

Full Length Research Paper

A characterization of H- strictly convex hypersurfaces in de Sitter space S_1^6

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In this study, we introduce a H-strictly convex hypersurface in the 6-dimensional unitary de Sitter space S_1^6 and give a lower bound approximation for the Ricci curvature of such hypersurfaces under some appropriate conditions.

Key words: Ricci curvature, H-convex submanifold, hypersurface, de Sitter space.

INTRODUCTION

Let L^7 be the 7-dimensional Lorentz-Minkowski space endowed with the Lorentzian metric tensor g given by

$$g(V, W) = \sum_{i=1}^6 v_i w_i - v_7 w_7$$

and let $S_1^6 \subset L^7$ be the 6-dimensional unitary de Sitter space S_1^6 , that is

$$S_1^6 = \{X \in L^7 : g(X, X) = 1\}$$

As is well known, the de Sitter space S_1^6 is the standard simply connected Lorentzian space form of positive constant sectional curvature. A smooth immersion, $M \rightarrow S_1^6 \subset L^7$ of an 5-dimensional connected manifold, M , is said to be spacelike hypersurface if the induced metric is a Riemannian metric on M which is denoted by g .

In the last years, the study of spacelike hypersurfaces in de Sitter space has been of substantial interest from both physical and mathematical points of view. In this

work, we obtain a result for a H-strictly convex space-like hypersurface in de Sitter space to be spherical in terms of a pinching condition for the Ricci curvature.

Let $T_x M^\perp$ be the normal space to M at x . We denote by ∇ (resp. ∇^\perp) the covariant differentiation on M (resp. S_1^6). Then, for tangent vector fields X, Y and the unit normal field ζ on M , as is well known, the formulas of Gauss and Weingarten are;

$$\nabla^\perp_x Y = \nabla_x Y + \sigma(X, Y), \quad (1)$$

$$\nabla^\perp_x \zeta = -A_\zeta(X) \quad (2)$$

Where; σ is the second fundamental form of M and satisfies $\sigma(X, Y) = \sigma(Y, X)$ and A is the symmetric linear transformation on each tangent space to M , which is called the shape operator. Since M is a hypersurface we may write

$$\sigma(X, Y) = h(X, Y)\zeta \quad (3)$$

Then it can be seen that

$$h(X, Y) = g(\sigma(X, Y), \zeta) = g(A_\zeta(X), Y). \quad (4)$$

The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_5$ of the shape operator A

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are called principal curvatures of M and an orthonormal basis $\{e_1, e_2, \dots, e_5\}$ such that:

$$A_{\zeta} e_i = \lambda_i e_i, \quad 1 \leq i \leq 5,$$

are called principal vectors on M . In this case, $\lambda_i = h(e_i, e_i)$, where $i = 1, 2, \dots, 5$.

Furthermore, the mean curvature vector of the hypersurface M is defined by $H = \frac{1}{n} \text{trace } \sigma$ and $K_5 = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5$ is called the Gaussian curvature of M .

The second fundamental form σ is said to be semi-definite at $x \in M$ if $\sigma(X, X) \geq 0$ or $\sigma(X, X) \leq 0$ for all non-zero vectors $X \in T_x M$, that is, h is either positive semidefinite or negative semidefinite. It is well known that if M is convex at $x \in M$, then the h is semidefinite at the point x . The second fundamental form σ is said to be definite at $x \in M$ if $\sigma(X, X) \neq 0$ for all non-zero vectors $X \in T_x M$, that is, h is either positive definite or negative definite. In this case the hypersurface M is said to be strictly convex at the point x . σ is said to be non-degenerate at x if h is non-degenerate at x . Taking the mean curvature vector H to M instead of the unit normal field ζ on M , an H -strictly convex submanifold, can be define in a Riemannian space form (Chen, 1999; Udriște, 1986).

Definition 1

A Riemannian submanifold is said to be H -strictly convex submanifold if the shape operator A_H is positive definite at each point of the submanifold.

Denote by R the Riemannian curvature tensor of M . Then the equation of Gauss is given by

$$R(X, Y; Z, W) = (g(X, W)g(Y, Z) - g(X, Z)g(Y, W)) + g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)) \quad (5)$$

For vector fields X, Y, Z, W tangent to M . For the hypersurface M , denote by $K(\pi)$ the sectional curvature of a 2-plane section $\pi \subset T_x M, x \in M$ and choose an orthonormal basis $\{e_1, e_2, \dots, e_5\}$ of $T_x M$ such that $e_1 = X$, then we may define the Ricci curvature of $T_x M$ at x by

$$Ric(X) = \sum_{j=2}^5 K_{1j}, \quad (6)$$

where; K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j . The scalar curvature τ of the hypersurface M is defined by

$$\tau(M) = \sum_{1 \leq i \leq j \leq 5} K_{ij} \quad (7)$$

RICCI CURVATURE OF A H-CONVEX HYPERSURFACE IN S_1^6

Now, following Chen (1999), a Riemannian invariant on the hypersurface M of the 6-dimensional de Sitter space S_1^6 of constant sectional curvature 1 is introduced, defined by

$$q_5(x) = \left(\frac{1}{4}\right) \inf_{x \in T_x M, \|X\|=1} Ric(X), \quad x \in M. \quad (8)$$

In this study, the following theorem for the hypersurface M of the 6-dimensional de Sitter space S_1^6 is proved.

Theorem 1

Let M be a hypersurface of the 6-dimensional de Sitter space S_1^6 , for any point $x \in S_1^6$ we have: i) If $q_5(x) \neq 1$, then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{4}{5}(q_5(x) - 1)I, \quad x \in M \quad (9)$$

ii) If $q_5(x) = 1$, then $A_H \geq 0$ at x .

Proof

Let $\{e_1, e_2, \dots, e_5\}$ be an orthonormal basis of $T_x M$. Considering Equation 6, 7 and 8, we have

$$\tau(x) \geq 10 \cdot \theta_5(x) \quad (10)$$

Then by following Chen (1999), the equation below is obtain

$$H^2(x) \geq \frac{1}{10} \tau(x) - 1 \quad (11)$$

Now, from Equation 10 and 11, $H^2(x) \geq q_5(x) - 1$ is obtain. This shows that only when $q_5(x) \neq 1, H(x) = 0$ and in this case i) and ii) is clearly satisfied, so it may be

assumed that $H(x) \neq 0$. Choose an orthonormal basis $\{e_1, e_2, \dots, e_6\}$ at x such that e_6 is in the direction of the mean curvature vector $H(x)$ and e_1, e_2, \dots, e_5 diagonalize the shape operator A_H . Then we have

$$A_H = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix} \quad (12)$$

From the equation of Gauss, the equation below is obtain $a_i a_j = K_{ij} - 1$ (13)

and from Equation 13, the Equation below is obtain $a_1(a_2 + \dots + a_5) = Ric(e_1) + 4$ (14)

If $e_1 = X$ in Equation 13, taking into account Equation 8, Equation 14 becomes

$$a_1(a_1 + a_2 + \dots + a_5)^3 - 4(q_5(x) - 1) + a_1^2 \quad (15)$$

In similar way, the following equalities for any j , $j = 1, 2, \dots, 5$, can be obtain

$$a_j(a_1 + a_2 + \dots + a_5)^3 - 4(q_5(x) - 1) + a_j^2 \quad (16)$$

Which yields

$$A_H \geq \frac{4}{5}(q_5(x) - 1)I \quad (17)$$

Here, the equality case only occurs when one of the vectors e_1, e_2, \dots, e_5 is in the null space, but for the hypersurfaces this is impossible, so the inequality (17) must be sharp, that is

$$A_H > \frac{4}{5}(q_5(x) - 1)I \quad (18)$$

From the Theorem 1, the following can be obtained.

Corollary 1

Let M be a hypersurface of the 6-dimensional de Sitter space S_1^6 of constant sectional curvature 1, if the Ricci curvature of M is positive, then M is a H -strictly convex hypersurface immersed in S_1^6 . All hypersurfaces in S_1^6 are bounded, that is, M is contained in a closed geodesic

ball of finite radius r . Without loss of generality, we take such a geodesic ball as the closed ball $B(a, r)$ with the center $a = (0, 0, \dots, 1)$. By simple trigonometry, it can be deduce that the distance in L^7 from the timelike direction a to a point of the geodesic sphere $S(a, r) = \partial B(a, r)$ is $t = 2 \sinh(r/2)$.

By the generalized extremum principle, the present author obtained some upper bound estimations for the Ricci curvatures of hypersurfaces in the sphere and the hyperbolic manifold (Erdoğan, 1996, 1998). However, Being motivated by (Erdoğan, 1996, 1998, 2009) Alias (2000) proved the following:

Theorem 2

Let M be a complete hypersurface in 6-dimensional de Sitter space S_1^6 whose sectional curvatures are bounded away from $-\infty$. If M is contained in the region $\Omega(a, \rho) = \{x \in S_1^6 : g(a, x) \leq -\sinh(\rho) < 0\}$ for the timelike direction $a \in L^7$ and a positive real number ρ and if r is assume to be less than $\rho/2$, then,

$$\liminf_{X \in T_p M, \|X\|=1, p \in M} Ric(X, X) \geq \frac{4}{\cosh^2(\rho)}. \quad (19)$$

Now, obtaining a sharp result for the best possible approximation of the Ricci curvature of a H -strictly convex hypersurface immersed in S_1^6 ; According to the Corollary 1, for such an hypersurface, the left side of the inequality (Equation 19) must be positive. On the other hand, to be a H -strictly convex hypersurface, M must satisfy the condition (Equation 18), that is,

$$A_H > \frac{4}{5}(q_5(x) - 1)I \quad \text{or}$$

$$A_H > \frac{1}{5} \inf ric - \frac{4}{5}.$$

Hence, M is definitely H -strictly convex if $\inf ric \geq 4$. Therefore, combining two conditions, the equation below is obtain

$$4 \leq \inf ric \leq \frac{4}{\cosh^2(\rho)} \quad (20)$$

Thus, considering that $\cosh^2(\rho) \geq 1$, from Equation 20,

the following conclusion can be drawn.

Theorem 3

Let M be a H -strictly convex hypersurface in 6-dimensional de Sitter space S_1^6 such that all sectional curvatures of M are bounded away from $-\infty$. If M is contained in the region $\Omega(a, \rho) = \{x \in S_1^6 : g(a, x) \leq -\sinh(\rho) < 0\}$ for the timelike direction $a \in L^7$ and a positive real number ρ , then for any point $x \in M$, the best possible approximation for the minimum Ricci curvature of M is 4 and M is a round 5-sphere of radius $\cosh(\rho)$.

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