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Full Length Research Paper

On Mannheim partner curves in E^3

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The aim of this paper is to study the Mannheim partner curves in Euclidean space E^3. We obtain the relationships between the curvatures and the torsions of the Mannheim partner curves with respect to each other.

Key words: Mannheim partner curves, curvature, torsion.

INTRODUCTION

As is well-known, a surface is said to be "ruled" if it is generated by moving a straight line continuously in Euclidean space E^3 (O'Neill, 1997). Ruled surfaces are one of the simplest objects in geometric modeling. One important fact about ruled surfaces is that they can be generated by straight lines. A practical application of this type surfaces is that they are used in civil engineering and physics (Guan et al., 1997).

Since building materials such as wood are straight, they can be considered as straight lines. The result is that if engineers are planning to construct something with curvature, they can use a ruled surface since all the lines are straight (Orbay et al., 2009).

The curves are a fundamental structure of differential geometry. An increasing interest of the theory of curves makes a development of special curves to be examined. Especially, Bertrand curves are well-studied classical curves (Carom, 1976). In this study, we introduce and study the Mannheim curves, which are other special curves and not well known.

In recent works, Liu and Wang (2007, 2008) are curious about the Mannheim curves in both Euclidean and Minkowski 3-space and they obtained the necessary and sufficient conditions between the curvature and the torsion for a curve to be the Mannheim partner curves. Meanwhile, the detailed discussion concerned with the Mannheim curves can be found in literature (Wang and Liu, 2007; Liu and Wang, 2008; Orbay and et al., 2009) and references therein.

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MATERIALS AND METHODS

Let E^3 denote Euclidean 3 space with an inner product<,> and a vector product$\wedge$. Consider two space curves C and C*: $I \to E^3$, where I is a real interval that has at least four continuous derivatives. If there exists a corresponding relationship between the space curves C and C* such that the principal normal lines of C coincides with the binormal lines of C* at the corresponding points of the curves, then C is called as a Mannheim curve and C* is called as a Mannheim partner curve of C. The pair of {C, C*} is said to be a Mannheim pair (Liu and Wang, 2008).

Let C: $\alpha(s)$ be the Mannheim curve in E^3 parameterized by its arc length s and C*: $\alpha^*(s^*)$ is the Mannheim partner curve of C with an arc length parameter s^*. Denote by $\{T(s), N(s), B(s)\}$ the Frenet frame field along C: $\alpha(s)$, that is; $T(s)$ is the tangent vector field, $N(s)$ is the normal vector field, and $B(s)$ is the binormal vector field of the curve C respectively. The famous Frenet equations and the derivative formulas are given by (O'Neill, 1997; Struik, 1988)

$$\langle T,T\rangle = \langle N,N\rangle = \langle B,B\rangle = 1 \ , \ \langle T,N\rangle = \langle N,B\rangle = \langle T,B\rangle = 0$$
$$T = N \wedge B, N = B \wedge T, B = T \wedge N \qquad (1)$$

and

$$\begin{aligned} \dot{T} &= \kappa N, \\ \dot{N} &= -\kappa T + \tau B. \\ \dot{B} &= -\tau N \end{aligned} \qquad (2)$$

Here, that's; in the above equations, and after that, we use the dot notation to denote the derivative with respect to the arc length parameter of a curve. In this paper, we study the Mannheim partner curves in E^3. We will obtain the relationships between the curvatures and the torsions.

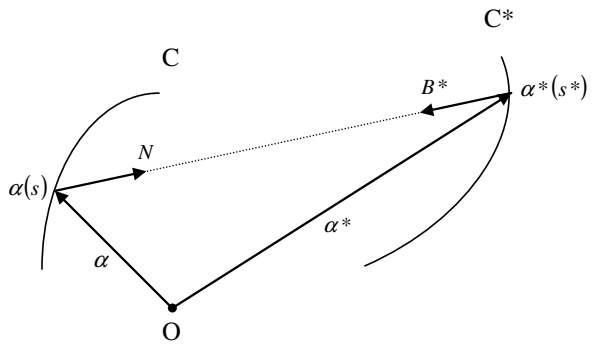

Figure 1. The Mannheim partner curves.

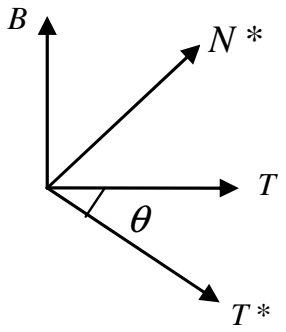

Figure 2. Frenet frame.

of the Mannheim partner curves with respect to each other. Using these relationships, we will comment again Bertrand's, Schell's and Mannheim's theorems.

RESULTS AND DISCUSSION

We can represent the Mannheim pair {C, C*} in the following Figure 1.

Theorem 1: The distance between corresponding points of the Mannheim partner curves in E^3 is constant.

Proof: From the figure1, we can write

$$\alpha(s)=\alpha^*(s^*)+\lambda(s^*)B^*(s^*) \tag{3}$$

for some function $\lambda(s^*)$. By taking the derivative of Equation (3) with respect to s^* and using Equation (2), we obtain

$$T\frac{ds}{ds^*}=T^*-\lambda\tau^*+\dot{\lambda}B^*. \tag{4}$$

Since N and B^* are linearly dependent, $\langle T, B^*\rangle = 0$, we get

$$\dot{\lambda}=0$$

This means that λ is a nonzero constant. On the other hand, from the distance function between two points, we have

$$\begin{aligned} d(\alpha^*(s^*),\alpha(s)) &= \|\alpha(s)-\alpha^*(s^*)\| \\ &= \|\lambda B^*\| \\ &= |\lambda| \ . \end{aligned}$$

Namely, $d(\alpha^*(s^*),\alpha(s)) = constant$.

Theorem 2: For a curve C in E^3, there is a curve C* so that (C, C*) is a Mannheim pair.

Proof: Since N and B are linearly dependent, Equation (3) can be re written as

$$\alpha^*=\alpha-\lambda N \tag{5}$$

Now that λ is a nonzero constant, there is a curve C* for all values of λ.

Theorem 3: Let {C, C*} be a Mannheim pair in E^3. The torsion of the curve C* is $\tau^*=\frac{\kappa}{\lambda\tau}$.

Proof: By considering λ is nonzero constant in Equation (4), we obtain

$$T=\frac{ds^*}{ds}T^*-\lambda\tau^*\frac{ds^*}{ds}N^* \tag{6}$$

Even so, we know that

$$\begin{aligned} T &= \cos\theta T^*+\sin\theta N^*, \\ B &= -\sin\theta T^*+\cos\theta N^* \end{aligned} \tag{7}$$

Where θ is the angle between T and T^* at the corresponding points of C and C* (Figure 2).

By taking into consideration Equations (6) and (7), we get

$$\cos\theta=\frac{ds^*}{ds} \ , \ \sin\theta=\lambda\tau^*\frac{ds^*}{ds}.$$

Besides, by taking the derivative of Equation (5) with respect to s and using Equation (2), we have

$$T^*=(1+\lambda\kappa)\frac{ds}{ds^*}T-\lambda\tau\frac{ds}{ds^*}B. \tag{8}$$

From Equation (7), we obtain

$$T^* = \cos\theta T - \sin\theta B,$$
$$N^* = \sin\theta T + \cos\theta B. \quad (9)$$

By applying Equations (8) and (9), we get

$$\cos\theta = (1+\lambda\kappa)\frac{ds}{ds^*}\ ,\ \sin\theta = \lambda\tau\frac{ds}{ds^*}. \quad (10)$$

From both values of $\cos\theta$ and $\sin\theta$, we see

$$\cos^2\theta = 1+\lambda\kappa\ ,\ \sin^2\theta = -\lambda^2\tau\tau^*.$$

Here, with the help of the fundamental equation $\cos^2\theta + \sin^2\theta = 1$, we reach the equation $\tau^* = \frac{\kappa}{\lambda\tau}$.

Result 1: Let {C, C*} be a Mannheim pair in E^3. Then the products of torsions τ and τ^* at the corresponding points of the Mannheim curves are not constant. Namely, Schell's theorem is invalid for the Mannheim curves.

By looking over again the equation $\sin^2\theta = -\lambda^2\tau\tau^*$ obtained from the proof of the theorem 4, we write easily the following result too.

Result 2: If {C, C*} is a Mannheim pair in E^3, then τ and τ^* have opposite signs.

Theorem 4: Let {C, C*} be a Mannheim pair in E^3. Between the curvature and the torsion of the curve C, there is the relationship

$$\mu\tau - \lambda\kappa = 1$$

Where λ and μ are nonzero real numbers.

Proof: From Equation (10), we have

$$\frac{\cos\theta}{1+\lambda\kappa} = \frac{\sin\theta}{\lambda\tau}$$

Arranging this equation, we obtain

$$\lambda\cos\theta\tau - \lambda\kappa = 1$$

$$\mu\tau - \lambda\kappa = 1$$

Result 3: Let {C, C*} be a Mannheim pair in E^3. Then there exists a linear relationship between κ and τ with constant coefficients. Namely, Bertrand's theorem is valid for the Mannheim curves.

Theorem 5: Let {C, C*} be a Mannheim pair in E^3. There are the following equations for the curvatures and the torsions of the curves C and C*:

i. $\kappa^* = -\frac{d\theta}{ds^*}$,

ii. $\tau^* = \sin\theta\,\kappa\frac{ds}{ds^*} - \cos\theta\,\tau\frac{ds}{ds^*}$,

iii. $\kappa = \sin\theta\,\tau^*\frac{ds^*}{ds}$,

iv. $\tau = -\cos\theta\,\tau^*\frac{ds^*}{ds}$.

Proof: i. By taking the derivative of the equation $\langle T, T^*\rangle = \cos\theta$ with respect to s^*, we have

$$\langle \kappa^* N^*, T\rangle + \left\langle T^*, \kappa N\frac{ds}{ds^*}\right\rangle = -\sin\theta\frac{d\theta}{ds^*}.$$

Furthermore, by considering N and B^* are linearly dependent and using Equations (9) and (1), we reach

$$\kappa^* = -\frac{d\theta}{ds^*}.$$

By considering the equations $\langle N, N^*\rangle = 0$, $\langle T, B^*\rangle = 0$ and $\langle B, B^*\rangle = 0$, we can do the proofs of ii, iii and iv of the theorem respectively as in the proof of i.

From iii and iv of the theorem 5, we obtain the following result.

Result 4: $\kappa^2 + \tau^2 = \frac{ds^*}{ds}\tau^{*2}$.

Theorem 6: Let {C, C*} be a Mannheim pair in E^3. For the points $\alpha(s)$ and $\alpha^*(s^*)$ are two corresponding points of {C, C*} and M and M* are the curvature centers at these points, the ratio

$$\frac{\|\alpha^*(s^*)M\|}{\|\alpha(s)M\|} : \frac{\|\alpha^*(s^*)M^*\|}{\|\alpha(s)M^*\|}$$

is not constant.

Proof: From Figure 1, we obtain the following equations:

$$\|\alpha(s)M\| = \frac{1}{\kappa} \qquad \|\alpha^*(s^*)M^*\| = \frac{1}{\kappa^*},$$

$$\|\alpha(s)M^*\| = \frac{1}{\kappa^*} - \lambda, \qquad \|\alpha^*(s^*)M\| = \frac{1}{\kappa} + \lambda.$$

So, we have

$$\frac{\|\alpha^*(s^*)M\|}{\|\alpha(s)M\|} : \frac{\|\alpha^*(s^*)M^*\|}{\|\alpha(s)M^*\|} = (1+\lambda\kappa)(1-\lambda\kappa^*) \neq const.$$

Result 5: Mannheim's theorem is invalid for the Mannheim curves.

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