

*Full Length Research Paper*

# Solutions of the Klein-Gordon equation for $I \neq 0$ with position-dependent mass for modified Eckart potential plus Hulthen potential

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In this paper we solve analytical the position-dependent effective mass Klein–Gordon equation for modified Eckart potential plus Hulthen potential with unequal scalar and vector potential for  $I \neq 0$ . The Nikiforov-Uvarov (NU) method is used to obtain the energy eigenvalues and wave functions. We also discuss the energy eigenvalues and wave functions for the constant-mass case. The wave functions of the system are taken in the form of the Laguerre polynomials. The results are the exact analytical. The energy eigenvalues and wave functions are interesting for experimental physicists.

**Key words:** Klein–Gordon equation, modified Eckart potential plus Hulthen potential, Nikiforov-Uvarov (NU) method, position-dependent mass.

## INTRODUCTION

The description of phenomena at higher energy requires the investigation of a relativistic wave equation. Therefore one of the interesting problems in nuclear and high energy physics is to obtain analytical solution of the Klein - Gordon, Duffin – Kemmer - Petiau and Dirac equations for mixed vector and scalar potentials (Oyewumi and Akoshile, 2010). The exact solutions of the wave equations (non-relativistic or relativistic) are very important since they contain all the necessary information regarding the quantum system under consideration. However, analytical solutions are possible only in a few simple cases such as the hydrogen atom and the harmonic oscillator (Schiff, 1955; Landau and Lifshitz, 1977).

If we consider the case where the interaction potential is not strong enough to create particle-antiparticle pairs,

we can apply the Klein-Gordon equation to the treatment of a zero-spin particle and apply the Dirac equation to that of a 1/2-spin particle (Cheng and Dai, 2007). Spin and pseudospin symmetries are  $SU(2)$  symmetries of a Dirac Hamiltonian with vector and scalar potentials. They are realized when the difference,  $\Delta(r)=V(r)-S(r)$ , or the sum,  $\Sigma(r)=V(r)+S(r)$ , are constants. The near realization of these symmetries may explain degeneracies in some heavy meson spectra (spin symmetry) or in single-particle energy levels in nuclei (pseudospin symmetry), when these physical systems are described by relativistic mean-field theories (RMF) with scalar and vector potentials (Ginocchio, 2005; Feizi et al., 2013; Alberto et al., 2013).

The kind of various methods have been used for the exact solutions of the Klein – Gordon equation and Dirac

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equation such as the Supersymmetry Quantum Mechanics (Jia et al., 2006; Astorga et al., 2012; Feizi et al., 2011). Asymptotic iteration method (AIM) (Ciftci et al., 2003; O'zer and Le'vai, 2012) and Nikiforov-Uvarov (NU) (Shojaei et al., 2014; Berkdemir et al., 2006; Rajabi and Hamzavi, 2013) and others.

The Klein – Gordon and Dirac wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics with some typical potential by using different methods (Ikot et al., 2011). For example, Kratzer potential (Qiang, 2004, 2003), Woods-Saxon potential (Berkdemir et al., 2006; Guo and Sheng, 2005). Scarf potential (Xue-Cai et al., 2005; Zhang et al., 2005). 16 Hartmann potential (Chen, 2005; de Souza Dutra and Hott, 2006), Rosen Morse potential, (Yi et al., 2004; Alhaidari, 2001) and Hulthen potential (Farrokh et al., 2013).

The problem of the non-relativistic and relativistic wave equations with spatially dependent masses has been attracting much attention in the literature. Systems with position-dependent mass have been found to be very useful in studying the physical properties of various microstructures, such as semiconductor heterostructure (VonRoos, 1983). Quantum liquids (Arias de Saavedra et al., 1994), quantum wells and quantum dots (Serra and Lipparini, 1997), 3He clusters (Barranco et al., 1997), compositionally graded crystals (Geller and Kohn, 1993; Jia et al., 2012) etc.

A lot of studies have been performed to obtain the solutions of the Schrodinger, Klein-Gordon and Dirac equations with position-dependent mass for different potentials (Arda et al., 2010, 2009). For example, Aygun et al. (2012), Jia et al. (2012), Antia et al. (2012) and Souza Dutra considered position-dependent effective mass (Jia and de Souza Dutra, 2006).

In this paper, we attempt to solve approximately Klein – Gordon equation for  $l \neq 0$  with modified Eckart potential plus Hulthen potential for the scalar and vector potential with a spatially dependent mass by using the Nikiforov – Uvarov (NU) method. We also discuss the limit of the scalar and vector potential with constant mass (Shojaei et al., 2014; Cheng and Dai, 2007).

## REVIEW of NIKIFOROV -UVAROV (NU) METHOD

The NU method is based on the solution of a generalized second order linear differential equation with special orthogonal functions. The NU method has been used to solve the Schrodinger, Dirac, and Klein-Gordon wave equations for a certain kind of potential. In this method the differential equations can be written as follows (Shojaei et al., 2014; Cheng and Dai, 2007):

$$\Psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \Psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \Psi(s) = 0 \quad (1)$$

Where  $\sigma(s)$  and  $\tilde{\sigma}(s)$  are words of second degree and

$\tilde{\tau}(s)$  is a first degree polynomials. In obtaining the exact solution to Equation (1) we set the wave function as:

$$\Psi(s) = \phi(s)y(s) \quad (2)$$

And on substituting Equation (2) into Equation (1) reduces Equation (1) into hyper geometric type,

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \quad (3)$$

Where  $\Phi(s)$  is defined as a logarithmic derivative

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \quad (4)$$

Where  $\pi(s)$  is most a first degree polynomial.

The other part,  $y(s)$ , is the hyper geometric-type function whose polynomial solutions are given by the Rodrigues relation.

$$y_n(s) = \frac{B_n}{\rho_n} \frac{d^n}{ds^n} (\sigma^n(s)\rho(s)) \quad (5)$$

Where  $B_n$  is the normalizing constant and the weight function  $\rho(s)$  must satisfy the following condition:

$$\frac{d\omega(s)}{ds} = \frac{\tau(s)}{\sigma(s)} \omega(s) \quad (6)$$

Where  $\omega(s) = \sigma(s)\rho(s)$ .

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \quad (7)$$

It is necessary that the classical orthogonal polynomials  $\tau(s)$  be equal to zero to some point of an interval  $(a, b)$  and its derivative at this interval at  $\sigma(s) > 0$  will be negative, that is

$$\frac{d\tau(s)}{ds} < 0 \quad (8)$$

Therefore, the function  $\pi(s)$  and the parameter  $\lambda$  required for the NU-method are defined as follows:

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)}, \quad (9)$$

$$\lambda = k + \pi'(s)$$

The  $k$ -values in the square-root of Equation (9) are possible to evaluate if the expression under the square root must be square of polynomials. This is possible if its discriminant is zero. Thus, a new eigenvalue equation

for the second-order differential equation becomes:

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad (n = 0, 1, 2, \dots) \quad (10)$$

Where  $\tau(s)$  is as defined in Equation (7) and on comparing Equations (9) and (10), we obtain the energy eigenvalues.

**SOLUTIONS OF THE KLEIN-GORDON EQUATION FOR  $l \neq 0$**

The three-dimensional radial arbitrary l-state K-G equation with position-dependent mass is written as follows (Greiner, 2000):

$$\frac{d^2U(r)}{dr^2} + \frac{1}{\hbar^2c^2} \{ [E - V(r)]^2 - [Mc^2 + S(r)]^2 - \frac{\ell(\ell+1)}{r^2} \hbar^2c^2 \} U(r) = 0 \quad (11)$$

Where M is the rest mass, E is the relativistic energy, c is the speed of light,  $\hbar$  is the reduced Planck's constant, V(r) and S(r) are vector and scalar potentials, respectively. From Equation (11), we have

$$\frac{d^2U(r)}{dr^2} + \frac{1}{\hbar^2c^2} \{ E^2 - M^2c^4 - 2EV(r) - 2Mc^2S(r) + V^2(r) - S^2(r) - \frac{\ell(\ell+1)}{r^2} \hbar^2c^2 \} U(r) = 0 \quad (12)$$

Vector and scalar the potential by investigation modified Eckart potential plus Hulthen potential are respectively written as:

$$V(r) = \coth(\alpha r) + \frac{v_0}{(1 - e^{-2\alpha r})}, \quad S(r) = \coth(\alpha r) + \frac{s_0}{(1 - e^{-2\alpha r})} \quad (13)$$

Where  $s_0, v_0$ , the potential depth, and  $\alpha$  are constant. We emblazon the position-dependent mass in the specific form:

$$M(r) = m_0 + m_1 \coth(\alpha r) + m_2 \frac{1}{(1 - e^{-2\alpha r})} \quad (14)$$

If we define a new variable  $U(r) = rR(r)$  and substituting it in to Equation (12), we obtain the radial equation of Klein – Gordon equation as

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \frac{1}{\hbar^2c^2} \{ E^2 - M^2c^4 - 2EV(r) - 2Mc^2S(r) + V^2(r) - S^2(r) - \frac{\ell(\ell+1)}{r^2} \hbar^2c^2 \} R(r) = 0 \quad (15)$$

We can evaluate the new improved approximation scheme by using the following pekeris-type approximation that is valid for  $\alpha \leq 1$ , (Hill, 1954).

$$\frac{1}{r^2} \approx \frac{4\alpha^2}{(e^{-2\alpha r} - 1)^2} \quad (16)$$

Using the transformation  $s = (1 - \exp(-2\alpha r))$  Equation (15) brings into the form

$$\frac{d^2R(r)}{dr^2} + \frac{(-4\alpha)}{s} \frac{dR(r)}{dr} + \frac{1}{\hbar^2c^2} \left\{ E^2 + \frac{(2-s)^2}{s^2} + \frac{v_0^2}{s^2} + \frac{2(2-s)v_0}{s^2} - \frac{2E(2-s)}{s} - \frac{2Ev_0}{s} - m_0^2c^4 - \frac{[m_1c^2(2-s) + m_2c^2]^2}{s^2} - \frac{2m_0c^2[m_1c^2(2-s) + m_2c^2]}{s} - \frac{(2-s)^2}{s^2} - \frac{s_0^2}{s^2} - \frac{2(2-s)s_0}{s^2} - \frac{2m_0c^2(2-s)}{s} - \frac{2m_1c^2(2-s)^2}{s^2} - \frac{2m_2c^2(2-s)}{s^2} - \frac{2m_0c^2s_0}{s} - \frac{2m_1c^2s_0(2-s)}{s^2} - \frac{2m_2c^2s_0}{s^2} - \frac{4\alpha^2\ell(\ell+1)}{s^2} \hbar^2c^2 \right\} R(r) = 0 \quad (17)$$

We can write the Equation (17) as summarized:

$$R'' + \frac{(-4\alpha)}{s} R' + \frac{1}{s^2} [A's^2 + B's + c'] R = 0 \quad (18)$$

Where the parameters A', B' and C' are defined as follows:

$$A' = \frac{1}{\hbar^2c^2} \{ (E^2 + 2E) - \gamma \}$$

$$B' = \frac{1}{\hbar^2c^2} (aE + b) \quad (19a)$$

$$C' = \frac{1}{\hbar^2c^2} \{ (v_0 + 2)^2 - (s_0 + 2)^2 - (2m_1c^2 + m_2c^2)(2m_1c^2 + m_2c^2 + 2s_0 + 4) \} - 4\alpha^2\ell(\ell+1)$$

And,

$$\gamma = (m_0c^2 - m_1c^2)(m_0c^2 - m_1c^2 - 2)$$

$$a = -(4 + 2v_0) \quad (19b)$$

$$b = 2m_1c^2(m_1c^2 + m_2c^2 - 2m_0c^2 + s_0 + 4) - 2m_0c^2(m_2c^2 + s_0 + 2) + 2(s_0 - v_0 + m_2c^2)$$

Now we use the NU method to solve the Equation (18). Comparing Equation (1) and (18) we get

$$\tilde{\tau} = -4\alpha, \quad \sigma(s) = s, \quad \tilde{\sigma}(s) = A's^2 + B's + C' \quad (20)$$

Substituting the expressions above into Equation (9), we have the function  $\pi(s)$  as

$$\pi(s) = 1 \pm [(-A')^{1/2} s \pm (1 - C')] \quad \text{for } k = B \pm 2\sqrt{A'(C' - 1)} \quad (21)$$

With respect to Condition of Equation (8), the best choice for k and  $\pi$  as follows:

$$\pi(s) = 1 - \left[ (-A')^{1/2} s - (1-C') \right] \text{ for } k = B - 2\sqrt{A'(C'-1)} \quad (22)$$

By following the equation  $\lambda = \lambda_n$  in the NU method, we have the energy eigenvalues equation

$$B'^2 = [(2n+1) + 2(1-C')^{1/2}]^2 (-A') \quad (23)$$

And use Equation (19a) and Equation (23)

$$\begin{aligned} a^2 E^2 + b^2 + 2abE &= -\alpha' \hbar^2 c^2 (E^2 + 2E - \gamma), \\ \alpha' &= [(2n+1) + 2(1-C')^{1/2}]^2 \end{aligned} \quad (24)$$

By solving the Equation (24) the exact energy eigenvalues of the K-G equation for this system are derived as:

$$E_{n,\ell} = \frac{-(ab + \alpha' \hbar^2 c^2) \pm \sqrt{\alpha' \hbar^2 c^2 [(1+\gamma)(\alpha' \hbar^2 c^2 + a^2) - (a-b)]}}{2(\alpha' \hbar^2 c^2 + a^2)} \quad (25)$$

Let us now find the corresponding eigenfunctions for this system. Firstly, we find the first part of the eigenfunctions by Using Equation (4)

$$\phi(s) = s^{[1+(1-c')^{1/2}]} \exp[-(-A')^{1/2} s] \quad (26)$$

Secondly, we calculate the weight function as

$$\rho(s) = s^{2[2\alpha + \frac{1}{2} + (1-c')^{1/2}]} \exp[-2(-A')^{1/2} s] \quad (27)$$

Which the second part of the wave function gives by Equation (5) as

$$y_n = B_n s^{-2[2\alpha + \frac{1}{2} + (1-c')^{1/2}]} \exp[2(-A')^{1/2} s] \frac{d^n}{ds^n} [s^{n+2[2\alpha + \frac{1}{2} + (1-c')^{1/2}]} \exp[-2(-A')^{1/2} s]] \quad (28)$$

By using terms of the generalized Laguerre polynomials

$$L_n^k(x) = \frac{e^x x^{-k}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+k}) \quad (\text{Auvil and Brown, 1978})$$

and using Equation (2) we have

$$R(s) = B_n s^{[1+(1-c')^{1/2}]} \exp[-(-A')^{1/2} s] n! L_n^k(2(-A')^{1/2} s) \quad (29)$$

By using  $U(s) = rR(s)$  we find

$$U(r) = Nr [1 - \exp(-2\alpha r)]^{[1+(1-c')^{1/2}]} \exp[-(-A')^{1/2} (1 - \exp(-2\alpha r))] n! L_n^k \quad (30)$$

Where N is the normalization constant. We have obtained the energy eigenvalues and the wave function of the radial K-G equation for modified Eckart potential plus Hulthen potential with scalar and vector potential for  $l \neq 0$ .

## A SOME SPECIAL CASES

Here, we consider some special cases of interest if we consider spatially independent mass and spin symmetry we have

$$M(r) = m_0, \quad m_1 = m_2 = 0 \quad (31)$$

$$V(r) = S(r), \quad v_0 = s_0 \quad (32)$$

We have from Equation (19a) and (19b) the following equations.

$$\begin{aligned} A'' &= E^2 + 2E - \gamma', \quad \gamma' = m_0(m_0 + 1) \\ B'' &= a'E + b', \quad a' = -2(v_0 + 2), \quad b' = -2m_0(v_0 + 2) \end{aligned} \quad (33)$$

$$C'' = -4\alpha^2 \ell(\ell + 1)$$

And we have energy eigenvalues by

$$\begin{aligned} E_{n,\ell} &= \frac{-(4m_0(2 + v_0)^2 + \alpha') \pm \alpha'(m_0 - 1)}{4(2 + v_0)^2 + \alpha'}, \\ \alpha' &= [(2n+1) + 2(1-C'')^{1/2}]^2 \end{aligned} \quad (34)$$

Thus, the wave function can be written as,

$$U(r) = Nr [1 - \exp(-2\alpha r)]^{[1+(1-c'')^{1/2}]} \exp[-(-A'')^{1/2} (1 - \exp(-2\alpha r))] n! L_n^k \quad (35)$$

We have obtained the energy eigenvalues and the wave function of the radial K-G equation for independent mass and spin symmetry for  $l \neq 0$ .

## CONCLUSIONS

In this paper, we have discussed approximately the solutions of the Klein - Gordon equation for modified Eckart potential plus Hulthen potential with scalar and vector potential for  $l \neq 0$  and position-dependent mass. We can obtain the energy eigenvalues and the wave function in terms of the generalized Laguerre polynomials functions via the Nikiforov-Uvarov method. We have also considered the limiting cases of spin symmetry and position-independent mass to obtain the energy eigenvalues and the wave function. We can conclude that our results are interesting for experimental physicists, because the results are exact, more general and useful to study nuclear scattering, nuclear and particle physics.

## Conflict of Interest

The authors have not declared any conflict of interest.

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