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Modified and a new spectral method for solving nonlinear ordinary differential equations

Z. Kalateh bojdi[1], S. Ahmadi-Asl[1] and A. Aminataei[2]*

[1]Department of Mathematics, Birjand University, Birjand, Iran.
[2]Department of Mathematics, K. N. Toosi University of Technology, P. O. Box 16315-1618, Tehran, Iran.

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In this paper, we present a modified and new version of spectral method which is based on minimization of obtained residual term in $\|.\|_{w^{(\alpha,\beta)}(x)}$ norm, where $w^{(\alpha,\beta)}(x)$ is a weight function with respect to Jacobi polynomials. Using this approach is efficient and effective rather than Tau and collocation methods. It reduces the nonlinear ordinary differential equations to the nonlinear programming problems which is an easy problem to solve. Hence, easy implementation of the method is the importance of our approach and some numerical test experiments show the accuracy and efficiency of this method.

Key words: A new spectral method, nonlinear ordinary differential equations.

INTRODUCTION

Spectral methods have been successfully applied in the approximation of differential boundary value problems (Gottlieb and Orszag, 1977; Boyd, 2000; Canuto et al., 1984, 2006; Trefethen, 2000). The most three widely used spectral versions are the Galerkin, Collocation, and Tau methods (Canuto et al., 1984, 2006; Trefethen, 2000). Their utilities are based on the fact that if the solution sought is smooth, usually only a few terms in an expansion of global basic functions are needed to represent it to high accuracy (Gottlieb and Orszag, 1977; Boyd, 2000; Canuto et al., 1984, 2006; Trefethen, 2000). The main advantage of these methods lies in their accuracy for a given number of unknowns. For smooth problems in simple geometries, they offer exponential rates of convergence (spectral accuracy) (Hesthaven et al., 2009). In contrast, finite difference and finite element methods yield only algebraic convergence rates (Ben-yu, 1996).

Approximating functions in spectral methods are related to polynomial solutions of eigenvalue problems in ordinary differential equations, known as Sturm-Liouville problems (Trefethen, 2000; Hesthaven et al., 2009; Ben-yu, 1996). On the non-periodic canonical interval $[-1,1]$, the Jacobi polynomials are the well known class of polynomials exhibiting spectral convergence, of which particular examples are Chebyshev polynomials of the first and second kinds, and Legendre polynomials (Gottlieb and Orszag, 1977; Boyd, 2000; Canuto et al., 1984, 2006; Trefethen, 2000; Hesthaven et al., 2009; Ben-yu, 1996; Imani et al., 2011). We must note at this point that, collocation methods have become

*Corresponding author. E-mail: ataei@kntu.ac.ir.

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increasingly popular for solving differential equations, rather than another spectral method such as Tau and Galerkin methods because of thier easy implementation (Imani et al., 2011). More so, the other importance of Collocation methods are providing very useful highly accurate solutions to nonlinear differential equations.

In this study, we present a new approach for the minimization of obtained residual term in $\|\cdot\|_{w^{(\alpha,\beta)}(x)}$ norm, where $w^{(\alpha,\beta)}(x)$ is a weight function with respect to Jacobi polynomials. We consider the class of nonlinear ordinary differential equations in the following form:

$$\sum_{i=0}^{s} P_i(x)\left(y''\right)^i + \sum_{i=0}^{l} Q_i(x)\left(y'\right)^i + \sum_{i=0}^{m} R_i(x)\left(y\right)^i = f(x), \tag{1}$$

with boundary conditions

$$\sum_{i=0}^{1} a_{i,j} y_j^{(i)}(a_i) = b_{i,j},\ j = 1,2,\ a_{i,j}, b_{i,j} \in R,\ a_i \in [-1,1]. \tag{2}$$

Where $s,l,m \geq 0, P_i(x), Q_i(x), R_i(x),$ and $f(x)$ are defined on the interval $-1 \leq x \leq 1$.Using our method, we reduced the nonlinear ordinary differential Equations (1) and (2), to the nonlinear programming problems which easily solves the problem. The remainder of the paper is organized as follows:

The properties of Jacobi polynomials and the basic formulation of them required for our subsequent development were introduced. The operational matrix of general Jacobi polynomials (product, derivative and moment) is devoted to with some useful theorems. Nonlinear programming problems and their models were prersented. The application of general Jacobi matrix method to the solution of Problems (1) and (2) were summarized. Thus, nonlinear programming problems are formed and the solutions of the considered problem are introduced.

The proposed method is also applied to several numerical experiments and a comparison is made with existing methods in the literature. Finally, we have monitored a brief conclusion in this work. Note that we have computed the numerical results by Matlab (version 2012) programming.

The Jacobi polynomials

The Jacobi polynomials associated with the real parameters $(\alpha > -1, \beta > -1),$ are the sequence of polynomials $P_n^{(\alpha,\beta)}(x), (n = 0,1,...)$ satisfying the orthogonality relation (Hesthaven et al., 2009; Ben-yu, 1996; Imani et al., 2011),

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)dx = \begin{cases} 0, & m \neq n, \\ h_i^{(\alpha,\beta)}, & m = n, \end{cases}$$

where

$$h_n = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.$$

These polynomials are eigenfunctions of the following singular Sturm-Liouville equations in Hesthaven et al. (2009)

$$(1-x^2)y''(x) + [\beta - \alpha - (\beta+\alpha+2)x]y'(x) + n(n+\alpha+\beta+1)y(x) = 0 \tag{4}$$

and produces a complete $L^2_{w^{(\alpha,\beta)}}(-1,1)-$ orthogonal system with the following inner product and norms:

$$< f, g >_{w^{(\alpha,\beta)}} = \int_{-1}^{1} f(x)g(x)w^{(\alpha,\beta)}(x)dx,$$

$$\left\| f \right\|_{w^{(\alpha,\beta)}} =< f, f >^{1/2}_{w^{(\alpha,\beta)}}. \tag{5}$$

A function $y(x) \in L^2_{w^{(\alpha,\beta)}}[-1,1]$, can be expressed in terms of general shifted Jacobi polynomials as

$$y(x) = \sum_{i=0}^{\infty} a_i P_i^{(\alpha,\beta)}(x), \tag{6}$$

where the coefficient a_i is given by

$$a_i = \frac{1}{h_i^{(\alpha,\beta)}} \int_a^b P_i^{(\alpha,\beta)}(x)y(x)w^{(\alpha,\beta)}(x)dx. \tag{7}$$

In practice, only the first $m+1$ terms shifted Jacobi polynomials are considered. Then we have:

$$y(x) \cong \sum_{i=0}^{m} \delta_t^{(0,0,1)} P_i^{(\alpha,\beta)}(x) = \left(P^{(\alpha,\beta)}(x)\right)^T \Delta^{(0,0,1)}, \tag{8}$$

where the shifted Jacobi coefficient vector $\Delta^{(0,0,1)}$ and the shifted Jacobi vector $P^{(\alpha,\beta)}(x)$ are given by

$$\Delta^{(0,0,1)} = \left[\delta_1^{(0,0,1)}, \delta_2^{(0,0,1)}, \ldots, \delta_m^{(0,0,1)}\right]^T,$$
$$P^{(\alpha,\beta)}(x) = \left[P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \ldots, P_m^{(\alpha,\beta)}(x)\right]^T. \tag{9}$$

OPERATIONAL MATRIX OF GENERAL SHIFTED JACOBI POLYNOMIALS (PRODUCT, DERIVATIVE AND MOMENT)

In this section, we present the operational matrices of general Jacobi polynomials (product, derivative and moment). The derivative and moment operational matrix with respect to classical Jacobi polynomials are obtained in Eslahchi et al. (2012) and we present them in the following theorem. To do this, first we introduce the concept of operational matrix.

Operational matrix

Definition 1. Suppose

$$\phi = [\phi_0, \phi_1, \ldots, \phi_n], \tag{10}$$

where $\phi_0, \phi_1, \ldots, \phi_n$ are the basis functions on the given interval $[a,b]$. The matrices $E_{n\times n}$ and $F_{n\times n}$ are named as the operational matrices of derivatives and integrals respectively if and only if

$$\frac{\mathrm{d}}{\mathrm{dt}}\phi(t) \cong E\phi(t),$$
$$\int_a^x \phi(t)dt \cong F\phi(t). \tag{11}$$

We further assume $g = [g_0, g_1, \ldots, g_n]$, named as the operational matrix of the product, if and only if

$$\phi(x)\phi^T(x) \cong G_g\phi(x). \tag{12}$$

In other words, to obtain the operational matrix of a product, it is sufficient to find $g_{i,j,k}$ in the following relation:

$$\phi_i(x)\phi_j(x) \cong \sum_{k=0}^{i+j} g_{i,j,k}\phi_k(x), \tag{13}$$

which is called the linearization formula (Eslahchi and Dehghan, 2011).

Operational matrices are used in several areas of numerical analysis and they hold particular importance in various subjects such as integral equations (Razzaghi and Ordokhani, 2001), differential and partial differential equations (Khellat and Yousefi, 2006), etc. Also many textbooks and papers have employed the operational matrices for spectral methods. Now we present the following theorem:

Theorem 1.

If we consider the Jacobi approximation

$$y(x) \cong \sum_{i=0}^{m} \delta_t^{(0,0,1)} P_i^{(\alpha,\beta)}(x) = \left(P^{(\alpha,\beta)}(x)\right)^T \Delta^{(0,0,1)}, \tag{14}$$

then Jacobi approximation of $x^i[y^{(j)}(x)]^k$ is in the following form

$$x^i\left(y^{(j)}(x)\right)^k \cong \sum_{t=0}^{n} \delta_t^{(i,j,k)} P_t^{(\alpha,\beta)}(x) = \left(\Delta^{(i,j,k)}\right)^T P^{(\alpha,\beta)}(x), \tag{15}$$

where

$$\Delta^{(i,j,k)} = \left(eG^iD^j\Delta^{(0,0,1)}\right)P^{k-1}L\left(\sum_{r=0}^{n} eG^rD^j\Delta^{(0,0,1)}\right)^{k-2}, (k \geq 2),$$

$$e = [1,1,\ldots,1], P = \sum_{r=0}^{n}\gamma_r, \gamma_r = \sum_{s=r}^{n} E_s^{(\alpha,\beta,r)}, L = \sum_{s=0}^{n} G^sD^j\Delta^{(0,0,1)},$$

$$E_s^{(\alpha,\beta,r)} = \sum_{k=0}^{s}(-1)^{s-k}B_k^{(\alpha,\beta,r)}\binom{s}{k}; \alpha,\beta > -1,\ 0 \leq s \leq r,$$

$$B_k^{(\alpha,\beta,r)} = 2^{-k}\binom{s+\alpha+\beta+k}{k}\binom{s+\alpha}{s-k}; k = 1,2,\ldots,n, \tag{16}$$

and

$$D = \begin{pmatrix} 0 & E^{-1} \\ 0 & 0 \end{pmatrix}_{(n+1)\times(n+1)}, E_{p,q} = \begin{cases} \dfrac{-2(p+\alpha+1)(p+\beta+1)}{(\alpha+\beta+p+1)(\alpha+\beta+2p+2)(\alpha+\beta+2p+3)}, & p = q-2, \\ \dfrac{2(\alpha+\beta+p)}{(\alpha+\beta+2p-1)(\alpha+\beta+2p)}, & p = q, \\ \dfrac{2(\alpha-\beta)}{(\alpha+\beta+2p)(\alpha+\beta+2p+2)}, & p = q-1, \\ 0, & otherwise, \end{cases} \tag{17}$$

$$G_{p,q} = \begin{cases} \dfrac{2(p+\alpha)(p+\beta)}{(\alpha+\beta+2p)(\alpha+\beta+2p+1)}, & p = q-1, \\ \dfrac{\beta^2-\alpha^2}{(\alpha+\beta+2p+1)(\alpha+\beta+2p+2)}, & p = q, \\ \dfrac{2p(\alpha+\beta+p)}{(\alpha+\beta+2p)(\alpha+\beta+2p-1)}, & p = q+1, \\ 0, & otherwise. \end{cases} \tag{18}$$

Eslahchi et al. (2012).

The nonlinear programming

Like linear programming problems, another optimization problem which can be solved in a finite number of steps is a nonlinear programming problem. These optimization problems are in the following forms:

$$\text{M}inimize \;\; q(x) = \frac{1}{2} X^T G X + g^T X, \tag{19}$$

subject to

$$\begin{aligned} a_i^T X &= b_i, \\ a_i^T X &\le b_i, \end{aligned} \tag{20}$$

wherein the objective function $q(x)$ is nonlinear and the constraint functions are linear. Thus the problem is to find a solution where it is always possible to arrange that the matrix G is symmetric (Mikosch et al., 2006). As in linear programming, the problem may be infeasible or the solution may be unbounded, however these possibilities are readily detected in the algorithms, so the most part of it is assumed that a solution x exist (Fletcher, 2000). If the Hessian matrix G is positive-semidefinite, x is a global solution, and if G is positive definite, x is also unique. When the Hessian G is indefinite, then the local solutions which are not global can occur, and a computation of any such local solution is of interest (Mikosch et al., 2006; Fletcher, 2000). Classical methods for solving these problems are the Lagrangian methods and the Active set methods (Mikosch et al., 2006; Fletcher, 2000).

THE METHOD OF NUMERICAL SOLUTION

In this section, we describe our new approach for solving the class of nonlinear ordinary differential Equations (1) with respect to the mixed conditions Equations (2). Our approach is based on approximating the exact solution of Equation (1) by truncated Jacobi expansion as:

$$y(x) \cong \sum_{i=0}^{m} \gamma_i^{(0,0,1)} P_i^{(\alpha,\beta)}(x) = \left(P^{(\alpha,\beta)}(x)\right)^T \Lambda^{(0,0,1)}, \tag{21}$$

where

$$\Lambda^{(0,0,1)} = \left[\gamma_0^{(0,0,1)}, \gamma_1^{(0,0,1)}, \ldots, \gamma_m^{(0,0,1)}\right]^T,$$

$$P^{(\alpha,\beta)}(x) = \left[P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \ldots, P_m^{(\alpha,\beta)}(x)\right]^T \tag{22}$$

Also we assume that the coefficients $P_j(x), Q_j(x), R_j(x)$ has Taylor series expansion in the following form:

$$\begin{aligned} P_j(x) &= \sum_{i=0}^{m_j} e_{j,i} x^i, \\ Q_j(x) &= \sum_{i=0}^{n_j} r_{j,i} x^i, \\ R_j(x) &= \sum_{i=0}^{p_j} f_{j,i} x^i. \end{aligned} \tag{23}$$

Now by substituting Equation (23) into Equation (1), we obtain

$$\sum_{k=1}^{s}\sum_{i=0}^{m_j} e_{k,i} x^i (y''(x))^k + \sum_{k=1}^{l}\sum_{i=0}^{n_j} e_{k,i} x^i (y'(x))^k + \sum_{k=1}^{m}\sum_{i=0}^{p_j} e_{k,i} x^i (y(x))^k \cong f(x). \tag{24}$$

So from Equation (24), we must simplify $x^i \left(y^{(j)}(x)\right)^k$ as the following

$$(x^i y^{(j)}(x))^k \cong \sum_{i=0}^{m} \gamma_i^{(i,j,k)} P_i^{(\alpha,\beta)}(x) = \left(P^{(\alpha,\beta)}(x)\right)^T \Lambda^{(i,j,k)}$$

Also we approximate the right hand side of Equation (24) as

$$f(x) = \sum_{i=0}^{m} b_i P_i^{(\alpha,\beta)}(x) = \left(P^{(\alpha,\beta)}(x)\right)^T B, \tag{26}$$

where

$$B = \left[b_0, b_1, \ldots, b_m\right]^T, \tag{27}$$

and

$$P^{(\alpha,\beta)}(x) = \left[P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \ldots, P_m^{(\alpha,\beta)}(x)\right]. \tag{28}$$

Using Equations (25) and (26) into Equation (24), we obtain

$$\begin{aligned} &\left(P^{(\alpha,\beta)}(x)\right)^T \left(\sum_{k=1}^{s}\sum_{i=0}^{m_j} e_{i,k} \Lambda^{(i,2,k)} + \sum_{k=1}^{l}\sum_{i=0}^{n_j} e_{i,k} \Lambda^{(i,1,k)} + \sum_{k=1}^{m}\sum_{i=0}^{p_j} e_{i,k} \Lambda^{(i,0,k)} \right) = \\ &\left(P^{(\alpha,\beta)}(x)\right)^T F \cong \left(P^{(\alpha,\beta)}(x)\right)^T B. \end{aligned} \tag{29}$$

From linear independency of Jacobi polynomials, we conclude that:

$$\text{Res(x)}=\left(P^{(\alpha,\beta)}(x)\right)^T\left(B-F\right)\cong 0, \tag{30}$$

where

$$F=[f_0, f_1,..., f_m]. \tag{31}$$

Therefore from Equation (30), we have a system of $m+1$ algebraic equations for the $m+1$ unknown coefficients $\gamma_i^{(0,0,1)}$. Finally, we must obtain the corresponding matrix form for the boundary conditions. For this purpose from Equation (2), the values $y^{(j)}(a)$ can be written as:

$$y^{(j)}(a)=\left(P^{(\alpha,\beta)}(x)\right)^T\Lambda^{(0,0,1)}, a\in[-1,1], \tag{32}$$

Substituting Equation (32) in the boundary conditions (2) and then simplifying it, we obtain the following matrix form

$$\sum_{i=0}^{j} b_{i,l}y^{(l)}(a_i)=\left(P^{(\alpha,\beta)}(x)\right)^T\left\{\sum_{i=0}^{j} b_{i,l}D^i\Lambda^{(0,0,1)}\right\}=\sigma_l, a\in[-1,1]. \tag{33}$$

Now from Equations (30) and (33), we have $m+j+1$ algebraic equations and $m+1$ unknown coefficients.

In ordinary spectral method such as Tau and Collocation methods for obtaining the unknown coefficients, we must eliminate j arbitrary equations from these $m+j+1$ equations. But because of the necessity of holding the boundary conditions, we eliminate the last j equations from Equation (30). Finally, replacing the last j equations of (30) by the j equations of (33), we obtain a system of $m+1$ equations and $m+1$ unknowns $\gamma_i^{(0,0,1)}$. The elimination process in the classical spectral methods (Tau and Collocation methods) decrease the accuracy of the spectral method but in this paper we present a new approach as the following:

$$\text{Minimization}\left\|\text{Res(x)}\right\|_2$$

$$\sum_{i=0}^{1} a_{i,j}y_j^{(i)}(a_i)=b_{i,j},\ j=1,2,\ a_{i,j},b_{i,j}\in R, \tag{34}$$

for minimization of an obtained residual term Equation (34) from nonlinear ordinary differential Equation (1).

Remark 1

As we can see from Equation (34), in our new approach we have not eliminated the process, so we can obtain better results than the Tau and collocation methods.

It is easy to check that Equation (30) is a nonlinear programming problem, because

$$\left\|\text{Re}\,s(x)\right\|_2^2=\left\|\sum_{i=0}^{n} b_iP_i^{(\alpha,\beta)}(x)\right\|_2^2=\int_{-1}^{1}\left(\sum_{i=0}^{n} b_iP_i^{(\alpha,\beta)}(x)\right)^2 w^{(\alpha,\beta)}(x)dx=$$
$$\sum_{i=0}^{n} b_i^2\left(P_i^{(\alpha,\beta)}(x)\right)^2 w^{(\alpha,\beta)}(x)dx+2\sum_{i<j} b_ib_j\int_{-1}^{1}P_i^{(\alpha,\beta)}(x)P_j^{(\alpha,\beta)}(x)w^{(\alpha,\beta)}(x)dx=$$
$$\sum_{i=0}^{n} b_i^2h_i^{(\alpha,\beta)}. \tag{35}$$

In Equation (35), b_i are the functions of unknown coefficients by order of nonlinearity s (mentioned in Equation (1). It should be noted that only the linear differential equation (when $s=1$) by using Jacobi polynomials (or each other orthogonal ones) by defining the Norm-2 of residual, could be converted to quadratic programming problem.

Remark 2

The nonlinear programming problem (optimization problem) Equation (34) has unique solution because its Hessian matrix is a positive definite.

THE TEST EXPERIMENTS

In this section, several numerical experiments are given to illustrate the properties of the method and all of them were performed on the computer using programmed written and optimization toolbox in Matlab 2012.

Experiment 1

Consider the following nonlinear boundary value problem (Eslahchi et al., 2012).

$$\begin{aligned}&4y''-2(y')^2+y=0,\\&y(0)=-1,\ y'(0)=-1.\end{aligned} \tag{36}$$

The exact solution is $y(x)=\frac{x^2}{8}-1$. We solved it with the above mentioned method for different values of α and β, and we show the results. To apply the method, we assume that the solution is in the following form:

$$y(x) \cong \sum_{i=0}^{10} \gamma_i^{(0,0,1)} P_i^{(\alpha,\beta)}(x) = \left(P^{(\alpha,\beta)}(x)\right)^T \Lambda^{(0,0,1)}, \tag{37}$$

where

$$\Lambda^{(0,0,1)} = \left[\gamma_0^{(0,0,1)}, \gamma_1^{(0,0,1)}, \ldots, \gamma_{10}^{(0,0,1)}\right]^T,$$

$$P^{(\alpha,\beta)}(x) = \left[P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \ldots, P_{10}^{(\alpha,\beta)}(x)\right] \tag{38}$$

Also we reduce the nonlinear ordinary differential Equation (36) to a system of algebraic equations as the following:

$$4\mathrm{D}^2\Lambda^{(0,0,1)} - 2\mathrm{D}\Lambda^{(0,1,2)} + \mathrm{I}_{11\times 11}\Lambda^{(0,0,1)} = 0, \tag{39}$$

and also the boundary conditions as

$$\sum_{i=0}^{10} \gamma_i^{(0,0,1)} P_i^{(\alpha,\beta)}(0) = \left(P^{(\alpha,\beta)}(0)\right)^T \gamma^{(0,0,1)} = -1, \tag{40}$$

and

$$\sum_{i=0}^{10} \gamma_i^{(0,0,1)} \left(P_i^{(\alpha,\beta)}\right)(0) = \left(P^{(\alpha,\beta)}(0)\right)^T D\gamma^{(0,0,1)} = -1. \tag{41}$$

By implementation of our method which is presented in section 5, we can obtain the exact solution.

Experiment 2

Consider the first-order nonlinear differential equation (Eslahchi et al., 2012).

$$y'(x) - 2y(x) + y^2(x) = 1,\ 0 \le x \le 1, \tag{42}$$

with boundary condition $y(0) = 0.$ The exact solution is

$$y(x) = 1 + \sqrt{2}\tanh\left[\sqrt{2}x + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]. \tag{43}$$

Now we approximate the exact solution of Equation (42) by:

$$y(x) \cong \sum_{i=0}^{10} \gamma_i^{(0,0,1)} P_i^{(\alpha,\beta)}(x) = \left(P^{(\alpha,\beta)}(x)\right)^T \Lambda^{(0,0,1)}, \tag{44}$$

where

$$\Lambda^{(0,0,1)} = \left[\gamma_0^{(0,0,1)}, \gamma_1^{(0,0,1)}, \ldots, \gamma_{10}^{(0,0,1)}\right]^T,$$

$$P^{(\alpha,\beta)}(x) = \left[P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \ldots, P_{10}^{(\alpha,\beta)}(x)\right] \tag{45}$$

Also we expand the right hand side of Equation (42) as

$$1 \cong \sum_{i=0}^{10} b_i P_i^{(\alpha,\beta)}(x) = \left(P^{(\alpha,\beta)}(x)\right)^T B, \tag{46}$$

where

$$b_i = \frac{\int_{-1}^{1} e^x P_i^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)dx}{\eta_n^{\alpha,\beta}}, \tag{47}$$

and

$$B = [b_0, 0, .., 0]. \tag{48}$$

First we reduce the Equation (42) into the following matrix form

$$\mathrm{D}\Lambda^{(0,0,1)} - 2\Lambda^{(0,0,1)} + \mathrm{I}_{11\times 11}\Lambda^{(0,0,2)} = \mathrm{B}, \tag{49}$$

and also its boundary condition as

$$\sum_{i=0}^{10} a_i P_i^{(\alpha,\beta)}(0) = \left(P^{(\alpha,\beta)}(1/2)\right)^T \gamma^{(0,0,1)} = 0, \tag{50}$$

By implementation of our method which is presented in this article we can obtain the numerical results. The obtained results and superiority of our method is presented in Table 1.

CONCLUSION

In this paper, we have presented a modified and new version of Spectral method. Our method reduces the nonlinear ordinary differential equations to the nonlinear programming problems. Also in our new method (opposite to the classical spectral methods which have the elimination process in obtaining unknowns), we do not have any elimination process. Therefore we can

Table 1. The comparison between the errors of our method and Tau and collocation methods when $\alpha = 1.5, \beta = 1.25, N = 10,$ of Experiment 1.

X	Tau method	Collocation method	Our method
-1.0	0.000000879	0.000000841	0.000000617
-0.8	0.000000789	0.000000720	0.000000540
-0.6	0.000000798	0.000000799	0.000000513
-0.4	0.000000678	0.000000689	0.000000504
-0.2	0.000000688	0.000000593	0.000000420
0.0	0.0000000000	0.000000000	0.000000000
0.2	0.000000631	0.000000610	0.000000530
0.4	0.000000680	0.000000690	0.000000543
0.6	0.000000356	0.000000410	0.000000355
0.8	0.000000340	0.000000349	0.000000214
1.0	0.000000523	0.000000541	0.000000314

obtain better results than Tau and collocation methods. In addition, for showing the accuracy and efficiency of our approach, we have presented two experiments. In Experiment 2, we obtained better results than the Tau and collocation methods.

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