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Full Length Research Paper

Solution of sixth-order boundary-value problems by collocation method using Haar wavelets

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A new method based on uniform Haar wavelets is proposed for the numerical solution of sixth-order two-point boundary value problems (BVPs) in ordinary differential equations. Numerical examples are given to illustrate the practical usefulness of present approach. Accuracy and efficiency of the suggested method is established through comparison with the existing spline based technique and variational iteration method. Haar wavelets have useful properties like simple applicability, orthogonality and compact support. In comparison the beauty of other wavelets like Walsh wavelet functions and wavelets of high order spline basis is overshadowed by computational cost of the algorithm. In the case of Haar wavelets, more accurate solutions can be obtained by increasing the level in the Haar wavelet. The main advantage of this method is its efficiency and simple applicability.

Key words: Sixth-order boundary-value problem (BVP), Haar wavelets, ordinary differential equations, dynamo action.

INTRODUCTION

In this paper, we consider linear and nonlinear sixth-order boundary-value problems (BVPs) of the form:

$$y^{(6)}(x) = f(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}); a < x < b, \quad (1)$$

subject to the following two types of boundary conditions

Type I

$$y(a) = \alpha_1,\ y(b) = \alpha_2,\ y^{(2)}(a) = \alpha_3,\ y^{(2)}(b) = \alpha_4,\ y^{(4)}(a) = \alpha_5,\ y^{(4)}(b) = \alpha_6. \quad (2)$$

Type II

$$y(a) = \beta_1,\ y(b) = \beta_2,\ y^{(1)}(a) = \beta_3,\ y^{(1)}(b) = \beta_4,\ y^{(2)}(a) = \beta_5,\ y^{(2)}(b) = \beta_6. \quad (3)$$

Where y and f are continuous functions defined in the interval $x \in [a, b]$, $f \in C^6[a, b]$ is real and $\alpha_i, \beta_i, i = 1, 2, ..., 6$ are finite real numbers. Such problems are known to arise in astrophysics, the narrow convecting layers bounded by stable layers, which are believed to surround A-type stars, may be modeled by sixth-order BVPs (Toomre et al., 1976). Dynamo action in some stars may be calculated by such equations (Glatzmaier, 1985). Chandrasekhar (1981) determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is at ordinary convection, the ordinary differential equation is sixth-order.

Theorems which list the conditions for the existence and uniqueness of solutions of sixth-order BVPs are thoroughly discussed in the book (Agarwal, 1986). The literature on the numerical solution of sixth-order BVP is sparse. Some of the methods include finite difference method (Twizell, 1988; Twizell and Boutayeb, 1990), various forms of splines (Siddiqi and Twizell, 1996;

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Siddiqi et al., 2007; Islam et al., 2008), variational iteration method (VIM) (Noor et al., 2009) and sinc-Galarkin method (Gamel et al., 2003). Decomposition and modified domain decomposition methods were applied by Wazwaz (2001) to find solution of such BVPs.

Recently, wavelet approach is becoming more popular in numerical approximations. Different types of wavelets have been used in this context. A short survey on Haar wavelets is given in Lepik (2005, 2007), Hsiao and Wang (2001) and Hsiao (2004). Recently, Islam et al. (2010) used Haar wavelets for the numerical solution of BVPs and Aziz et al. (2011) used them for numerical integration. Haar wavelets have useful properties like simple applicability, orthogonality and compact support. In comparison, the beauty of other wavelets like Walsh wavelet functions and wavelets of high order spline basis is overshadowed by computational cost of the algorithm.

HAAR WAVELETS

The Haar wavelet family for $x \in [0,1)$ is defined as:

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\alpha, \beta) \\ -1 & \text{for } x \in [\beta, \gamma) \\ 0 & \text{elsewhere} \end{cases} \tag{4}$$

Where

$$\alpha = \frac{k}{m}, \beta = \frac{k+0.5}{m}, \gamma = \frac{k+1}{m} \tag{5}$$

In the above definition, integer $m = 2^j,\ j = 0,1,2,\dots,J$, J indicates the level of the wavelet and integer $k = 0,1,2,\dots,m-1$ is the translation parameter. Maximum level of resolution is J. The index *i* in Equation 4 is calculated using the formula $i = m + k + 1$. In case of minimal values $m = 1, k = 0,$ we have $i = 2$. The maximal value of i is, $i = 2M = 2^{J+1}$. For $i = 1$, the function $h_1(x)$ is the scaling function for the family of Haar wavelets which is defined as:

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0,1) \\ 0 & \text{elsewhere} \end{cases} \tag{6}$$

We introduce the notations

$$P_{i,1}(x) = \int_0^x h_i(z)\,dz \tag{7}$$

$$P_{i,\nu+1}(x) = \int_0^x P_{i,\nu}(z)\,dz,\ \ \nu = 1,2,\dots \tag{8}$$

Evaluating these integrals using Equation 4, we obtain

$$P_{i,1}(x) = \begin{cases} x-\alpha & \text{for } x \in [\alpha, \beta) \\ \gamma - x & \text{for } x \in [\beta, \gamma) \\ 0 & \text{elsewhere} \end{cases} \tag{9}$$

$$P_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2 & \text{for } x \in [\alpha, \beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2 & \text{for } x \in [\beta, \gamma) \\ \frac{1}{4m^2} & \text{for } x \in [\gamma, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{10}$$

$$P_{i,3}(x) = \begin{cases} \frac{1}{6}(x-\alpha)^3 & \text{for } x \in [\alpha, \beta) \\ \frac{1}{4m^2}(x-\beta) - \frac{1}{6}(\gamma - x)^3 & \text{for } x \in [\beta, \gamma) \\ \frac{1}{4m^2}(x-\beta) & \text{for } x \in [\gamma, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{11}$$

$$P_{i,4}(x) = \begin{cases} \frac{1}{24}(x-\alpha)^4 & \text{for } x \in [\alpha, \beta) \\ \frac{1}{8m^2}(x-\beta)^2 - \frac{1}{24}(\gamma - x)^4 + \frac{1}{192m^4} & \text{for } x \in [\beta, \gamma) \\ \frac{1}{8m^2}(x-\beta)^2 + \frac{1}{192m^4} & \text{for } x \in [\gamma, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{12}$$

$$P_{i,5}(x) = \begin{cases} \frac{1}{120}(x-\alpha)^5 & \text{for } x \in [\alpha, \beta) \\ \frac{1}{24m^2}(x-\beta)^3 + \frac{1}{120}(\gamma - x)^5 + \frac{1}{192m^4}(x-\beta) & \text{for } x \in [\beta, \gamma) \\ \frac{1}{24m^2}(x-\beta)^3 + \frac{1}{192m^4}(x-\gamma) + \frac{1}{384m^5} & \text{for } x \in [\gamma, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{13}$$

$$P_{i,6}(x) = \begin{cases} \frac{1}{720}(x-\alpha)^6 & \text{for } x \in [\alpha, \beta) \\ \frac{1}{96m^2}(x-\beta)^4 - \frac{1}{720}(\gamma - x)^6 + \frac{1}{384m^4}(x-\beta)^2 + \frac{1}{23040m^6} & \text{for } x \in [\beta, \gamma) \\ \frac{1}{96m^2}(x-\beta)^4 + \frac{1}{192m^4}(x-\gamma)^2 + \frac{1}{384m^5}(x-\gamma) + \frac{1}{1440m^6} & \text{for } x \in [\gamma, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{14}$$

We also introduce the following notation:

$$C_{i,\nu}(x) = \int_0^1 P_{i,\nu}(x)\,dx\,,\ \nu = 1,2,\dots \tag{15}$$

Any function $f(x)$ which is square integrable in the interval (0,1) can be expressed as an infinite sum of Haar wavelets as:

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x) \tag{16}$$

The above series terminates at finite terms if $f(x)$ is piecewise constant or can be approximated as piecewise constant during each subinterval.

METHOD OF SOLUTION

We assume that

$$y^{(6)}(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{17}$$

Equation 17 is integrated repeatedly with suitable limits of integration depending upon the boundary conditions. In this way, we express the solution $y(x)$ and its first six derivatives in terms of Haar functions and their integrals. We consider the collocation points

$$x_j = \frac{j-0.5}{2M},\ j = 1, 2, \ldots, 2M. \tag{18}$$

The expressions of $y(x), y^{(1)}(x), \ldots, y^{(6)}(x)$ are substituted in the given differential equation and discretization is applied using the collocation points given in Equation 18. Thus, we obtain a system of $2M$ equations in $2M$ unknowns. The Haar coefficients $a_i,\ i = 1, 2, \ldots, 2M$ are calculated by solving this system. The approximate solution can be easily recovered with the help of Haar coefficients. The method is further explained with the help of specific boundary conditions. We will consider two different sets of boundary conditions here. The other types of boundary conditions can be handled in a similar manner. In this study, we take $a = 0$ and $b = 1$.

Type I: Boundary conditions

$$y(0) = \alpha_1, \quad y(1) = \alpha_2, \quad y^{(2)}(0) = \alpha_3,$$

$$y^{(2)}(1) = \alpha_4, \quad y^{(4)}(0) = \alpha_5, \quad y^{(4)}(1) = \alpha_6.$$

We assume that

$$y^{(6)}(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{19}$$

Integrating and using boundary conditions, we obtain the following:

$$y^{(5)}(x) = \alpha_6 - \alpha_5 + \sum_{i=1}^{2M} a_i (P_{i,1}(x) - C_{i,1}), \tag{20}$$

$$y^{(4)}(x) = \alpha_5 + (\alpha_6 - \alpha_5)x + \sum_{i=1}^{2M} a_i (P_{i,2}(x) - xC_{i,1}) \tag{21}$$

$$y^{(3)}(x) = \alpha_4 - \alpha_3 - \tfrac{1}{3}\alpha_5 - \tfrac{1}{6}\alpha_6 + \alpha_5 x + (\alpha_6 - \alpha_5)\frac{x^2}{2} + \sum_{i=1}^{2M} a_i \left(P_{i,3}(x) - \left(\frac{x^2}{2} - \frac{1}{6}\right)C_{i,1} - C_{i,3}\right) \tag{22}$$

$$y^{(2)}(x) = \alpha_3 + \left(\alpha_4 - \alpha_3 - \tfrac{1}{3}\alpha_5 - \tfrac{1}{6}\alpha_6\right)x + \alpha_5\frac{x^2}{2} + (\alpha_6 - \alpha_5)\frac{x^3}{6} + \sum_{i=1}^{2M} a_i \left(P_{i,4}(x) - \left(\frac{x^3}{6} - \frac{1}{6}x\right)C_{i,1} - xC_{i,3}\right) \tag{23}$$

$$y^{(1)}(x) = \alpha_2 - \alpha_1 - \tfrac{1}{3}\alpha_3 - \tfrac{1}{6}\alpha_4 + \tfrac{1}{45}\alpha_5 + \tfrac{7}{360}\alpha_6 + \alpha_3 x + \left(\alpha_4 - \alpha_3 - \tfrac{1}{3}\alpha_5 - \tfrac{1}{6}\alpha_6\right)\frac{x^2}{2} + \alpha_5\frac{x^3}{6} + (\alpha_6 - \alpha_5)\frac{x^4}{24} + \sum_{i=1}^{2M} a_i \left(P_{i,5}(x) - \left(\frac{x^4}{24} - \frac{x^2}{12} + \frac{7}{360}\right)C_{i,1} - \left(\frac{x^2}{2} - \frac{1}{6}\right)C_{i,3} - C_{i,5}\right), \tag{24}$$

$$y(x) = \alpha_1 + \left(\alpha_2 - \alpha_1 - \tfrac{1}{3}\alpha_3 - \tfrac{1}{6}\alpha_4 + \tfrac{1}{45}\alpha_5 + \tfrac{7}{360}\alpha_6\right)x + \alpha_3\frac{x^2}{2} + \left(\alpha_4 - \alpha_3 - \tfrac{1}{3}\alpha_5 - \tfrac{1}{6}\alpha_6\right)\frac{x^3}{6} + \alpha_5\frac{x^4}{24} + (\alpha_6 - \alpha_5)\frac{x^5}{120} + \sum_{i=1}^{2M} a_i \left(P_{i,6}(x) - \left(\frac{x^5}{120} - \frac{x^3}{36} + \frac{7}{360}x\right)C_{i,1} - \left(\frac{x^3}{6} - \frac{1}{6}x\right)C_{i,3} - xC_{i,5}\right). \tag{25}$$

Type II: Boundary conditions

$$y(0) = \beta_1, y(1) = \beta_2,\ y^{(1)}(0) = \beta_3,\ y^{(1)}(1) = \beta_4, \quad y^{(2)}(0) = \beta_5,\ y^{(2)}(1) = \beta_6.$$

The numerical solution $y(x)$ and its derivatives in this case can be expressed as:

$$y^{(6)}(x) = \sum_{i=1}^{2M} a_i h_i(x), \tag{26}$$

$$y^{(5)}(x) = y^{(5)}(0) + \sum_{i=1}^{2M} a_i P_{i,1}(x), \tag{27}$$

$$y^{(4)}(x) = y^{(4)}(0) + xy^{(5)}(0) + \sum_{i=1}^{2M} a_i P_{i,2}(x), \tag{28}$$

$$y^{(3)}(x) = y^{(3)}(0) + xy^{(4)}(0) + \frac{x^2}{2}y^{(5)}(0) + \sum_{i=1}^{2M} a_i P_{i,3}(x), \tag{29}$$

$$y^{(2)}(x) = \beta_5 + xy^{(3)}(0) + \frac{x^2}{2}y^{(4)}(0) + \frac{x^3}{6}y^{(5)}(0) + \sum_{i=1}^{2M} a_i P_{i,4}(x), \tag{30}$$

$$y^{(1)}(x) = \beta_3 + \beta_5 x + \frac{x^2}{2}y^{(3)}(0) + \frac{x^3}{6}y^{(4)}(0) + \frac{x^4}{24}y^{(5)}(0) + \sum_{i=1}^{2M} a_i P_{i,5}(x), \tag{31}$$

$$y(x) = \beta_1 + \beta_3 x + \beta_5\frac{x^2}{2} + \frac{x^3}{6}y^{(3)}(0) + \frac{x^4}{24}y^{(4)}(0) + \frac{x^5}{120}y^{(5)}(0) + \sum_{i=1}^{2M} a_i P_{i,6}(x), \tag{32}$$

The unknown values $y^{(3)}(0), y^{(4)}(0)$ and $y^{(5)}(0)$ can be calculated using boundary conditions and are given by:

$$y^{(3)}(0) = 20\beta_1 - 20\beta_2 + 12\beta_3 + 8\beta_4 + 3\beta_5 - \beta_6 - \sum_{i=1}^{2M} a_i (C_{i,3} - 8C_{i,4} + 20C_{i,5}), \tag{33}$$

$$y^{(4)}(0) = 30\beta_1 - 30\beta_2 + 16\beta_3 + 14\beta_4 + 3\beta_5 - 2\beta_6 - 12\sum_{i=1}^{2M} a_i (2C_{i,3} - 14C_{i,4} + 30C_{i,5}), \tag{34}$$

$$y^{(5)}(0) = 12\beta_1 - 12\beta_2 + 6\beta_3 + 6\beta_4 + \beta_5 - \beta_6 - 60\sum_{i=1}^{2M} a_i (C_{i,3} - 6C_{i,4} + 12C_{i,5}). \tag{35}$$

NUMERICAL VALIDATION

In order to demonstrate the efficiency and applicability of the new method developed in the previous section, we apply it to a number of problems from the literature. For the sake of comparison, we have taken problems from the work of Siddiqi et al. (2007) and Noor et al. (2009). Double precision arithmetic is used to reduce the round-off errors to minimum.

Example 1

Consider the linear BVP

$$y^{(6)}(x) - y(x) = -6\,e^x, \qquad 0 \le x \le 1, \tag{36}$$

subject to the boundary conditions

$$y(0) = 1,\ y^{(1)}(0) = 0,\ y^{(2)}(0) = -1, y(1) = 0,\ y^{(1)}(1) = -e,\ y^{(2)}(1) = -2e. \tag{37}$$

The exact solution is given by:

$$y(x) = (1 - x)e^x. \tag{38}$$

Table 1. Maximum absolute errors for Example 1 with boundary conditions of Type I.

2M	Present method	Siddiqi et al. (2007)
8	1.7187E-07	3.6463E-6
16	4.4137E-08	3.0209E-07
32	1.1101E-08	2.1369E-08
64	2.7778E-09	1.2289E-09
128	6.9440E-10	1.4821E-09
256	1.7361E-10	-
512	4.3404E-11	-
1024	1.0851E-11	-

Table 2. Maximum absolute errors for Example 2 with boundary conditions of Type I.

2M	Present method	Siddiqi et al. (2007)
8	9.6894E-08	1.8429E-6
16	2.4988E-08	1.3951E-07
32	6.2943E-09	9.4848E-09
64	1.5761E-09	5.6293E-10
128	3.9413E-10	6.4848E-10
256	9.8531E-11	-
512	24633E-11	-
1024	6.1582E-12	-

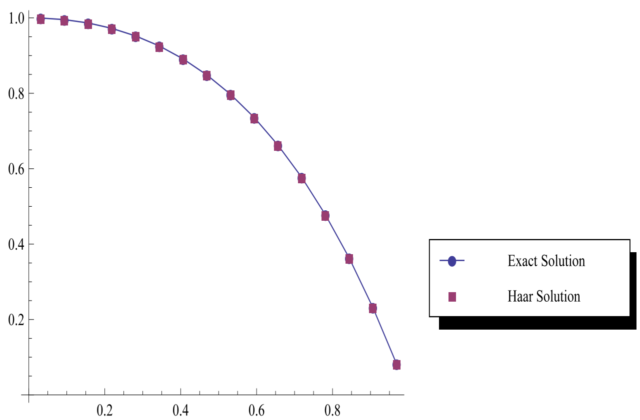

Figure 1. Uniform Haar solution of Example 1 for 2*M* = 16.

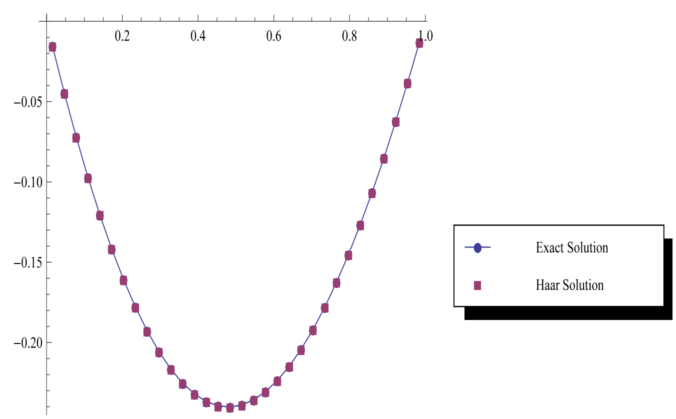

Figure 2. Uniform Haar solution of Example 2 for 2*M* = 32.

We have compared our results with quintic spline method (Siddiqi et al., 2007); the results are shown in Table 1. It is clear from the table that accuracy of the quintic spline scheme degrades at and beyond 128 number of mesh points. On the other hand, our scheme produces stable results and performs better when the number of points is increased. Figure 1 shows exact and approximate solution for $2M = 16$.

Example 2

Consider the linear BVP

$$y^{(6)}(x) - y(x) = 6\cos(x), \qquad 0 \le x \le 1, \tag{39}$$

subject to the boundary conditions

$$y(0) = 0,\ y^{(1)}(0) = -1,\ y^{(2)}(0) = 2,\ y(1) = 0,\ y^{(1)}(1) = \sin(1),\ y^{(2)}(1) = 2\cos(1). \tag{40}$$

The exact solution is given by

$$y(x) = (1 - x)\sin(x). \tag{41}$$

We have again compared our results with quintic spline method (Siddiqi et al., 2007) and are shown in Table 2. We observe similar comparison as in the case of Example 1. Figure 2 shows exact and approximate solution for $2M = 32$.

Example 3

Consider the linear BVP

$$y^{(6)}(x) = (1 + c)y^{(4)}(x) - cy^{(2)}(x) + cx, \quad 0 \le x \le 1, \tag{42}$$

subject to boundary conditions

$$y(0) = 1, y^{(1)}(0) = 1,\ y^{(2)}(0) = 0,\ y(1) = \tfrac{7}{6} + \sinh(1), y^{(1)}(1) = \tfrac{1}{2} + \cosh(1), y^{(2)}(1) = 1 + \sinh(1). \tag{43}$$

The exact solution is given by

$$y(x) = 1 + \frac{x^3}{6} + \sinh(x). \tag{44}$$

Table 3 exhibits maximum absolute errors for various

Table 3. Maximum absolute errors for various values of c in Example 3.

	Methods							
Present	***c*=1**	***c*=10**	***c* =10^2**	***c*=10^3**	***c* =10^6**	***c* =10^8**	***c* =10^{10}**	***c* =10^{12}**
2M = 2	1.6833E-7	1.7910E-7	5.0573E-7	2.1862E-7	1.7972E-7	1.7969E-7	1.7969E-7	1.7969E-7
2M = 4	2.7863E-8	2.7034E-8	2.6853E-8	1.3595E-8	3.3547E-9	3.3484E-9	3.3483E-9	3.3483E-9
2M = 8	7.1515E-9	6.6044E-9	4.0645E-9	2.5490E-9	5.6348E-10	5.6393E-10	5.6394E-10	5.6394E-10
2M = 16	1.8213E-9	1.6584E-9	9.0835E-10	2.3478E-10	5.6208E-11	5.6146E-11	5.6146E-11	5.6146E-11
2M = 32	4.5667E-10	4.1435E-10	2.2036E-10	4.6002E-11	4.1498E-12	4.1152E-12	4.1145E-12	4.1138E-12
2M = 64	1.4112E-10	1.0345E-10	5.4631E-11	1.0772E-11	2.8688E-13	2.7622E-13	2.1757E-13	2.7578E-13
2M = 128	2.8536E-11	2.5864E-11	1.3634E-11	2.6477E-12	2.1094E-14	1.7764E-14	1.7986E-14	1.8208E-14
VIM	8.6E-8	3.4E-5	9.2E-3	1.0E-1	6.5E-4	-	-	-

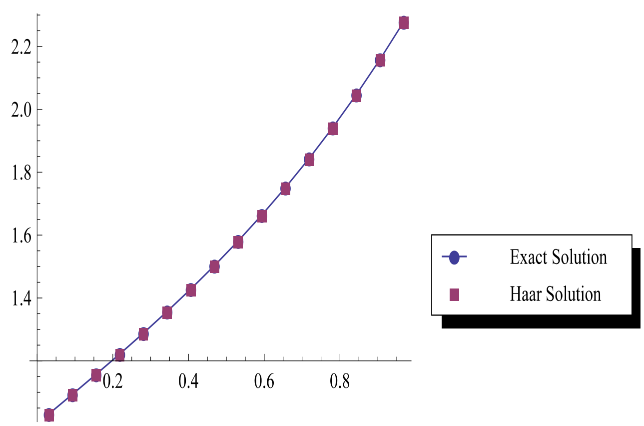

Figure 3. Uniform Haar solution of Example 3 for 2 *M* = 16 when *c* = 10^6.

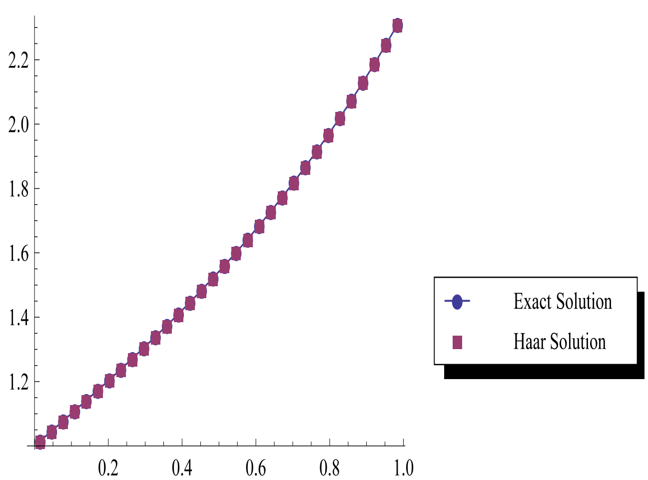

Figure 4. Uniform Haar solution of Example 3 for 2*M* = 32 when *c* = 10^{12}.

values of the parameter *c*. From the table, it is clear that the solution obtained by the semi-analytical method previously mentioned is dependent on the parameter *c* and the method is valid only for *c* < 10^6. The main reason for this failure is that this method lacks well established theoretical convergence analysis according to the value of *c*. Figures 3 and 4 show exact and approximate solutions for $2M = 16, 32$ when $c = 10^6, 10^{12}$, respectively.

Example 4

Consider the nonlinear BVP

$$y^{(6)}(x) = e^{-x}y^2, \quad 0 \le x \le 1, \tag{45}$$

subject to the boundary conditions

$$y(0) = 1, y^{(2)}(0) = 1, y^{(4)}(0) = 1, y(1) = e, y^{(2)}(1) = e, y^4(1) = e. \tag{46}$$

The exact solution is given by

$$y(x) = e^x. \tag{47}$$

We again compare our results with VIM (Noor et al., 2009). The point wise errors are shown in Table 4 and better performance of present method is obvious. Figure 5 shows exact and approximate solution for $2M = 16$.

Example 5

Consider the linear BVP

$$y^{(6)}(x) = -6e^x + y(x) + y^2(x), \quad 0 \le x \le 1, \tag{48}$$

subject to the boundary conditions

$$y(0) = 1, y^{(2)}(0) = -1, y^{(4)}(0) = -3, y(1) = 0, y^{(2)}(1) = -2e, y^4(1) = -4e. \tag{49}$$

The exact solution of the problem does not exist

Table 4. Point wise errors for Example 4.

Errors in present method									Errors in reported method
x	*2M = 2*	*2M = 4*	*2M = 8*	*2M = 16*	*2M = 32*	*2M = 64*	*2M = 128*	*2M = 256*	*2M = 512*	Noor et al. (2009)
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	5.0960E-6	1.6633E-8	4.3865E-7	1.1106E-7	2.7852E-8	6.9684E-9	1.7424E-9	4.3563E-10	1.0891E-10	-1.233E-4
0.2	9.6537E-6	3.1657E-7	8.3490E-7	2.1138E-7	5.3011E-8	1.3263E-8	3.3164E-9	8.2914E-10	2.0729E-10	-2.354E-4
0.3	1.3259E-5	4.3631E-6	1.1505E-6	2.9128E-7	7.3047E-8	1.8276E-8	4.5699E-9	1.1425E-9	2.8563E-10	-3.257E-4
0.4	1.5618E-5	5.1359E-6	1.3547E-6	3.4229E-7	8.6017E-8	2.1521E-8	5.3813E-9	1.3454E-9	3.3635E-10	-3.855E-4
0.5	1.6489E-5	5.4104E-6	1.4274E-6	3.6143E-7	9.0642E-8	2.2678E-8	5.6707E-9	1.4177E-9	3.5444E-10	-4.086E-4
0.6	1.5690E-5	5.1557E-6	1.3608E-6	3.4461E-7	8.6425E-8	2.1623E-8	5.4069E-9	1.3518E-9	3.3795E-10	-3.919E-4
0.7	1.3242E-5	4.3930E-6	1.1605E-6	2.9390E-7	7.3712E-8	1.8443E-8	4.6116E-9	1.1530E-9	2.8824E-10	-3.361E-4
0.8	9.4764E-6	3.1977E-6	8.4498E-7	2.1404E-7	5.3682E-8	1.3431E-8	3.3585E-9	8.3967E-10	2.0992E-10	-2.459E-4
0.9	4.9038E-6	1.6815E-6	4.4493E-6	1.1271E-7	2.8271E-8	7.0734E-9	1.7687E-9	4.4220E-10	1.1055E-10	-1.299E-4
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	2.000E-9

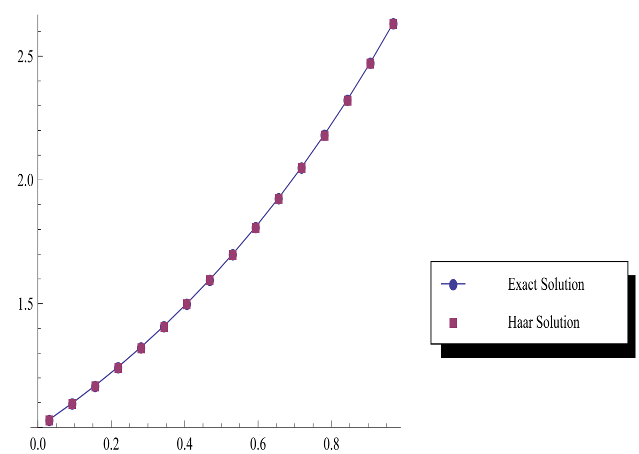

Figure 5. Uniform Haar solution of Example 4 for 2*M* = 16.

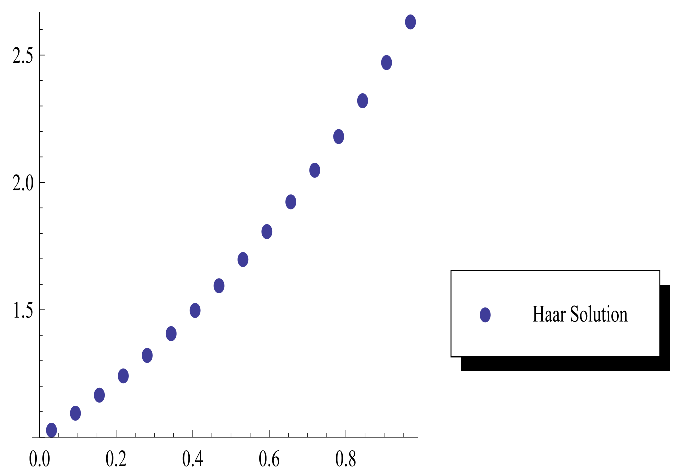

Figure 6. Uniform Haar solution of Example 5 for 2*M* = 16.

Table 5. Numerical results for Example 5.

	Solution				
x	*2M = 2*	*2M = 4*	*2M = 8*	*2M = 16*	*2M = 32*
0.0	1.00000	1.00000	1.00000	1.00000	1.00000
0.1	1.05480	1.05490	1.05500	1.05500	1.05500
0.2	1.22197	1.22201	1.22202	1.22202	1.22202
0.3	1.35063	1.35067	1.35069	1.35069	1.35070
0.4	1.49270	1.49275	1.49277	1.49278	1.49278
0.5	1.64961	1.64967	1.64969	1.64970	1.64970
0.6	1.82294	1.82299	1.82301	1.82302	1.82302
0.7	2.01443	2.01447	2.01449	2.01445	2.01445
0.8	2.22602	2.22605	2.22606	2.22607	2.22607
0.9	2.45985	2.45987	2.45987	2.45988	2.45988
1.0	2.71828	2.71828	2.71828	2.71828	2.71828

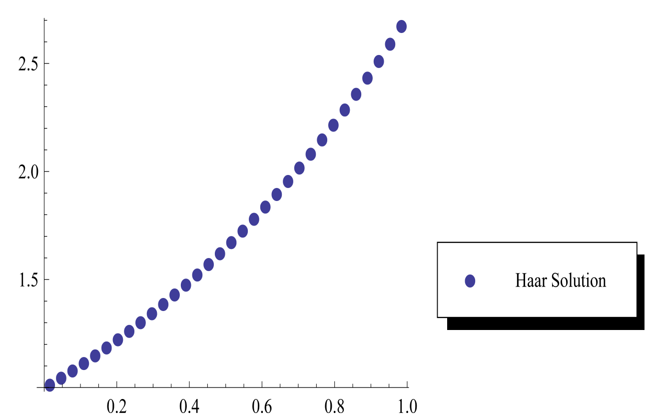

Figure 7. Uniform Haar solution of Example 5 for 2*M* = 32.

Noor et al., 2009). The approximate solution is shown in Table 5 for various values of *x*. Figures 6 and 7 show (approximate solution for $2M = 16$ and $2M = 32$, respectively.

Conclusion

Uniform Haar wavelets are used to develop numerical method for solving linear and nonlinear sixth-order BVPs. The method is computationally efficient and the algorithm can easily be implemented on a computer. Comparison was made with quintic spline based scheme and VIM method in the case of linear and nonlinear BVPs.

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