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Improvement of the homotopy perturbation method to nonlinear problems

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This paper proposes an effective improvement of the homotopy perturbation method (HPM) by using Jacobi and He's polynomials to solve some nonlinear ordinary differential equations. With this method, the source terms of ordinary differential equations can be expanded in series of shifted Jacobi polynomials. Numerical results are given in this paper to illustrate the reliability of this method with nonlinear ordinary differential equations.

Key words: Homotopy perturbation method (HPM), shifted Jacobi polynomials, nonlinear ordinary differential equations.

INTRODUCTION

In recent years, the subject of differential calculus received attention in regards to effective numerical methods for solving linear and nonlinear differential equations. Examples of these methods are the Adomian decomposition method (ADM) (Wazwaz et al., 2015; Hosseinzadeh et al., 2017), the variational iteration method (Akter and Chowdhury, 2017; Wazwaz, 2015; Glowinski, 2015; Ghorbani and Bakherad, 2017), the pseudospectral method (Bhrawy et al., 2015; Wei et al., 2017; Borluk and Muslu, 2015) and the reproducing kernel Hilbert space method (Arqub et al., 2016).

In 1999, He (1999) proposed the HPM which combines the standard homotopy in topology and perturbation techniques. The HPM is a powerful and effective tool for solving a wide range of problems that arise in various fields. With this method, numerical solutions are

expressed as sums of infinite series. The sums converge rapidly to find solutions.

The HPM can be applied to integro-differential equation (Elbeleze et al., 2016), linear and nonlinear Newell-Whitehead-Segel equations (Nourazar et al., 2017), nonlinear optimal control problems (Jafari et al., 2016), integral equations (Elzaki and Alamri, 2016; Hasan and Matin, 2017), nonlinear wave-like equations with variable coefficients (Gupta et al., 2013), boundary value problems (Opanuga et al., 2017), the quadratic Riccati differential equation (Aminikhah and Hemmatnezhad, 2010), Boussinesq-like equations (Fernández, 2014) and others (Sakar et al., 2016; Soori et al., 2015; Qureshi et al., 2017; Zhang et al., 2014; Najafi and Edalatpanah, 2014; Roy et al., 2015; Abou-Zeid, 2016).

In an overview of approximations of nonlinear ordinary

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differential equations, adomian decomposition method with orthogonal polynomials was proposed as a method for solving nonlinear problems (Liu, 2009). Chun (2010) proposed an efficient modification of the HPM that used Chebyshev's and He's polynomials to solve nonlinear differential equations. Behrooz and Ebadi (2011) further developed the HPM using Legendre polynomials. Recently, Novin and Dastjerd (2015) improved the adomian decomposition method to obtain solutions for the Duffing equation.

This article applies the HPM to the shifted Jacobi polynomials of the right-side function $f(x)$ to solve nonlinear differential equations. The advantage of this approach is that such polynomials are simple and do not require small parameters. Moreover, with a few iterations one can find accurate solutions. To the best of the authors' knowledge, this approach was not employed to solve linear and nonlinear differential equations in the past.

This manuscript is arranged as follows: First, various properties of shifted Jacobi polynomials are presented, followed by a discussion of He's HPM. Thereafter, the proposed HPM is presented along with solutions to three numerical examples and with comparisons of the solutions and results found with other methods; therein, the validity and accuracy of the proposed method is considered. Additionally, the results of the numerical simulation using Maple 17 are given, and the study is concluded.

PROPERTIES OF SHIFTED JACOBI POLYNOMIALS

The well-known standard Jacobi polynomials, $P_k^{(\alpha,\beta)}(x)$ $(\alpha > -1, \beta > -1)$ are defined on the interval $[-1,1]$. The standard Jacobi polynomials of degree k $(P_k^{(\alpha,\beta)}(x), k = 0,1\dots)$ satisfy the following Rodrigue's formula:

$$P_k^{(\alpha,\beta)}(x) = \frac{(-1)^k}{2^k k!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^k}{dx^k}\left[(1-x)^{\alpha+k}(1+x)^{\beta+k}\right]. (1)$$

For $\alpha = \beta$, one recovers the ultraspherical polynomials (symmetric Jacobi polynomials) and for $\alpha = \beta = \pm\frac{1}{2}, \alpha = \beta = 0$ is the Chebyshev polynomial of the first and second kinds and Legendre polynomials respectively; and for the non-symmetric Jacobi polynomials, the two important special cases $\alpha = -\beta = \pm\frac{1}{2}$ (Chebyshev polynomials of the third and fourth kinds) are also recovered.

The Jacobi polynomials (Bhrawy et al., 2016) satisfy the orthogonality relation.

$$\left(P_k^{(\alpha,\beta)}(x), P_\ell^{(\alpha,\beta)}(x)\right)_{\omega^{(\alpha,\beta)}(x)} = \int_{-1}^{1} P_k^{(\alpha,\beta)}(x) P_\ell^{(\alpha,\beta)}(x)\omega^{(\alpha,\beta)}(x)dx = h_k\delta_{\ell k}, (2)$$

where $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, $h_k = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)}$.

In order to use these polynomials on the interval $[0,L]$, we define the so-called shifted Jacobi polynomials by introducing the change of variable $x = \frac{2x}{L} - 1$ Let the shifted Jacobi polynomials $P_i^{(\alpha,\beta)}\left(\frac{2x}{L} - 1\right)$ be denoted by $P_{L,i}^{(\alpha,\beta)}(x)$. Then $P_{L,i}^{(\alpha,\beta)}(x)$ can be obtained with the aid of the following recurrence formula:

$$P_{L,i+1}^{(\alpha,\beta)}(x) = (a_i^{(\alpha,\beta)}(\frac{2x}{L}-1) - b_i^{(\alpha,\beta)})P_{L,i}^{(\alpha,\beta)}(x) - c_i^{(\alpha,\beta)}P_{L,i-1}^{(\alpha,\beta)}(x), \quad i \geq 1, (3)$$

$$P_{L,0}^{(\alpha,\beta)}(x) = 1, \quad P_{L,1}^{(\alpha,\beta)}(x) = \frac{1}{L}(\alpha+\beta+2) x - (\beta+1),$$

where

$$a_i^{(\alpha,\beta)} = \frac{(2i+\alpha+\beta+1)(2i+\alpha+\beta+2)}{2(i+1)(i+\alpha+\beta+1)}$$

$$b_i^{(\alpha,\beta)} = \frac{(\beta^2-\alpha^2)(2i+\alpha+\beta+1)}{2(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)}$$

$$c_i^{(\alpha,\beta)} = \frac{(i+\alpha)(i+\beta)(2i+\alpha+\beta+1)}{(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)}.$$

The analytic form of the shifted Jacobi polynomials $P_{L,i}^{(\alpha,\beta)}(x)$ of degree i is given by

$$P_{L,i}(x) = \sum_{k=0}^{i}(-1)^{i+k}\frac{\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!k!L^k}x^k, (4)$$

and the orthogonality condition is

$$\int_0^L P_{L,j}^{(\alpha,\beta)}(x) P_{L,k}^{(\alpha,\beta)}(x)\omega^{(\alpha,\beta)}(x)dx = \hbar_{L,k}^{(\alpha,\beta)}\delta_{jk}, (5)$$

where $\omega_L^{(\alpha,\beta)}(x) = x^\beta(L-x)^\alpha$ and $\hbar_{L,k}^{(\alpha,\beta)} = \frac{L^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)}.$

A function $f(x)$, square integrable in $[0,L]$, may be expressed in terms of shifted Jacobi polynomials as

$$f(x) = \sum_{j=0}^{\infty}c_j P_{L,j}^{(\alpha,\beta)}(x),$$

where the coefficients c_j are given by

$$c_j = \frac{1}{\hbar_{L,k}^{(\alpha,\beta)}}\int_0^L f(x)P_{L,j}^{(\alpha,\beta)}(x)\omega_L^{(\alpha,\beta)}(x)dx, \qquad j = 0,1,2,\dots (6)$$

HE'S HPM

Here, we will present HPM used by He (1999, 2006) to solve nonlinear differential equations that take the following form

$$L(u) + R(u) + N(u) = f(x), \qquad x \in \Omega, \qquad (7)$$

with boundary conditions

$$B\left(u \frac{\partial u}{\partial x}\right) = 0, \qquad x \in \Gamma, \qquad (8)$$

where L is a linear operator of highest order, R is a linear operator of lower order than L, N is a nonlinear operator, B is a boundary operator, $f(x)$ is the source term and Γ is the boundary of the domain Ω. He (1999) defines the homotopy technique as $v(r,p): \Omega \times [0,1] \rightarrow \mathbb{R}$, which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + R(v) + N(v) - f(x)] = 0, \qquad (9)$$

or

$$H(v,p) = L(v) - L(u_0) + p[L(u_0) + L(v) + R(v) + N(v) - f(x)] = 0, \qquad (10)$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial estimated approximation of Equation 7 which satisfies the boundary conditions. Obviously, we have

$$H(v,0) = L(v) - L(u_0) = 0,$$
$$H(v,1) = L(v) + R(v) + N(v) - f(x) = 0. \qquad (11)$$

The expansion of p from 0 to 1 is the same as that for $H(v,p)$ from $L(v) - L(u_0)$ to $L(v) + R(v) + N(v) - f(x)$. In topology, this is called deformation and $L(v) - L(u0)$, $L(v) + R(v) + N(v) - f(x)$ are called homotopic. Using the parameter p, we expand the solution of Equation 9 in the following form:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \qquad (12)$$

When $p \rightarrow 1$, Equation 12 becomes the approximate solution of Equation 7, that is,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots \qquad (13)$$

METHODOLOGY OF HPM BASED ON SHIFTED JACOBI POLYNOMIALS

When implementing the previous HPM on some problems we find that the source term $f(x)$ is not easy to integrate. So, in this paper, for an arbitrary natural number N, $f(x)$ can be expressed in the

shifted Jacobi series

$$f(x) \approx f_{(J,N)}(x) = \sum_{i=0}^{N} a_i P_{L,i}^{(\alpha,\beta)}(x) \qquad (14)$$

To deal with the nonlinear term $N(u)$, we will use He's polynomials

$$N(v_0, v_1, \ldots, v_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{j=0}^{n} p^j v_j\right)_{p=0}, \qquad (15)$$

and satisfy the following relation

$$N(v) = N(v_0) + pN(v_0, v_1) + p^2 N(v_0, v_1, v_2) + \ldots + p^n N(v_0, v_1, v_2, \ldots, v_n) + \ldots \qquad (16)$$

Substituting Equations 12, 14 and 16 into 10, and equating coefficients of like powers of p, we get

$$p^0 : L(v_0) - L(u_0) = 0,$$
$$p^1 : L(v_1) + L(u_0) + R(v_0) + N(v_0) - f_{(J,N)}(x) = 0,$$
$$p^2 : L(v_2) + R(v_1) + N(v_0, v_1) = 0, \qquad (17)$$
$$p^3 : L(v_3) + R(v_2) + N(v_0, v_1, v_2) = 0,$$
$$\vdots$$
$$p^{n+1} : L(v_{n+1}) + R(v_n) + N(v_0, v_1, v_2, \ldots, v_n) = 0,$$

and so on. By solving the above set equations with suitable initial conditions, $v_0, v_1, v_2, \ldots, v_n$ can be determined and the series solution (12) will be entirely determined. The N-term approximation solution of Equation 7 can be considered as follows

$$U_N = \sum_{k=0}^{N-1} v_k. \qquad (18)$$

NUMERICAL SIMULATION AND COMPARISONS

Here, several numerical examples to demonstrate the high accuracy and applicability of the proposed methods for solving nonlinear ordinary differential equations are presented. We also compare the results given from our method and those reported in the literature. The comparisons reveal that our methods are very effective and convenient.

Example 1

We consider the following equation (Liu, 2009; Behrooz and Ebadi, 2011)

$$u'' + xu' + x^2 u^3 = (2 + 6x^2)e^{x^2} + x^2 e^{3x^2}, \qquad 0 \le x \le 1, \quad (19)$$

$$u(0) = 1, \; u'(0) = 0, \qquad (20)$$

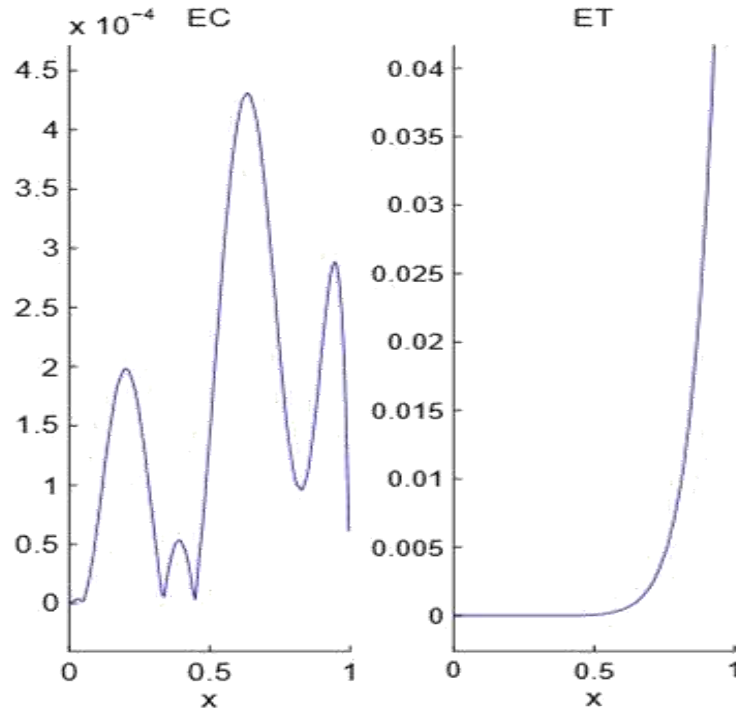with exact solution $u(x) = e^{x^2}$.

Figure 1. AEs of HPM by Chebyshev (EC) and Taylor (ET) polynomials at
$N = 6$ for Example 1.
Source: Behrooz and Ebadi (2011).

In an operator form, Equation 19 can be written as:

$$L(u) + R(u) + N(u) = f(x), \tag{21}$$

where $L = \frac{d}{dx^2}$, $R(u) = x \frac{d}{dx} u$, $N(u) = x^2 u^3$ and $f(x) = (2 + 6x^2) e^{x^2} + x^2 e^{3x^2}$.

Behrooz and Ebadi (2011) introduced this problem and presented Figure 1 to show the absolute errors (AEs) of HPM with Chebyshev and Taylor polynomials at $N = 6$. Moreover, Liu (2009) applied the ADM with Legendre, Chebyshev and Taylor polynomials to this problem and presented the absolute errors (AEs) in Figures 2, 3 and 4. Now, we apply our method for this problem.

The homotopy equation is

$$v'' - L(u_0) + pL(u_0) + p[R(v) + N(v) - f(x)] = 0, \tag{22}$$

where

$$v = \sum_{i=0}^{\infty} p^i v_i(x). \tag{23}$$

According to Equation 15, He's polynomials are found to be:

$$
\begin{aligned}
N(v_0) &= x^2 v_0^3, \\
N(v_0, v_1) &= x^2 (3v_0^2 v_1), \\
N(v_0, v_1, v_2) &= x^2 (3v_0^2 v_2 + 3v_0 v_1^2), \\
N(v_0, v_1, v_2, v_3) &= x^2 (3v_0^2 v_3 + 6v_0 v_1 v_2 + v_1^3), \\
N(v_0, v_1, v_2, v_3, v_4) &= x^2 (3v_1^2 v_2 + 3v_0 v_2^2 + 6v_0 v_1 v_3 + 3v_0^2 v_4), \\
N(v_0, v_1, v_2, v_3, v_4, v_5) &= x^2 (3v_1 v_2^2 + v_1^2 v_3 + 6v_0 v_2 v_3 + 6v_0 v_1 v_4 + 3v_0^2 v_5), \\
&\vdots
\end{aligned}
\tag{24}
$$

Substituting relations Equation 24 in Equation 16, gives the following relation

$$N(v) = x^2 v_0^3 + x^2 (3v_0^2 v_1) p + x^2 (3v_0^2 v_2 + 3v_0 v_1^2) p^2 + x^2 (3v_0^2 v_3 + 6v_0 v_1 v_2 + v_1^3) p^3 + \dots \tag{25}$$

Now, if $\alpha = \beta = \dfrac{1}{20}$ and $N = 6$ the expansions of $f(x)$ in shifted Jacobi polynomials are obtained by

$$f_{(J,6)}(x) \approx 2.075653430 - 3.81896575x + 55.6619634x^2 - 229.6216407x^3 \\ + 544.8167168x^4 - 595.0415693x^5 + 267.6207961x^6. \tag{26}$$

Substituting Equation 25 and Equation 23 into the homotopy (22) and equating the terms with identical powers of p, gives:
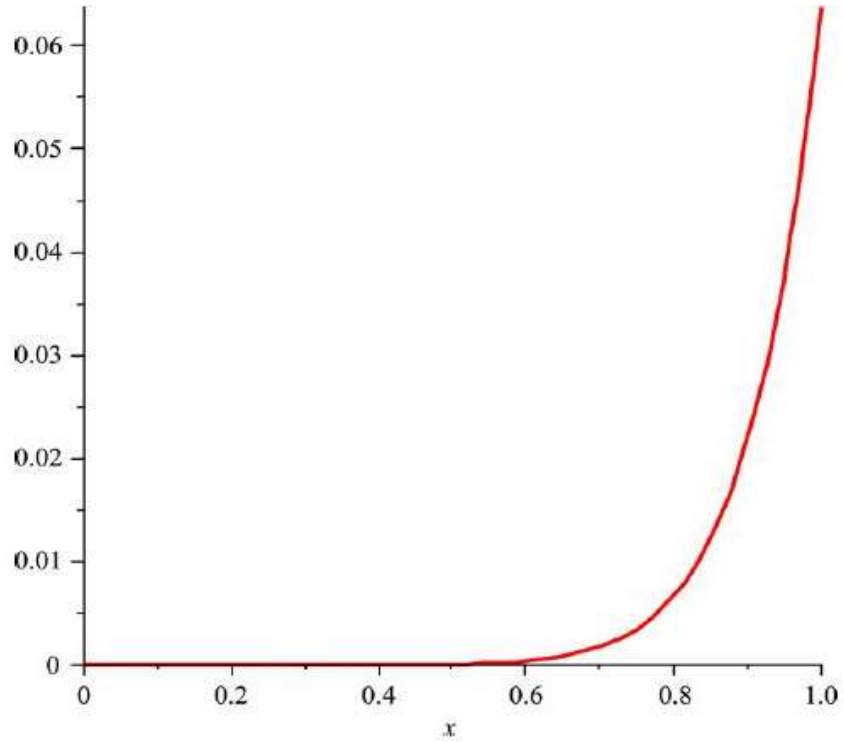
Figure 2. AEs of ADM by Taylor polynomials at $N = 6$ for Example 1.
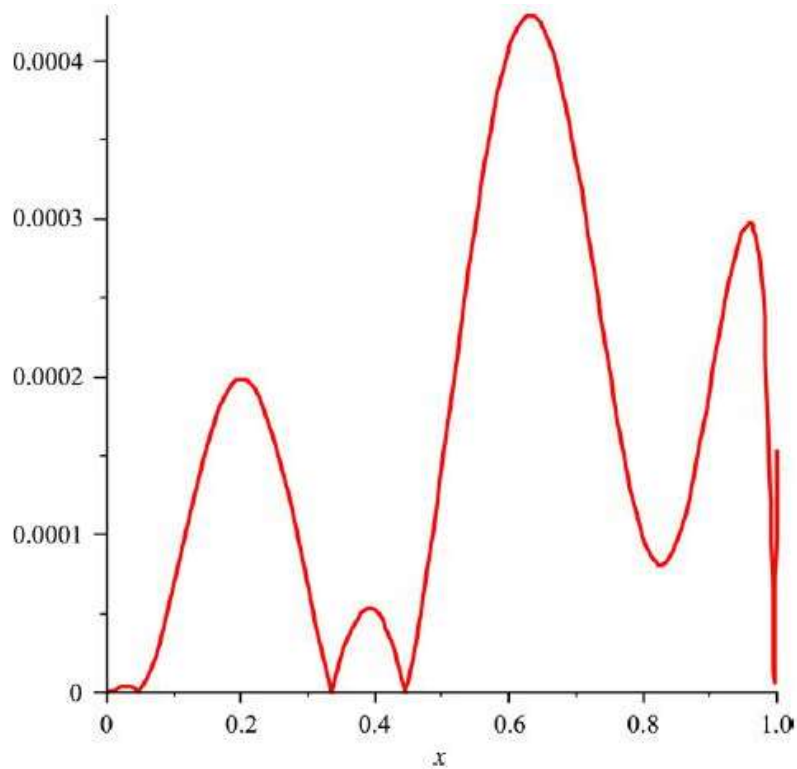Source: Behrooz and Ebadi (2011).

Figure 3. AEs of ADM by Chebyshev polynomials at $N = 6$ for Example 1.
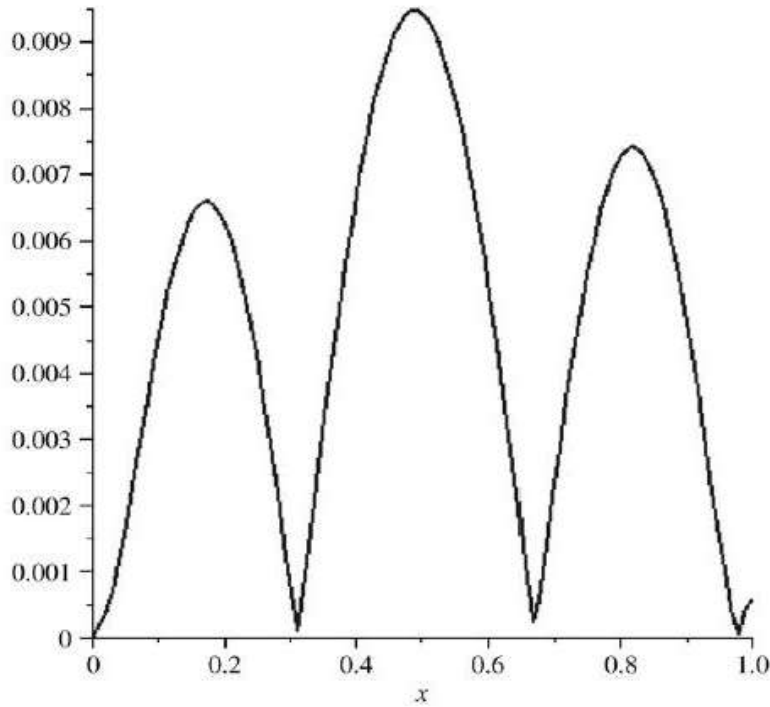Source: Liu (2009).

Figure 4. AEs of ADM by Legendre polynomials at $N = 6$ for Example 1.
Source: Liu (2009).

$$p^0 : \begin{cases} v'' - 2 = 0, \\ v_0(0) = 1, \quad v_0'(0) = 0, \end{cases}$$

$$p^1 : \begin{cases} v_1'' + 2 + x v_0' + x^2 v_0^3 - f_{(J,6)}(x) = 0, \\ v_1(0) = 0, \quad v_1'(0) = 0, \end{cases}$$

$$p^2 : \begin{cases} v_2'' + 3x^2 v_0^2 v_1 + x v_1' = 0, \\ v_2(0) = 0, \quad v_2'(0) = 0, \end{cases}$$

$$p^3 : \begin{cases} v_3'' + 3x^2 (v_0^2 v_2 + v_0 v_1^2) + x v_2' = 0, \\ v_3(0) = 0, \quad v_3'(0) = 0, \end{cases}$$

$$p^4 : \begin{cases} v_4'' + x v_3' + x^2 (3v_0^2 v_3 + 6v_0 v_1 v_2 + v_1^3) = 0, \\ v_4(0) = 0, \quad v_4'(0) = 0, \end{cases}$$

$$p^5 : \begin{cases} v_5'' + x v_4' + x^2 (3v_0^2 v_4 + 6v_0 v_1 v_3 + 3v_0 v_2^2 + 3v_1^2 v_2) = 0, \\ v_5(0) = 0, \quad v_5'(0) = 0, \end{cases}$$

$$p^6 : \begin{cases} v_6'' + x v_5' + x^2 (3v_0^2 v_5 + 6v_0 v_1 v_4 + 6v_0 v_2 v_3 + 3v_1^2 v_3 + 3v_1 v_2^2) = 0, \\ v_6(0) = 0, \quad v_6'(0) = 0. \end{cases} \quad (27)$$

By substituting Equation 26 in $f_{(J,6)}(x)$ at Equation 27, and solving the above equations by the help of Maple, we obtain

$$u(x) = 1 + 1.037826715x^2 - 0.6364942917x^3 + 4.382192498x^4 + \dots$$
$$+ 8.685292246 \times 10^{-7} x^{46} + 5.764802911 \times 10^{-9} x^{47} - 1.642452152 \times 10^{-9} x^{48}$$
$$+ 1.024528074 \times 10^{-12} x^{50}. \quad (28)$$

The absolute error of HPM with shifted Jacobi polynomials at $N = 6$ is plotted in Figure 5.

The accuracy of this method is validated by comparing to the exact $u(x)$. By comparing Figures 1 to 5, it is found that the absolute errors (AEs) generated using our method are smaller than the errors caused by HPM with Chebyshev (EC) and Taylor (ET) polynomials and by ADM with Legendre, Chebyshev and Taylor polynomials. This means that the method here is more accurate than previous methods.

Example 2

We consider the following problem (Behrooz and Ebadi, 2011):

$$u'' + u u' = x \sin(2x^2) - 4x^2 \sin(x^2) + 2 \cos(x^2), \quad (29)$$

$$u(0) = 0, \quad u'(0) = 0, \qquad 0 \leq x \leq 1, \quad (30)$$

with exact solution $u(x) = \sin(x^2)$. In an operator form, Equation 29 can be written as:

$$L(u) + N(u) = f(x), \quad (31)$$

where $L = \dfrac{d}{dx^2}$, $N(u) = uu'$ and $f(x) = x \sin(2x^2) - 4x^2 \sin(x^2) + 2 \cos(x^2)$.
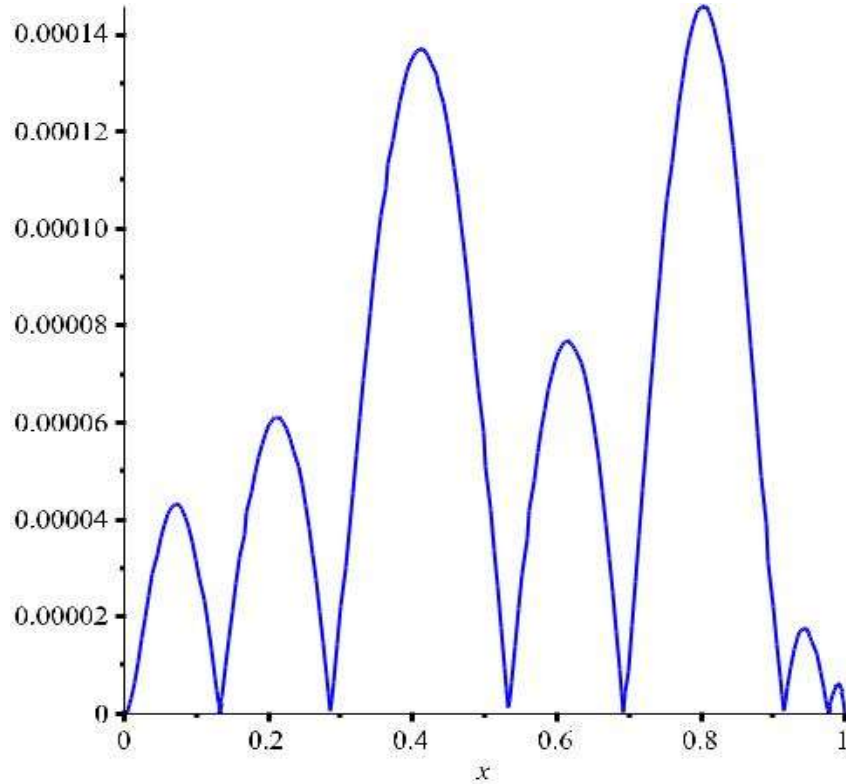
The homotopy equation is:

Figure 5. AEs of HPM by shifted Jacobi polynomials at $\alpha = \beta = \frac{1}{20}$ and $N = 6$ for Example 1.

$$v'' - L(u_0) + p L(u_0) + p[N(v) - f(x)] = 0. \qquad (32)$$

He's polynomials for the nonlinear term $N(u) = uu'$ are found to be

$$
\begin{aligned}
N(v_0) &= v_0 v_0', \\
N(v_0, v_1) &= v_0 v_1' + v_1 v_0', \\
N(v_0, v_1, v_2) &= v_0 v_2' + v_1 v_1' + v_2 v_0', \\
N(v_0, v_1, v_2, v_3) &= v_0 v_3' + v_1 v_2' + v_2 v_1' + v_3 v_0', \\
N(v_0, v_1, v_2, v_3, v_4) &= v_0 v_4' + v_1 v_3' + v_2 v_2' + v_3 v_1' + v_4 v_0', \\
&\vdots
\end{aligned}
\qquad (33)
$$

Now, if $\alpha = \beta = \frac{1}{20}$ and $N = 20$ the expansions of $f(x)$ in shifted Jacobi polynomials are obtained by

$$
\begin{aligned}
f_{(J,10)}(x) &\approx 1.999997823 + 0.26993648 \times 10^{-3} x - 0.8403435 \times 10^{-2} x^2 + 2.112845335 x^3 \\
&\quad - 5.80766750 x^4 + 3.416658954 x^5 - 8.977727745 x^6 + 13.49519614 x^7 \\
&\quad - 14.20329364 x^8 + 8.284749236 x^9 - 1.688604332 x^{10}.
\end{aligned}
\qquad (34)
$$

Substituting Equation 23 and using Equation 16 with relations

33 into the homotopy Equation 32 and equating the terms with identical powers of p, gives

$$
\begin{aligned}
p^0 &: \begin{cases} v_0'' - 2 = 0, \\ v_0(0) = 0, \quad v_0'(0) = 0, \end{cases} \\[4pt]
p^1 &: \begin{cases} v_1'' + 2 + v_0 v_0' - f_{(J,10)}(x) = 0, \\ v_1(0) = 0, \quad v_1'(0) = 0, \end{cases} \\[4pt]
p^2 &: \begin{cases} v_2'' + v_0' v_1 + v_0 v_1' = 0, \\ v_2(0) = 0, \quad v_2'(0) = 0, \end{cases} \\[4pt]
p^3 &: \begin{cases} v_3'' + v_1 v_1' + v_0' v_2 + v_0 v_2' = 0, \\ v_3(0) = 0, \quad v_3'(0) = 0, \end{cases} \\[2pt]
&\vdots \\[2pt]
p^{10} &: \begin{cases} v_{10}'' + v_1' v_8 + v_2' v_7 + v_3' v_6 + v_4' v_5 + v_4 v_5' + v_3 v_6' + v_2 v_7' + v_1 v_8' + v_0 v_9' + v_0 v_9', \\ v_{10}(0) = 0, \quad v_{10}'(0) = 0. \end{cases}
\end{aligned}
\qquad (35)
$$

By substituting Equation 34 in $f_{(J,10)}(x)$ at Equation 35, and solving the above equations by the help of Maple, we obtain

$$
\begin{aligned}
u(x) &= 0.9999989115 x^2 + 0.4498941333 \times 10^{-4} x^3 - 0.7002862500 \times 10^{-3} x^4 \\
&\quad + 0.5642484450 \times 10^{-2} x^5 + \cdots - 2.229469520 \times 10^{-13} x^{64} + 2.357695514 \times 10^{-14} x^{65} \\
&\quad - 1.572944876 \times 10^{-15} x^{66} + 4.949801810 \times 10^{-17} x^{67}.
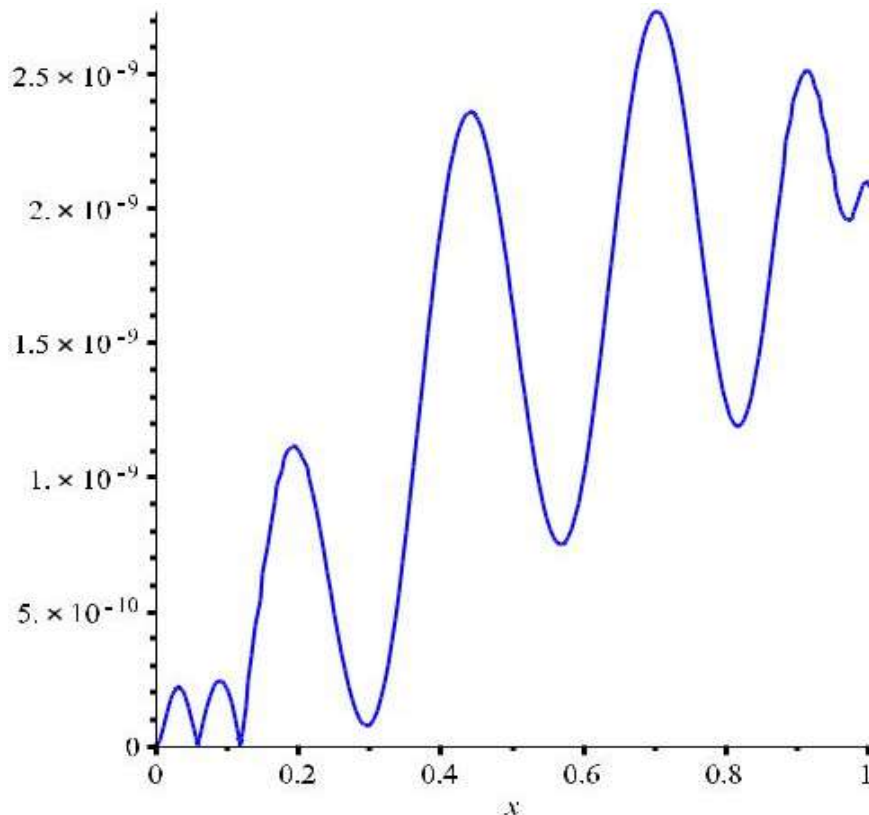\end{aligned}
\qquad (36)
$$

Figure 6. AEs of HPM by shifted Jacobi polynomials at $N = 10$ and $\alpha = \beta = \frac{1}{20}$ for Example 2.

The absolute errors (AEs) at $\alpha = \beta = \dfrac{1}{20}$ and $N = 10$ are presented in Figure 6 which showed that the results of HPM with shifted Jacobi polynomials to be more accurate results than the results of HPM with Legendre, Chebyshev and Taylor polynomials (Behrooz and Ebadi, 2011) represented in Figure 7.

Example 3

Consider the equation (Behrooz and Ebadi, 2011; Novin and Dastjerd, 2015),

$$u'' + 3u - 2u^3 = \cos(x)\sin(2x), \tag{37}$$

$$u(0) = 0, \quad u'(0) = 1, \quad 0 \le x \le 1. \tag{38}$$

The exact solution of this problem is $u(x) = sin(x)$. In an operator form, Equation 37 can be written as:

$$L(u) + R(u) + N(u) = f(x), \tag{39}$$

where $L = \dfrac{d}{dx^2}$, $R(u) = 3u$, $N(u) = -2u^3$ and $f(x) = \cos(x)\sin(2x)$.

The homotopy equation is:

$$v'' - L(u_0) + p\,L(u_0) + p[R(v) + N(v) - f(x)] = 0. \tag{40}$$

Similar to the previous two examples, if $\alpha = -0.01, \beta = -0.005$ and $N = 7$, we get

$$
\begin{aligned}
f_{(J,7)}(x) \approx & \ 0.5196100000 \times 10^{-5} + 2.000385742x - 0.6852219400 \times 10^{-2} x^2 \\
& - 2.282885057 x^3 - 0.1876329186 x^4 + 1.396156963 x^5 \\
& - 0.4115735108 x^6 - 0.01630481208 x^7.
\end{aligned} \tag{41}
$$

The approximate solutions

$$
\begin{aligned}
u(x) = & \ x - 0.2598050000 \times 10^{-5} x^2 - 0.1666023763 x^3 - 0.5703687707 - 10^{-3} x^4 \\
& + \cdots - 4.353096806 \times 10^{-20} x^{47} - 6.612018552 \times 10^{-22} x^{48} - 3.742005363 \times 10^{-24} x^{49}.
\end{aligned} \tag{42}
$$

In Table 1, we compare the AEs achieved using our method with those obtained using the ADM with Legendre and Taylor polynomials (Novin and Dastjerd, 2015) at $N = 7$. Figure 8 plot the AEs of the HPM with shifted Jacobi polynomials, which show that our method
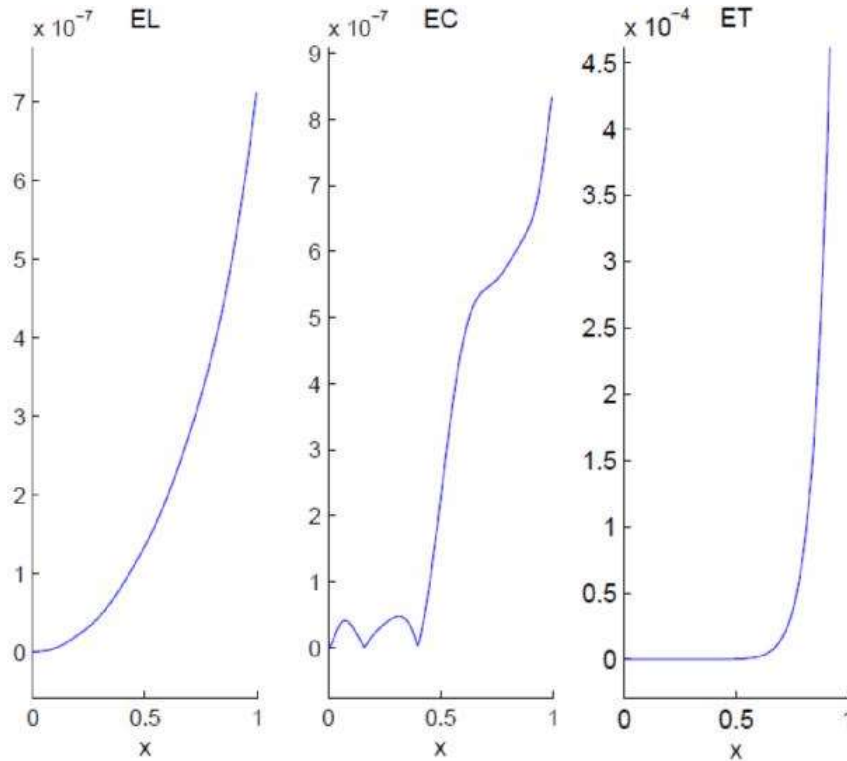
Figure 7. AEs of HPM by Legendre (EL), Chebyshev (EC) and Taylor (ET) polynomials for Example 2.
Source: Behrooz and Ebadi (2011).

Table 1. Comparison of our method with the ADM with Legendre and Taylor polynomials (Novin and Dastjerd, 2015) at $\alpha = -0.01$, $\beta = -0.005$ and $N = 7$ for Example 3.

x	Our method	ADM with Legendre polynomials	ADM with Taylor polynomials
0.2	1.9×10^{-9}	2.012×10^{-9}	5×10^{-12}
0.4	1×10^{-10}	1.16×10^{-10}	1.0224×10^{-8}
0.6	6×10^{-10}	1.951×10^{-9}	8.70197×10^{-7}
0.8	2.7×10^{-9}	4.3981×10^{-7}	1.9931859×10^{-5}
1	2.50×10^{-8}	1.36581×10^{-6}	$2.32196948 \times 10^{-4}$

is more accurate than HPM with Chebyshev and Taylor polynomials (Behrooz and Ebadi, 2011) shown in Figure 9.

From Table 1, Figures 8 and 9 listed above, it is shown that the method here is surpassed than ADM with Legendre and Taylor polynomials introduced by Novin and Dastjerd (2015) and the HPM with Chebyshev and Taylor polynomials introduced by Behrooz and Ebadi (2011).

perturbation algorithm has been proposed to find an accurate numerical solution for the nonlinear ordinary differential equations. The core of the proposed method was the source term that can be expressed in the shifted Jacobi series. By comparing the approximate solutions of the problems in this research with their exact solutions and with the approximate solutions achieved by other methods, the validity and accuracy of the scheme of this research is confirmed.

Conclusions

In this work, a generalization to the homotopy

CONFLICT OF INTERESTS

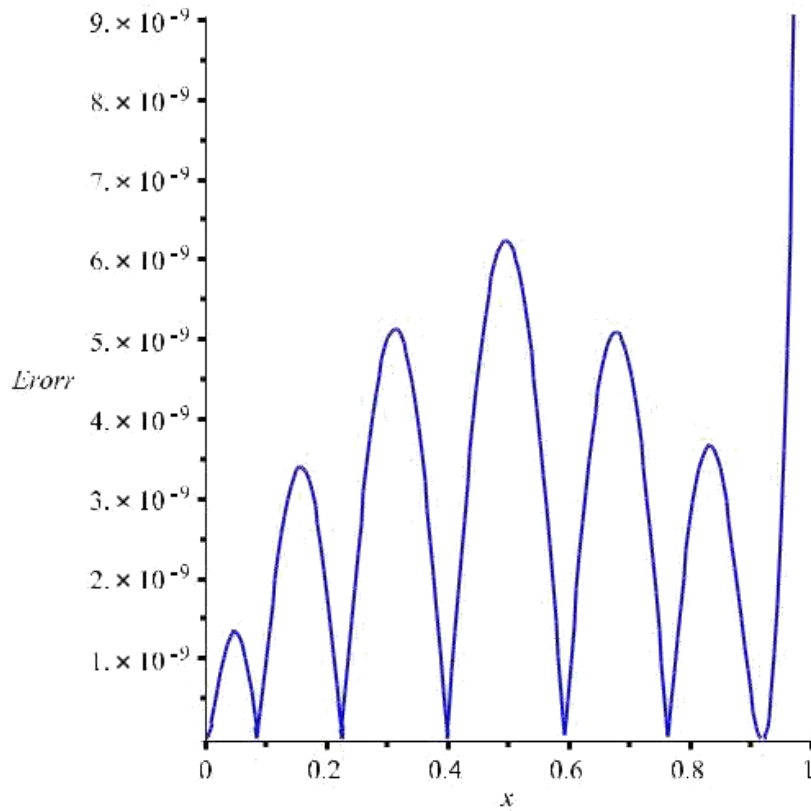The authors have not declared any conflict of interests.

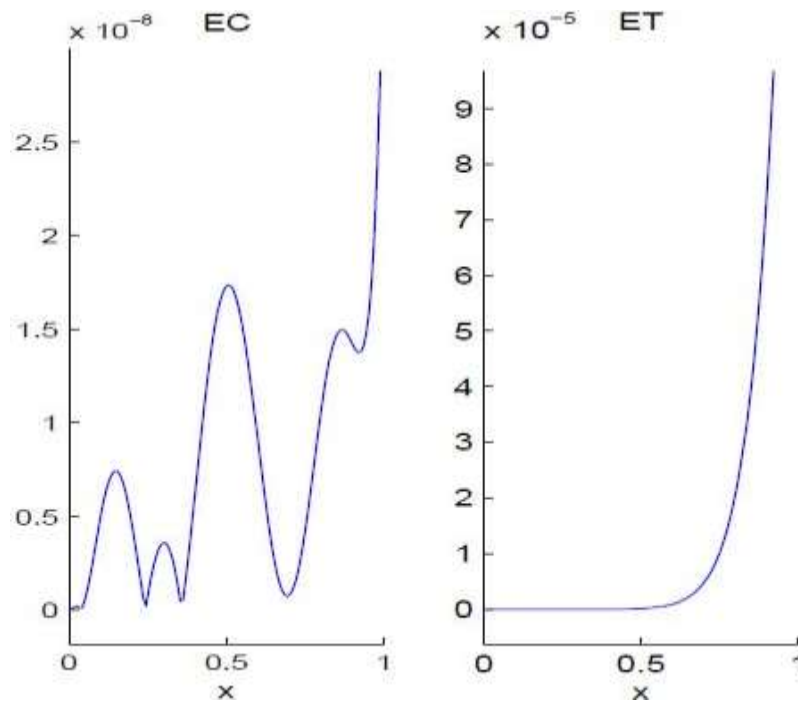**Figure 8.** AEs of HPM by shifted Jacobi polynomials at $\alpha = -0.01, \beta = -0.005$ and $N = 7$ for Example 3.

Figure 9. AEs of HPM by Chebyshev (EC) and Taylor (ET) polynomials for Example 3.
Source: Behrooz and Ebadi (2011).

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