{D^*(x_n, x_n, Tx_n), D^*(x_n, x, Tx)\} \}$

$= \lim_{n\to\infty} \{ a_1 D^*(x_n, x_n, x) + a_2 \max \{D^*(x_n, x_{n+1}, x_{n+1}), D^*(x_n, x_{n+1}, Tx)\} + a_3 \max \{D^*(x_n, x_n, x_{n+1}), D^*(x_n, x, Tx)\} \}$

$= a_1 (0) + a_2 D^*(x, x, Tx) + a_3 D^*(x, x, Tx)$
$< D^*(x, x, Tx)$, which is contradiction.
Thus Tx = x.
Uniqueness:
Suppose $y \neq x$ such that Ty = y.
Now D*(x, y, y) = D*(Tx, Ty, Ty)

$\leq \{ a_1 D^*(x, y, y) + a_2 \max \{D^*(x, Tx, Ty), D^*(y, Ty, Ty)\} + a_3 \max \{D^*(x, y, Ty), D^*(y, y, Ty)\} \}$

$= \{ a_1 D^*(x, y, y) + a_2 \max \{D^*(x, x, y), D^*(y, y, y) + a_3 \max \{D^*(x, y, y), D^*(y, y, y)\} \}$

$= a_1 D^*(x, y, y) + a_2 D^*(x, y, y) + a_3 D^*(x, y, y)$
$= (a_1+a_2+a_3) D^*(x, y, y)$
$< D^*(x, y, y)$, which is contradiction.
Hence T has a unique fixed point.

Theorem 8

Let (X, D*) be a complete D* - metric space and $T: X \to X$ be a map such that

$D^*(Tx, Ty, Tz) \leq a_1 D^*(x, y, z) + a_2 \max \{\left[\dfrac{D^*(x,Tx,Ty) + D^*(y,Ty,Tz)}{2}\right], \left[\dfrac{D^*(x,y,Ty) + D^*(y,z,Tz)}{2}\right]\}$ for all $x, y, z \in X$ and $0 \leq a_1 + 3\dfrac{a_2}{2} < 1$. Then T has a unique fixed point.

Proof

Let $x_0 \in X$ be any fixed arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for n = 0, 1, 2 . . .
For $n \geq 0$, we have

$D^*(x_n, x_n, x_{n+1}) = D^*(Tx_{n-1}, Tx_{n-1}, Tx_n)$
$\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max\{\left\{\dfrac{D^*(x_{n-1},Tx_{n-1},Tx_{n-1}) + D^*(x_{n-1},Tx_{n-1},Tx_n)}{2}\right\}, \left\{\dfrac{D^*(x_{n-1},x_{n-1},Tx_{n-1}) + D^*(x_{n-1},x_n,Tx_n)}{2}\right\}\}$

$\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{ \left\{\dfrac{D^*(x_{n-1},x_n,x_n) + D^*(x_{n-1},x_n,x_{n+1})}{2}\right\}, \left\{\dfrac{D^*(x_{n-1},x_{n-1},x_n) + D^*(x_{n-1},x_n,x_{n+1})}{2}\right\}\}$

$\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \left\{\dfrac{D^*(x_{n-1},x_n,x_n) + D^*(x_n,x_n,x_{n+1}) + D^*(x_{n-1},x_{n-1},x_n)}{2}\right\}$

$\left(1 - \dfrac{a_2}{2}\right) D^*(x_n, x_n, x_{n+1}) \leq \left(a_1 + a_2\right) D^*(x_{n-1}, x_{n-1}, x_n)$

$$D^*(x_n, x_n, x_{n+1}) \leq \frac{a_1 + a_2}{1 - \frac{a_2}{2}} D^*(x_{n-1}, x_{n-1}, x_n)$$

$\leq a D^*(x_{n-1}, x_{n-1}, x_n)$ for all $n \geq 0$ and $a = \dfrac{a_1 + a_2}{1 - \frac{a_2}{2}} < 1$

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$\leq a^n D^*(x_0, x_0, x_1) \to 0$ as $n \to \infty$.

Now we shall prove that $\{x_n\}$ is a cauchy sequence in X.
For $m > n \geq 0$, we have

$$D^*(x_n, x_n, x_m) \leq \sum_{k=n}^{m-1} D^*(x_k, x_k, x_{k+1}) \to 0 \text{ as } m, n \to \infty.$$

Thus $\{x_n\}$ is a D* - Cauchy sequence in X.
Since X is complete D*- metric space and $x_n \to x$ in X.
Now we prove that x = Tx suppose $x \neq Tx$.

$$D^*(x, x, Tx) = \lim_{n\to\infty} D^*(x_{n+1}, x_{n+1}, Tx) \leq \lim_{n\to\infty} \{a_1 D^*(x_n, x_n, x) + \frac{a_2}{2} \max \{D^*(x_n, Tx_n, Tx_n) + D^*(x_n, Tx_n, Tx)\},$$
$$\{D^*(x_n, x_n, Tx_n) + D^*(x_n, x, Tx)\}\}$$
$$= \lim_{n\to\infty}\{a_1 D^*(x_n, x_n, x) + \frac{a_2}{2} \max \{D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_n, x_{n+1}, Tx)\},$$
$$\{D^*(x_n, x_n, x_{n+1}) + D^*(x_n, x, Tx)\}\}$$
$$= a_1 (0) + \frac{a_2}{2} \max \{D^*(x, x, Tx), \{D^*(x, x, Tx)\}$$
$< D^*(x, x, Tx)$, which is contradiction. Thus Tx = x.

Uniqueness:
Suppose $y \neq x$ such that Ty = y.
Now D*(x, y, y) = D*(Tx, Ty, Ty)

$$\leq \left\{ a_1 D^*(x, y, y) + \frac{a_2}{2} \max\{D^*(x, Tx, Ty) + D^*(y, Ty, Ty)\}, \{D^*(x, y, Ty) + D^*(y, y, Ty)\} \right\}$$

$$= a_1 D^*(x, y, y) + \frac{a_2}{2} \max \{\{D^*(x, x, y) + D^*(y, y, y)\}, \{D^*(x, y, y) + D^*(y, y, y)\}$$

$$\leq \left\{ a_1 D^*(x, y, y) + \frac{a_2}{2} D^*(x, y, y) = (a_1 + \frac{a_2}{2}) D^*(x, y, y) \right\}$$

$< D^*(x, y, y)$, which is contradiction.
Hence T has a unique fixed point.

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