

Full Length Research Paper

Mathematical profiles and problem solving abilities of mathematically promising students

Ibrahim Budak

Department of Elementary Education, School of Education, Erzincan University, Turkey. E-mail: ibudak@gmail.com.
Tel: +90(446)224 0089. Fax: +90(446) 223 1901.

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Mathematically promising students are defined as those who have the potential to become the leaders and problem solvers of the future. The purpose of this research is to reveal what problem solving abilities mathematically promising students show in solving non-routine problems and type of profiles they present in the classroom and during problem solving. The students participating in this study were chosen on the basis of mathematics contest scores. Accordingly, four students from an urban private school, taking the first, ninth, tenth, and fourteenth place on the contest formed the sample of the study. The data were collected through non-routine problem solving activities, administered to the participating students after school hours at their school. After examining the students' responses to the questions, students were interviewed on those same questions. The results suggest that mathematically promising students were very determined, spent a long time in thinking, reflecting and planning. They attempted to solve the challenging problems multiple times, and looked for alternate ways if the one they tried did not work. They found an authentic solution when they do not know or remember the general algorithm.

Key words: mathematically gifted students, mathematical problem solving, students' mathematical profiles, elementary students.

INTRODUCTION

No child is identical is a known fact. When gifted and non-gifted children are taken into account, the difference increases a lot more. A gifted child's motive to create their own world is much stronger than any other child. What makes this motive so strong are the gifted students' curiosity and their other features. These features not only separate this group of students from their peers, but also help the families, educators, and researchers to identify and thus, to take more action towards their educational careers. Although the gifted children differ from other children in many ways, they show many similar features with other gifted children. Some of those features are listed as high-level language development and verbal ability, advanced comprehension, extraordinary quantity of information and unusual retentiveness, unusual intensity, unusually varied interests and curiosity, ability to generate original ideas and solutions, personel sensitivity, and idealism and sense of justice (Song and Porath, 2005). Each one of these abilities and characteristics emerges in a different way in the fields

they belong to. For example, gifted students in mathematics make use of the "sensitivity to problems" characteristic differently than the gifted students in the field of social sciences. While mathematically gifted students use this feature to focus on a mathematics problem, gifted students in social sciences, use it to focus on social problems (Koshy et al., 2004). It is not intended to say that the set of characters listed earlier cannot be seen at non-gifted children. The difference between the two is that the density, diversity, and the speed of development and usage of those characters (Al-Hroub, 2010; Sriraman, 2009).

Various definitions were used for mathematically "gifted" or "talented" in the literature (Krutetskii, 1976; Miller, 1990).

However, the 'Task Force on Promising Students' chaired by Linda Sheffield uses a much broader definition of giftedness, which they call, "mathematically promising". Their definition of giftedness includes the students who have been traditionally identified as gifted, talented, genius,

prodigy, precocious, etc., and the students who have been traditionally excluded from previous definitions of gifted and talent. They define mathematically promising students as those who have the potential to become the leaders and problem solvers of the future (Scheffield, 1999). In this paper, I prefer to use the definition “mathematically promising” for the students participated in the study, rather than “mathematically gifted”.

Mathematically gifted children are distinguished from the average children through their mathematical reasoning (Krutetskii, 1976). Indeed, mathematical giftedness is defined as the ability to solve difficult mathematical problems (Niederer et al., 2003). These kids solve mathematical problems in a different way than the average child. They tend to take more time to orient to a problem, to utilize a wider range of problem solving strategies, and to evaluate their progress both during and after completing a problem.

Researchers working on the identification or education of mathematically gifted students; list several personal characteristics and problem solving abilities distinguishing these children from their “regular” peers (Davis and Rimm, 1994; Heinze, 2005; Krutetskii, 1976; Renzulli et al., 2009). In comparison to “normal” elementary students, the mathematically gifted elementary students are able to perform the following tasks: switch from one strategy to another with ease, display flexibility in mathematical thinking, use a variety of representations to solve problems, need significantly less time to deal with unsolvable puzzles and sums, recognize patterns and formal structures, transfer recognised mathematical structures, and reverse operations and processes (Krutetskii, 1976; Renzulli et al., 2009). Threlfall and Hargreaves (2008) compared problem-solving abilities of gifted and non-gifted students. Although they did find similar abilities or characteristics found in earlier studies, they also found that the gifted students have a broader and more inter-connected knowledge base, are quicker at solving problems, while spending more time planning, prefer complex and challenging problems, and are more sophisticated in their meta-cognition, including self regulation.

In addition to these behaviors and abilities, Heinze (2005) listed three more abilities or characteristics that elementary gifted students display in addition to the ones listed above: the ability for logical thought and logical analysis, the high ability to verbalise and to explain their solutions, and the ability to use the insight in the mathematical structure of a problem in order to solve it by deducing or calculating the solution. Knowing the characteristics and problem solving abilities of mathematically gifted students is very vital, especially for mathematics educators and researchers, to identify and know what to expect from them. After the identification process, what is more important is to know the type of tasks and how these tasks should be presented to them.

The educators of gifted students need to be strong advocates for practices that best serve these academically

capable students. The best guidance on what is important often comes from the students themselves. The focus was on how mathematically gifted students understand why certain mathematical approaches or algorithms are used, as opposed to merely how these approaches are greatly appreciated and recommended by these students (Niederer et al., 2003). Selecting appropriate tasks is an issue for the teachers of this group of students. Krutetskii (1976) proposed a model that will provide direction for teachers in selecting problems. The model is comprised of four levels of mathematical tasks. Level 1 is mathematical exercises, Level 2 is word or a story problem, Level 3 is mathematical problems, and Level 4 is authentic mathematical problem-solving tasks. These Level 4 type of problems was also discussed in Threlfall and Hargreaves (2008) as the type of problems that the gifted students prefer.

Presentation of non-routine problems in elementary mathematics curriculum of Turkish schools is also something that is not routine. Therefore, the students are not well used to solving math problems that take longer time than they used to. The purpose of this research is to reveal what problem solving abilities mathematically promising students show in solving three non-routine mathematics problems and what type of profiles they present in the classroom and during problem-solving interviews.

METHODOLOGY

Sample

This study was conducted in one of the cities in Eastern Anatolia Region of Turkey. The city's population is fairly small, around 200,000 in 2008 census (TUIK, Turkish Statistics Institute). The city's population is getting smaller with a migration rate of 12 per thousand. The city's economy mostly depends on the agriculture and animal husbandry.

The students participating in this study were chosen on the basis of mathematics contest scores. The mathematics contest was organized by the Provincial Directorate of National Education (PDNE) in the city. The contest was announced to all public and private schools in the urban and suburban of the city by the PDNE. The students participating in this study were chosen both on the basis of their success on the mathematics contest and their easy access. Accordingly, four students (three boys and one girl, all in sixth grade) from an urban private school taking the first, ninth, tenth, and fourteenth place on the contest formed the sample of the study. These students were successful among the hundreds of students who were nominated either by their school administration or by their teachers to participate in the contest.

Instruments

The data were collected through problem solving activities and interviews. The elimination and mathematics contest tests were collaboratively administered by the PDNA and the private elementary school.

- Mathematics contest test: The mathematics contest test includes 40 questions from various topics- number sense, probability, number patterns, geometry, etc. For example, “how

many simple fractions can be written by using two counting numbers whose sum is 100?" and "There are three blue and six red balls in a bag. What is the probability of randomly picked ball to be red or blue?" For every three wrong answers, one right answer was eliminated. Two hours were given for students to complete the test.

- Mathematical problem solving activities: Problem solving skills can be best measured through problem solving activities (Cai and Lester, 2005). For this reason, the problem solving activities were chosen from the related literature (Krutetskii, 1976; Masingila et al., 2002; Span and Overtoom-Corsmit, 1986) to reveal students' problem solving abilities. Taking the National Mathematics curriculum for sixth grade and mathematics teacher of the participants into account, three questions were selected for the study. These questions measure students' probabilistic, spatial, abstract thinking abilities, reasoning, and generalization abilities.

In assessing problem solving tasks, students' solution methods give more information about their mathematical abilities rather than the number of their right and wrong answers in solving problems. The inadequacy of problem solving abilities even if a student gets higher scores on mathematics achievements tests, this is a sufficient reason to accept the student as nongifted. In assessing students problem solving abilities of mathematically promising students, Niederer and Irwin (2003) report that the researcher's decision would be enough.

Data collection

- Administration of mathematical problem solving activities: Problem solving activities were administered to the participating students after school hours at their school. Students completed the activities in an hour.

- Interviewing students and the mathematics teacher: After researchers' examining the students' responses to the questions, students were interviewed on those same questions. Also, this interview occurred after school hours in a silent classroom environment. During this interview, students were asked to clarify inexplicit parts of their solutions, to generalize the *hand-shake problem*, and to generate alternate solutions to the problems. The purpose of the interview was to identify the problem solving approaches, strategies and skills of the students who were more promising in mathematics than their peers.

The mathematics teacher of the participants was interviewed to gather more information on the personal characteristics and problem solving skills of the students during mathematics activities in the classroom.

RESULTS

Students' mathematical profiles

Ali is the student who was on the first place on the mathematics contest administered to all nominated sixth grade elementary school students in the urban and suburbs of the Anatolian city. The mathematics teacher noted that *Ali* highly involves in class discussions, does not like rutin problems and occasionally gives original solutions to non-routine problems. *Ali* seems very quiet in nature. He attempted to solve each problem quietly and did not make much interpretation.

Muhsin is the student on the ninth place on the contest. His mathematics teacher noted that he is successful both in multiple choices and in open-ended assessments. He looks for different strategies and ways in solving problem.

He is also academically one of the top students among his classmates. He gave explanations to his problem solutions during the interview.

Alp takes the tenth place on the mathematics contest. He was identified by his mathematics teacher as quiet, shy, and avoiding exposition of his ideas compared with other children in the classroom. He preferred to talk rarely during the interview.

Nisa was on the fourteenth place on the mathematics contest. She was identified as very determined, diligent, and hardworking student by her mathematics teacher. During the interview, she was observed as the student persisted in finding a solution, interested in brain-teaser problems, and evaluating problems from a broader perspective. *Nisa* made the interview easier by "thinking aloud" in solving the problems even though she was not asked to. This "think aloud" technique was used in identifying expert problem solvers (Lester, 1980). *Nisa* used this technique unconsciously, and shed a light to what was passing through her mind. As *Nisa*, herself and her teacher stated, she really is obstinate with the question, and does not feel comfortable until she reaches a solution. She made this very clear during the interview. *Nisa* attempted to solve the handshake problem for at least 13 times, and the goat problem at least 5 times (excluding the ones that the researcher missed while taking notes).

However, *Nisa* did not show the stubbornness that she showed for her first attempt in her second attempt. It does not seem to concern her whether or not a question has more than one answer. For example, since she found an answer to the vase problem on pre-interview trial, during the interview after giving some thought and a few suggestions, she chose to switch to other questions. After the interview, when asked what she thought about the interview questions, *Nisa*: "I am solving for the first time in my life...it requires logic...it does not happen by working but thought. Need to get used to it, many questions need to be solved."

Problem solving activities

Handshake problem

A total of three married couples meet at a party in which one of the couples, Ahmet-Emine Sen, is the host. Each person shakes hands with the person they see the first time. Excluding Ahmet-Emine Sen couple, each person shakes hands with different number of people from zero to four. In this case, how many people would Sen couple shake hand?

Before the interview, when *Ali* attacked this problem, he explained in his writing that Ahmet and Emine Sen couple would shake hands with two people and supported his explanations with a drawing. But, he missed some of the information that each person should shake hands with different number of people. In his reponse, while he shows

one other person shakes hands with two people (including the Mr. and Mrs. Sen couple), he also did not consider that one person should shake hand with no one. Similarly, he made the same mistakes during the interview. After his fifth attempt, the following dialogue occurred between the researcher and Ali:

Researcher--- *Are you sure you have read the question right? At other times, how many times would you read a question?*

Ali--- *I solve while reading it.* [He read the question again, and drew little squares for each person and wrote numbers from zero to four and decided that one couple should meet four people each.] This trial did not take him to success, either. On his eighth attempt, *"Nothing comes to my mind"* he said.

Researcher--- *Would you like to give up?* [He wanted to think a little more]. After his eleventh attempt, he said, *"Teacher, I could not find it."*

It was investigated that Ali did not fully understand the *handshake problem* and his solution methods were not so different from the other students, in other words, his solution method was not so authentic. Ali's biggest problem was that he was not reading and using the information given in the problem carefully. The cause of this carelessness may be the hastiness Ali showed in attempting to solve a problem, as he stated he solves the questions before even finishes reading.

In his solution before the interview, unlike Ali and Nisa, Muhsin took into account that one person should shake hands with no one. However, just like Ali and Nisa, Muhsin made two couples shake hands with two other people. This result suggests that Muhsin did not carefully evaluate the givens in the problem. Although he did realize that only Ahmet-Emine Sen couple could shake hands with two people, this time he made a different error that he was supposed to make one person shake hands with four people.

The following dialogue occurred between Muhsin and the researcher:

Researcher--- *What happens if there were 4 couples?*

Muhsin--- *The hosts would shake hands with 3 people* [Kept thinking]

Muhsin--- *No, cannot happen* [Kept thinking again].

Researcher--- *Were you able to reach a solution?*

Muhsin--- *I did something but they happened to be equal. Could not solve it, but the rest of the people should shake hands with people from 0 to 6, that means we need to add 5 and 6 to the problem.*

Researcher--- *What could the hosts be?*

Muhsin--- *I think it is 3.*

Researcher--- *Can you make a generalization?*

Muhsin--- *When there are three couples at the party, the hosts would each shake two hands. For four couples, the hosts might shake 3. For 5 couples, the hosts might shake 4 or 5. When the number of couples increases the number of hand shakes that the hosts increases, too.*

As it is seen from the dialogue between the researcher and Muhsin, Muhsin moves around the different thinking processes. Although he is not quite sure about the number of handshakes that occur between the hosts and the guests, he reaches some form of generalization (except for 5 couples). He also understands that when a variable or a value is changed in the problem, that affects other variables or the values given in the problem.

Alp like other three students did not use the fact that only the hosts can shake hands with an equal number of people and the fact that one person needs to shake hands with no one. Although he did find the right answer for the hosts, as two handshakes each, his solution to get the right answer was wrong. During the interview, he was asked if he could generalize for any number of couples. His response to this question showed that he was not quite sure about his thinking.

Nisa showed great efforts to solve this problem. At her initial attempt, she thought that there were three other people in addition to the hosts. Not evaluating the data presented in the problem, like other students in the study, caused Nisa to have an incorrect solution. Many times, she said *"Yes, found it"* and smiles appeared on her face, soon she discovered that she was wrong and leaned back to her chair and kept attacking to the problem. Her behavior was very similar to the behaviors of students in Krutetskii (1969) study.

The only child who tried to generalize this problem was Muhsin. However, all stated that they kept thinking about the problem when they went home. For instance, by stating, *"I get very obstinate with the question and don't feel relieved until I reach a solution"*, Nisa had shown, insistently, a feature of the master problem solvers in solving the problem.

In short, the handshake problem was more challenging than the other two for the promising students. However, this problem had many variables to consider such as only one couple should shake hands with the same number of people; the rest of the people should shake hands with different number of people, and the fact that the handshaking is mutual. All four students in the study either forgot or did not take into account that only the hosts need to shake hands with the equal number of people, or the fact that one person should not shake hands with any body.

The goat problem

A goat was tethered to one corner of a square barn with a four-meter rope. If the goat can reach only to the two corners of the barn, how much of an area can it graze?

When looking at Ali's solution to the *goat problem* in his paper, it was obvious that Ali was very good with geometry and measurement content areas; his drawing was very detailed; and he use the mathematical terms and symbols properly.

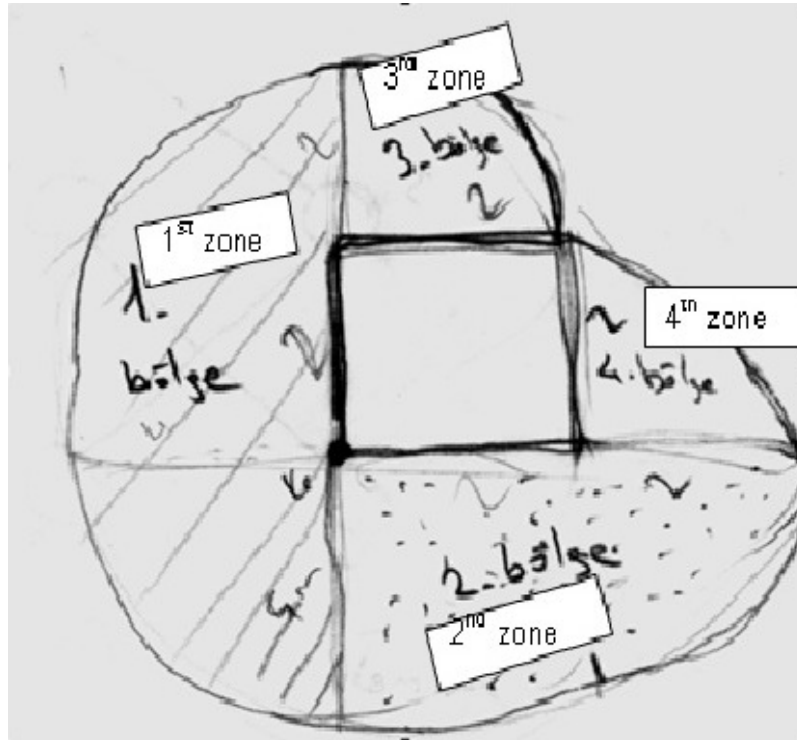


Figure 1. Muhsin's drawing for an extension of the goat problem.

Before the interview, in his solution to this problem, Muhsin drew a full circle with a radius of four without taking into account that there is a square barn forbidding to have a full circle, thus forbidding the goat to graze that area.

During the interview, Muhsin was asked an extension problem to the original problem. The extension problem was: What if the length of each side of the square is 2 and the rope is still 4 m, how much of an area would the goat graze? This time, he was asked to show his solution only with drawings. He spent very little time and said "we would calculate the area of one fourth circles and then add them all. We would find the area like this". Ali's drawing for the solution of this problem was very similar to Muhsin's drawing in Figure 1. Since he did not know the area formula of a circle, he could not reach a numerical value. However, one can definitely say that he has a good grasp of geometry and measurement knowledge.

Before the interview, Nisa could not find a numerical solution to the problem, but showed the grazed area by shading. Since she did not provide any numerical solution in her paper, during the interview, she was asked more questions to elaborate her thinking.

Researcher--- *Can you shade the area?* [Nisa draws a rectangular shape]

Nisa--- *The goat can graze so much space, in short, the goat can graze a square area with one side 4m....* [keeps

thinking]... but, then that becomes 8 m...it grazes the upper part, too. ...yes, yes, 8m seems reasonable. ... In fact, it forms a circle. That is more reasonable...if this is the center, and 4m is the radius [she puts a center point to the corner of the barn however she could not place the center point in the middle of the circle. [She draws several radius lines, but most have a different length]

Researcher--- *Do these radius lines have same length?*

[She starts drawing better lines] *Is your drawing circle?*

Nisa--- *But, it cannot eat the whole circle....In fact, it could.*

Researcher--- *Can you shade the grazed part?*

Nisa--- *Cannot shade....indeed, it can eat them all...It ate them all.*

Researcher--- *Ok, how can we find the area? Do not calculate just tell me how.*

Nisa--- *First, I would place the barn. The radius is 4 m...First, I would find the area of the circle. The area is πr^2 , and I would deduct the area of the square.*

In this dialogue, the three points (...) show the chain of thought from one after another. Nisa's response to the researcher's questions were sometimes related to the topic, sometimes were related to her own thinking process. She would just speak out what is passing through her mind. I see that Nisa does know the area formula of the circle. But, most importantly, she actually explains the solution right. However, she does not discuss why the goat cannot graze the full circle anymore knowing

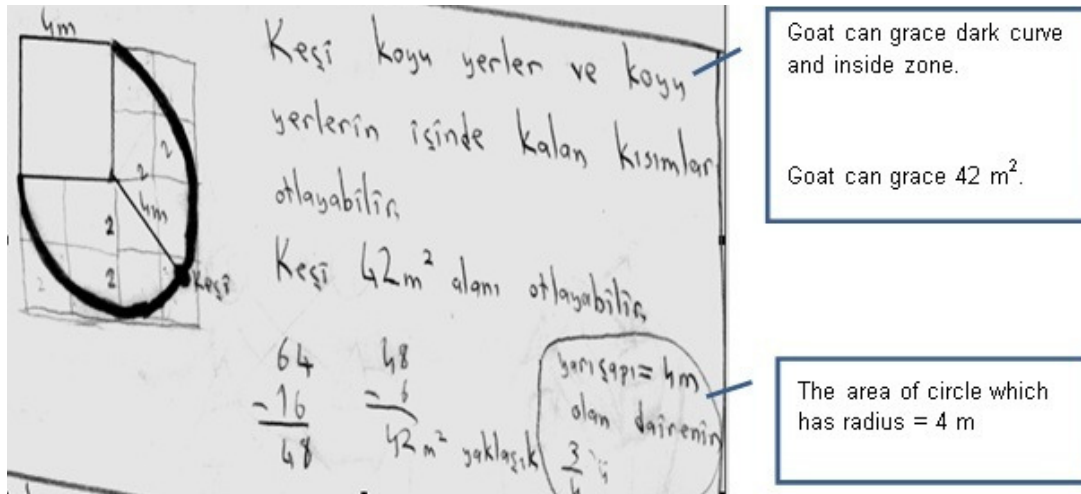


Figure 2. Alp's response to the goat problem.

the fact that the area of the square was subtracted from the area of a circle.

An alternative solution to this problem came from Alp (Figure 2). Although his solution does not provide us the exact numerical solution to the problem, his drawings, shows that he understands the concept of area, but does not know or remember the formal algorithm.

First, Alp finds the area of the larger square and then takes out the area of the squared barn. Then, he estimates the area between the outside of the $\frac{3}{4}$ circle and the square, as 6. Finally, he finds an estimated area of 42 m^2 . His strategy is especially useful for finding an area of an irregular polygons and his strategy definitely shows that he is very flexible in his thinking and have a good grasp of area.

The vase problem

How can you measure 6 liter water using 9 litered and 4 litered vases? (Assume that you are close to a sink; and can empty and refill the vases.)

This problem can either be solved by pouring water from 9 litered vase to 6 litered vase or from 6 litered to 9 litered vase. All three boys solved this problem by pouring from larger vase to the smaller one. While Ali did not attempt to solve it the other way around, Muhsin and Alp proposed to solve it by pouring water from smaller vase to the larger. Nisa's solution to the problem was wrong and did not want to attempt to solve it during the interview. The students did not find the vase problem as complicated as the other two problems.

DISCUSSION

In this article, four mathematically promising students' profiles and problem solving abilities on three non-routine

problems were discussed. Profiles of the students showed that while two of the students reported to be shy or quiet by their maths teacher and observed during the interview, the other two students were reported and observed to be more talkative. The observation of students showed that there seem to be no relationship between being talkative or quiet and their success in reaching a solution or in their level of interest in problem solving. However, the talkative students make the interview easier and thus, have the researcher understand and reveal their thought processes.

The results show that mathematically promising students were very determined to solve each of the non-routine mathematical problems. All students spent a long time in thinking, reflecting and planning. They attempted to solve the challenging problems multiple times, and looked for alternate ways if the one they tried did not work. They tried to find an authentic solution when they do not know or remember the general procedure. These findings confirm the findings of Kruteskii (1976).

When a complicated problem with many factors was given, the students had greater difficulty to take each one of those factors into consideration. This result confirms the findings of Diezmann and Watters (2002) that some students lacked particular skills to solve more complex tasks independently. This result suggests that mathematically promising students need more non-routine problem solving experiences during their regular math classrooms. Diezmann and Watters emphasized that in classrooms where appropriately problematised tasks were used, students displayed greater persistence and demonstrated flexibility in thinking. The development of these attributes should be fundamental goals in the education of mathematically gifted students.

From this research, a suggestion on the education of the gifted can be drawn that in teaching problem solving, students should be given opportunities to solve challenging problems with many factors, requiring logical

reasoning, and have multiple ways of solution, and those types of problems should be an integral part of teachers' lesson plans. In addition, students should be taught the techniques to read the questions carefully and take all the elements of the givens into account when attempting to solve a problem. Future research is needed to investigate abilities of mathematically promising students on different types of (requiring different levels of higher order thinking) problems compared to their regular peers in the same classroom. Additional research is needed to compare the relationship between the students' personal characteristics and problem solving abilities with larger samples of students.

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