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Full Length Research Paper

Certain subclasses of analytic functions

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The aim of present paper is to define certain subclasses of analytic functions in connection with the convolution operator. Moreover, some inclusion relationships, radii problems and a sharp coefficient bound have been successfully derived. These innovations are of extreme importance for a wide range of physical problems. Results are very encouraging.

Key words: Bounded boundary and bounded radius rotation, Salagean operator.

INTRODUCTION

This paper witnesses the exploration of some new classes of analytic functions in connection with the convolution operator. We consider A_n as the class of functions of the form

$$f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j, \ \left(n \in \mathrm{N} = \{1, 2, 3, ...\}\right), \tag{1}$$

which are analytic in the unit disc $E = \{z \in \mathrm{C} : |z| < 1\}$. The class A_n is closed under the convolution, denoted and defined by

$$(f * g)(z) = z + \sum_{j=n+1}^{\infty} a_j b_j z^j,$$

where $f(z)$ is given by Equation 1, and

$$g(z) = z + \sum_{j=n+1}^{\infty} b_j z^j, \ n \in \mathrm{N}. \tag{2}$$

Here, we list some classes of analytic functions (Noor, 2008). Let

$$p(z) = 1 + \sum_{j=n}^{\infty} a_j z^j, \ z \in E, \tag{3}$$

then $p(z) \in P(\gamma, n)$ if and only if $\operatorname{Re} p(z) > \gamma, \ 0 \le \gamma < 1, \ z \in E.$ It can be observed that $P(\gamma,1) = P(\gamma)$ is the class of functions with real part greater than γ and $P(0,1) = P$ is the well known class of functions with positive real part. Next, we have the class $P_k(\gamma, n)$ for $k \geq 2$,

$$P_k(\gamma,n) = \left\{ p(z) \in A_n \ : \ p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \right\}$$

with $p_1(z), \quad p_2(z) \in P(\gamma, n).$ For $n = 1,$ we have $P_k(\gamma,1) = P_k(\gamma)$. It is to be highlighted that this class was introduced by Padmanabhan and Parvatham (1975). Moreover for $n = 1, \ \gamma = 0,$ we obtain the class $P_k(0,1) = P_k$ defined by Pinchuk (1971) and for $k = 2, \ P_2(\gamma, n) = P(\gamma, n)$ defined earlier. It is easy to see that $p(z) \in P_k(\gamma, n),$ if and only if there exists $p_1(z) \in P_k(0, n)$ such that

$$p(z) = (1 - \gamma)\, p_1(z) + \gamma, \ z \in E.$$

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Further in (Noor, 2008), the following subclasses have been studied:

$$R_k(\gamma,n)=\left\{f(z):\ f(z)\in A_n \text{ and } \frac{zf'(z)}{f(z)}\in P_k(\gamma,n)\right\}.$$

$$V_k(\gamma,n)=\left\{f(z):\ f(z)\in A_n \text{ and } \frac{(zf'(z))'}{f'(z)}\in P_k(\gamma,n)\right\}.$$

We note that $R_k(0,1)=R_k$, the class of bounded radius rotation and $V_k(0,1)=V_k$, the class of bounded boundary rotation. For $k=2,\ \gamma=0,\ n=1,$ these classes reduce to the well known classes of starlike and convex univalent functions. It is given in (Noor, 2008) that $f(z)\in V_k(\gamma,n)\Leftrightarrow zf'(z)\in R_k(\gamma,n).$ With the help of convolution, we consider an operator $D_\lambda^m:\ A_n\to A_n\ (\lambda\geq 0,\ m\in \mathrm{N}_0=\mathrm{N}\cup\{0\})$ as follows:

$$D_\lambda^0(f*g)(z)=(f*g)(z),$$

$$D_\lambda^1(f*g)(z)=(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z),$$

and in general, we have

$$D_\lambda^m(f*g)(z)=D_\lambda\left(D_\lambda^{m-1}(f*g)(z)\right),\ (\lambda\geq 0,\ m\in \mathrm{N}). \quad (4)$$

If $f(z)$ and $g(z)$ are given by Equations 1 and 2 respectively, then from Equation 4 we have

$$D_\lambda^m(f*g)(z)=z+\sum_{j=n+1}^{\infty}\left[1+\lambda(j-1)\right]^m a_j b_j z^j,\ (\lambda\geq 0,\ m\in \mathrm{N}_0). \quad (5)$$

From Equation 5, it can be easily verified that

$$\lambda z D\left(_\lambda^m(f*g)(z)\right)'=D_\lambda^{m+1}(f*g)-(1-\lambda)D_\lambda^m(f*g),\ (\lambda>0). \quad (6)$$

For $n=1,$ this operator was introduced by Aouf and Seoudy (2010). For $n=1$ and $g(z)=\frac{z}{1-z},$ we have $D_\lambda^m(f*g)(z)=D_\lambda^m(f)(z),$ where D_λ^m is the generalized Salagean operator (Al-Oboudi, 2004), which yields the Salagean operator (Salagean, 1983) D^m for $\lambda=1.$ This operator was earlier studied by several authors in (Carlson and Shaffer, 1984; Dzoik and Srivastava, 1999) under specific conditions.

Furthermore, for $c>-1$ the generalized Bernardi operator (Bernard, 1969) for analytic functions is defined as:

$$J_c f(z)=\frac{c+1}{z^c}\int_0^z t^c f(t)\,dt. \quad (7)$$

After a simple calculation, Equation 7 can be written as:

$$cJ_c f(z)+z(J_c f(z))'=(c+1)f(z),\ z\in E. \quad (8)$$

Using the operator $D_\lambda^m,$ we define some new classes of analytic functions as:

Definition 1

Let $f(z),\ g(z)\in A_n,\ m>0,\ \lambda>0,\ 0\leq\gamma<1,\ z\in E,$ then $f(z)\in R_k^g(\lambda,m,n,\gamma)$ if and only if

$$D_\lambda^m(f*g)\in R_k(\gamma,n).$$

Definition 2

Let $f(z),\ g(z)\in A_n,\ m>0,\ \lambda>0,\ 0\leq\gamma<1,\ z\in E,$ then $f(z)\in V_k^g(\lambda,m,n,\gamma)$ if and only if

$$D_\lambda^m(f*g)\in V_k(\gamma,n).$$

Remark

For special values of parameters λ,m,k and $g(z)=\frac{z}{1-z^n}$, we have many known classes of analytic functions (Malik, 2010; Miller, 1975).

RESULTS AND DISCUSSION

Preliminary results

Lemma 1

Let $p(z)\in P(0,n)=P_n$ *for* $z\in E$ (Bernardi, 1974; MacGergor, 1963; Shah, 1972). Then

(i) $\left|\frac{h'(z)}{h(z)}\right|\leq\frac{2n|z|^{n-1}}{1-|z|^{2n}}$ (MacGergor, 1963)

(ii) $\left|zh'(z)\right| \leq \frac{2n|z|^n \operatorname{Re} h(z)}{1-|z|^{2n}}$ (Bernardi, 1975)

(iii) $\frac{1-|z|^n}{1+|z|^n} \leq \operatorname{Re} h(z) \leq |h(z)| \leq \frac{1+|z|^n}{1-|z|^n}$ (Shah, 1972)

Lemma 2

Let $u = u_1 + iu_2$ *and* $v = v_1 + iv_2$ and let $\Psi(u,v)$ be a complex valued function satisfying the conditions (Miller, 1975):

(i) $\Psi(u,v)$ *is continuous in* $D \subset C^2$.
(ii) $(1,0) \in D$ *and* $\operatorname{Re}\Psi(1,0) > 0$.
(ii) $\operatorname{Re}\Psi(iu_2, v_1) \leq 0$ *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

If $h(z)$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ with $\operatorname{Re}\Psi(h(z), zh'(z)) > 0$ for $z \in E,$ then $\operatorname{Re} h(z) > 0$ in $E.$

Main results

Theorem 1

Let $f(z) \in R_k^g(\lambda, m, n, \gamma).$ Then $f(z) \in R_k^g(\lambda, m+1, n, \gamma)$ for $|z| < r_0^n$, where r_0^n is given by

$$r_0^n = \frac{2\lambda - \lambda\gamma - 1}{\lambda(1-\gamma+n) + \sqrt{\lambda^2(1-\gamma+n)^2 - (1-\lambda\gamma+\gamma)(2\lambda-\lambda\gamma-1)}}. \quad (9)$$

Proof: Let $f(z) \in R_k^g(\lambda, m, n, \gamma)$. Then $D_\lambda^m(f*g) \in R_k(\gamma, n)$. Equivalently

$$\frac{z\left(D_\lambda^m(f*g)\right)'}{D_\lambda^m(f*g)} = H(z) \in P_k(\gamma, n), \quad (10)$$

where $H(z)$ is analytic in E and $H(0) = 1$. Using Equations 6 and 10, we obtain

$$\frac{D_\lambda^{m+1}(f*g)'}{D_\lambda^m(f*g)} = \lambda H(z) + 1 - \lambda.$$

Logarithmic differentiation yields

$$\frac{z\left(D_\lambda^{m+1}(f*g)\right)'}{D_\lambda^{m+1}(f*g)} = H(z) + \frac{zH'(z)}{H(z) + \frac{1-\lambda}{\lambda}}. \quad (11)$$

Since $H(z) \in P_k(\gamma, n),$ we can write

$$H(z) = (1-\gamma)\left\{\left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z)\right\} + \gamma, \quad (12)$$

where $h_1(z), h_2(z) \in P(0,n) = P_n.$ Then from Equations 11 and 12 we have

$$\frac{1}{(1-\gamma)}\left\{\frac{z\left(D_\lambda^{m+1}(f*g)\right)'}{D_\lambda^{m+1}(f*g)} - \gamma\right\} = \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z) + \frac{zh_1'(z)}{\{(1-\gamma)h_1(z)+\gamma\} + \frac{1-\lambda}{\lambda}}\right\}$$

$$-\left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_2(z) + \frac{zh_2'(z)}{\{(1-\gamma)h_2(z)+\gamma\} + \frac{1-\lambda}{\lambda}}\right\}.$$

Now, for $i = 1, 2,$ we use Lemma 1, with $|z| = r,$ to have

$$\operatorname{Re}\left\{h_i(z) + \frac{\lambda z h_i'(z)}{\lambda\{(1-\gamma)h_i(z)+\gamma\} + 1 - \lambda}\right\} \geq \operatorname{Re} h_i(z)\left\{1 - \frac{\frac{2\lambda n r^n}{1-r^{2n}}}{\lambda\left\{(1-\gamma)\left(\frac{1-r^n}{1+r^n}\right)+\gamma\right\} + 1 - \lambda}\right\}.$$

After some simplifications, we obtain

$$\operatorname{Re}\left\{h_i(z) + \frac{\lambda z h_i'(z)}{\lambda\{(1-\gamma)h_i(z)+\gamma\}+1-\lambda}\right\} \geq \operatorname{Re} h_i(z)\left\{\frac{1-2\lambda(1-\gamma+n)r^n + (2\lambda-2\lambda\gamma-1)r^{2n}}{1-2\lambda(1-\gamma)r^n + (2\lambda-2\lambda\gamma-1)r^{2n}}\right\}.$$

The right side of inequality is positive if $|z| < r_0^n$, where r_0^n is given by Equation 9. As a special case, when $\lambda = 1,\ n = 1,\ \gamma = 0,$ we obtain $r_0^1 = 2 - \sqrt{3}$ which is the well known radius of convexity for starlike functions.

Theorem 2

Let $f(z)\in V_k^g(\lambda,m,n,\gamma)$. Then $f(z)\in V_k^g(\lambda,m+1,n,\gamma)$ for $|z|<r_0^n$, where r_0^n is given by Equation 9.

Proof: Let $f(z)\in V_k^g(\lambda,m,n,\gamma)$. Then

$$\begin{aligned}
&D_\lambda^m(f*g)\in V_k(\gamma,n),\ z\in E\\
&\Leftrightarrow z\left(D_\lambda^m(f*g)\right)'\in R_k(\gamma,n),\ z\in E\\
&\Leftrightarrow D_\lambda^m\left(z(f*g)'\right)\in R_k(\gamma,n),\ z\in E\\
&\Leftrightarrow z(f*g)'\in R_k^g(\lambda,m,n,\gamma),\ \ z\in E\\
&\Leftrightarrow z(f*g)'\in R_k^g(\lambda,m+1,n,\gamma),\ |z|<r_0^n\\
&\Leftrightarrow D_\lambda^{m+1}\left(z(f*g)'\right)\in R_k(\gamma,n),\ |z|<r_0^n\\
&\Leftrightarrow z\left(D_\lambda^{m+1}(f*g)\right)'\in R_k(\gamma,n),\ |z|<r_0^n\\
&\Leftrightarrow D_\lambda^{m+1}(f*g)\in V_k(\gamma,n),\ |z|<r_0^n\\
&\Leftrightarrow D_\lambda^{m+1}(f*g)\in V_k(\gamma,n),\ |z|<r_0^n\\
&\Leftrightarrow f\in V_k^g(\lambda,m+1,n,\gamma),\ |z|<r_0^n.
\end{aligned}$$

This completes the proof.

Theorem 3

Let $f(z)\in A_n$. Then

$$R_k^g(\lambda,m+1,n,\gamma)\subset R_k^g(\lambda,m,n,\gamma_1).$$

where γ_1 is given by

$$\gamma_1=-\frac{1}{4}\left[(2\eta-2\gamma+1)-\sqrt{(2\eta-2\gamma+1)^2+8(2\eta\gamma+1)}\right]$$

Proof: Let $f(z)\in R_k^g(\lambda,m+1,n,\gamma)$ and set

$$\frac{z\left(D_\lambda^m(f*g)\right)'}{D_\lambda^m(f*g)}=H(z), \tag{13}$$

where $H(z)$ is analytic in E and $H(0)=1$. Using Equations 6 and 13, we obtain

$$\frac{D_\lambda^{m+1}(f*g)'}{D_\lambda^m(f*g)}=\lambda H(z)+1-\lambda.$$

Logarithmic differentiation yields

$$\frac{z\left(D_\lambda^{m+1}(f*g)\right)'}{D_\lambda^{m+1}(f*g)}=H(z)+\frac{zH'(z)}{H(z)+\eta}, \tag{14}$$

where $\eta=\dfrac{1-\lambda}{\lambda}$. Let

$$H(z)=\left(\frac{k}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)h_2(z). \tag{15}$$

From Equations 14 and 15, we have

$$\frac{z\left(D_\lambda^{m+1}(f*g)\right)'}{D_\lambda^{m+1}(f*g)}=\left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z)+\frac{zh_1'(z)}{h_1(z)+\eta}\right\}-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z)+\frac{zh_2'(z)}{h_2(z)+\eta}\right\}.$$

Since $f(z)\in R_k^g(\lambda,m+1,n,\gamma)$, we have

$$h_i(z)+\frac{zh_i'(z)}{h_i(z)+\eta}\in P(\gamma),\ \text{for } i=1,2.$$

Let $h_i(z)=\gamma_1+(1-\gamma_1)p_i(z)$ for $i=1,2$. Then

$$(\gamma_1-\gamma)+(1-\gamma_1)p_i(z)+\frac{(1-\gamma_1)zp_i'(z)}{\gamma_1+(1-\gamma_1)p_i(z)+\eta}\in P,\ \text{for } i=1,2.$$

We formulate a functional $\Psi(u,v)$ by taking $u=u_1+iu_2=p_i(z)$ and $v=v_1+iv_2=zp_i'(z)$, then

$$\Psi(u,v)=(\gamma_1-\gamma)+(1-\gamma_1)u+\frac{(1-\gamma_1)v}{(\eta+\gamma_1)+(1-\gamma_1)u}.$$

The first two conditions of Lemma 2 are obvious. For the third condition, we proceed as follows:

$$\operatorname{Re}\Psi(iu_2,v_1)=(\gamma_1-\gamma)+\frac{(1-\gamma_1)(\eta+\gamma_1)v_1}{(\eta+\gamma_1)^2+(1-\gamma_1)^2u_2^2}.$$

From $v_1 \le -\frac{1}{2}(1+u_2^2)$, we have

$$\operatorname{Re}\Psi(iu_2, v_1) \le \frac{A+Bu_2^2}{2C},$$

where

$$\begin{aligned} A &= 2(\gamma_1-\gamma)(\eta+\gamma_1)^2-(1-\gamma_1)(\eta+\gamma_1), \\ B &= 2(\gamma_1-\gamma)(1-\gamma_1)^2-(1-\gamma_1)(\eta+\gamma_1), \\ C &= (\eta+\gamma_1)^2+(1-\gamma_1)^2u_2^2. \end{aligned}$$

We note that $\operatorname{Re}\Psi(iu_2, v_1) \le 0$ if $A \le 0$ and $B \le 0$. From $A \le 0$, we obtain

$$\gamma_1 = -\frac{1}{4}\left[(2\eta-2\gamma+1)-\sqrt{(2\eta-2\gamma+1)^2+8(2\eta\gamma+1)}\right] \qquad (16)$$

By virtue of Lemma 2, we see that $p_i(z) \in P_n = P(0,n)$, for $i = 1,2$ and $z \in E$. Hence, $h_i(z) \in P(\gamma_1, n)$ which implies $H(z) \in P_k(\gamma_1, n)$ and consequently $f(z) \in R_k^g(\lambda, m, n, \gamma_1)$. This completes the proof.

Theorem 4

Let $f(z) \in A_n$. Then

$$V_k^g(\lambda, m+1, n, \gamma) \subset V_k^g(\lambda, m, n, \gamma_1).$$

where γ_1 is given by (3.8)

Proof: Let $f(z) \in V_k^g(\lambda, m+1, n, \gamma)$ and set

$$\frac{\left(z\left(D_\lambda^m(f*g)\right)'\right)'}{\left(D_\lambda^m(f*g)\right)'} = H(z), \qquad (17)$$

where $H(z)$ is analytic in E with $H(0)=1$. Using Equations 6 and 17, we obtain

$$\frac{D_\lambda^{m+1}(f*g)'}{D_\lambda^m(f*g)} = \lambda H(z)+1-\lambda.$$

Logarithmic differentiation yields

$$\frac{z\left(D_\lambda^{m+1}(f*g)\right)'}{D_\lambda^{m+1}(f*g)} = H(z)+\frac{zH'(z)}{H(z)+\eta}.$$

Now using the same steps as in Theorem 3, we obtain the required result.

Theorem 5

If $f(z) \in R_k^g(\lambda, m, 1, \gamma)$ and $f(z) = z + \sum\limits_{j=n+1}^{\infty} a_j z^j$ then

$$|a_j| \le \frac{[k(1-\gamma)]_{j-1}}{(j-1)![1+\lambda(j-1)]^m |b_j|}.$$

Proof: Let $f(z) \in R_k^g(\lambda, m, 1, \gamma)$. Then $D_\lambda^m(f*g) \in R_k(\gamma)$, or equivalently

$$\frac{z\left(D_\lambda^m(f*g)\right)'}{D_\lambda^m(f*g)} = H(z) \in P_k(\gamma), \qquad (18)$$

where $H(z)$ is analytic in E with $H(0)=1$. Let $H(z)$ be of the form

$$H(z) = 1+\sum_{j=1}^{\infty} c_j z^j, \; z \in E. \qquad (19)$$

From Equations 5, 18 and 19, we obtain

$$\begin{aligned} z+\sum_{j=2}^{\infty} j[1+\lambda(j-1)]^m a_j b_j z^j &= \left[z+\sum_{j=2}^{\infty}[1+\lambda(j-1)]^m a_j b_j z^j\right]\left[1+\sum_{j=1}^{\infty} c_j z^j\right], \\ &= \left[\sum_{j=1}^{\infty}[1+\lambda(j-1)]^m a_j b_j z^j\right]\left[1+\sum_{j=1}^{\infty} c_j z^j\right], \; a_1 = b_1 = 1 \\ &= \sum_{j=1}^{\infty}[1+\lambda(j-1)]^m a_j b_j z^j \\ &\quad + \left[\sum_{j=1}^{\infty}[1+\lambda(j-1)]^m a_j b_j z^j\right]\left[\sum_{j=1}^{\infty} c_j z^j\right]. \end{aligned}$$

By using Cauchy's product formula (Goodman, 1983) for the power series, we obtain

$$\sum_{j=1}^{\infty}(j-1)[1+\lambda(j-1)]^m a_j b_j z^j = \sum_{j=1}^{\infty}\left[\sum_{i=1}^{j-1}[1+\lambda(i-1)]^m a_i b_i c_{j-i}\right]z^j.$$

Equating the coefficients of z^j on both sides, we have

$$(j-1)\left[1+\lambda(j-1)\right]^m a_j b_j = \sum_{i=1}^{j-1}\left[1+\lambda(i-1)\right]^m a_i b_i c_{j-i}.$$

Since $H(z) \in P_k(\gamma)$, we have $\left|c_{j-i}\right| \leq k(1-\gamma)$. This implies

$$\left|a_j b_j\right| \leq \frac{k(1-\gamma)}{(j-1)\left[1+\lambda(j-1)\right]^m}\sum_{i=1}^{j-1}\left[1+\lambda(i-1)\right]^m \left|a_i b_i\right|.$$

By using induction on j, we obtain

$$\left|a_j\right| \leq \frac{\left[k(1-\gamma)\right]_{j-1}}{(j-1)!\left[1+\lambda(j-1)\right]^m \left|b_j\right|}.$$

This bound is sharp and the equality occurs for $f_0(z) \in A$ such that

$$\frac{z\left(D_\lambda^m(f*g)\right)'}{D_\lambda^m(f*g)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{1+(1-2\gamma)z}{1-z}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{1-(1-2\gamma)z}{1+z}\right).$$

Theorem 6

Let $f(z) \in R_k^g(\lambda,m,n,\gamma)$ and J_c is defined by Equation 7, then $J_c f(z) \in R_k^g(\lambda,m,n,\gamma)$.

Proof: Let $f(z) \in R_k^g(\lambda,m,n,\gamma)$ and set

$$\frac{z\left[J_c\left(D_\lambda^m(f*g)\right)\right]'}{J_c\left(D_\lambda^m(f*g)\right)} = H(z), \qquad (20)$$

where $H(z)$ is analytic in E and $H(0)=1$. Using Equation 8 and 20, we obtain

$$\frac{(c+1)D_\lambda^m(f*g)}{J_c\left(D_\lambda^m(f*g)\right)} = H(z)+c.$$

Logarithmic differentiation yields

$$\frac{z\left(D_\lambda^m(f*g)\right)'}{D_\lambda^m(f*g)} = H(z)+\frac{zH'(z)}{H(z)+c}. \qquad (21)$$

Now following the same steps as in theorem 3, we obtain the required result.

Theorem 7

Let $f(z) \in V_k^g(\lambda,m,n,\gamma)$, then $J_c f(z) \in V_k^g(\lambda,m,n,\gamma)$.

Proof: Let $f(z) \in V_k^g(\lambda,m,n,\gamma)$ and set

$$\frac{\left(z\left[J_c\left(D_\lambda^m(f*g)\right)\right]'\right)'}{\left[J_c\left(D_\lambda^m(f*g)\right)\right]'} = H(z),$$

where $H(z)$ is analytic in E and $H(0)=1$. Using Equations 8 and 20, we obtain

$$\frac{(c+1)\left(D_\lambda^m(f*g)\right)'}{\left[J_c\left(D_\lambda^m(f*g)\right)\right]'} = H(z)+c.$$

Logarithmic differentiation yields

$$\frac{\left[z\left(D_\lambda^m(f*g)\right)'\right]'}{\left[D_\lambda^m(f*g)\right]'} = H(z)+\frac{zH'(z)}{H(z)+c}.$$

Now following the same steps as in Theorem 3, we obtain the required result.

Theorem 8

Let $f(z) \in V_k^g(\lambda,m,n,\gamma)$ and $h(z) \in R_k^g(\lambda,m,n,\gamma)$, we define

$$G(z) = \int_0^z \left[\left(D_\lambda^m(f*g)(t)\right)'\right]^{\delta_1}\left[\frac{\left(D_\lambda^m(h*g)(t)\right)}{t}\right]^{1-\delta_1} dt, \qquad (22)$$

where $0 \leq \delta_1 \leq 1$. Then $G(z) \in V_k(\gamma,n)$.

Proof: From Equation 22, we have

$$zG'(z) = \left[\left(D_\lambda^m(f*g)(z)\right)'\right]^{\delta_1}\left[\left(D_\lambda^m(h*g)(z)\right)\right]^{1-\delta_1}.$$

Logarithmic differentiation yields

$$\frac{\left(zG'(z)\right)'}{G'(z)} = \delta_1 \frac{z\left[\left(D_\lambda^m\left(f*g\right)\right)'\right]'}{\left[\left(D_\lambda^m\left(f*g\right)\right)'\right]} + \left(1-\delta_1\right)\frac{z\left[\left(D_\lambda^m\left(h*g\right)\right)\right]'}{\left[\left(D_\lambda^m\left(h*g\right)\right)\right]}.$$

Since $f(z) \in V_k^g(\lambda, m, n, \gamma)$ and $h(z) \in R_k^g(\lambda, m, n, \gamma)$, we have

$$\frac{\left(zG'(z)\right)'}{G'(z)} = \delta_1 h_1(z) + \left(1-\delta_1\right) h_2(z),$$

where $h_1(z), h_2(z) \in P_k(\gamma, n)$. Since $P_k(\gamma, n)$ is a convex set, we have

$$\frac{\left(zG'(z)\right)'}{G'(z)} \in P_k(\gamma, n)$$

and this implies that $G(z) \in V_k(\gamma, n)$, which completes the proof.

CONCLUSIONS

Some new classes of analytic functions in connection with the convolution operator have been explored. Moreover, some inclusion relationships, radii problems and a sharp coefficient bound have also been successfully derived. It was also observed that the proposed innovations are of extreme importance for a wide range of physical problems.

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