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Bayesian analysis of two parameter Pareto mixture using censoring

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The Pareto distribution is a commonly used model for heavy tailed data. Pareto distribution is a useful modeling and predicting tool in a wide variety of socioeconomic contexts. In the present study, we model heterogeneous population by using two component mixture model and two parameter Pareto mixture is considered for this purpose. The expressions for Bayes estimators and their posterior risks (using squared error loss function and weighted loss function) are derived. System of non-linear equations that lead to obtain estimate for MLE's along with component of Fisher information matrix, are also constructed. An extensive simulation at different parameter points using probabilistic mixing is carried out to get censored simulated data. These data are then used to obtain Bayes estimates for parameters of the Pareto mixture and their respective posterior risks. Bayes estimates are obtained by non-informative as well as informative prior. Posterior predictive distribution and predictive intervals are derived using Gamma prior to obtain hyper-parameters. Finally, Pareto mixture is fitted to a real data example which includes estimation and testing of parameters.

Key words: Heavy tailed data, probabilistic mixing, predictive intervals, informative prior, hyper-parameters.

INTRODUCTION

Mixture models are frequently used to describe heterogeneous populations, and have wide statistical applications. Mixture models are useful to describe heterogeneous populations that can be thought of consisting of k subpopulations. Each sub-population has the same parametric form but different parameter value. Mendenhall and Hader (1958) gave an account on finite mixtures. Parameters of a finite mixture can be estimated by two main approaches. The likelihood based technique or the Bayesian approach. In present study we will focus on the later approach.

The two parameter Pareto distribution is a distribution whose shape parameter determines the degree of heaviness of the tail, so that it can be adapted to data with different features. Importance of two parameter Pareto model lies in the fact that it shifts the one parameter Pareto model in such a manner that the Pareto random variable can assume all positive values, so that the domain of the study is simplified.

A great account of research can be found on Pareto model. Krishnaji (1970) presents characterization of Pareto distribution through a model which considers the under-reporting of incomes to tax authorities. Jupp and Mardia (1982) provided a simple characterization of the multivariate Pareto distribution. Reed (2002) discussed the rank size distribution and employed the double Pareto log normal distribution for this purpose. Nadarajah and Kotz (2005) estimated the information matrix for a mixture of two Pareto distributions. Ali and Nadarajah (2006) discussed the truncated Pareto distribution. Zisheng and Chi (2006) obtained exact threshold for generalized Pareto distribution and fit it to the medical insurance claim data. Bee et al. (2009) presented a maximum likelihood estimation of a Pareto mixture.

The problem of censoring is more commonly encountered in life time data because no experiment may remain continued for infinite time. A typical example is the case when some individuals are still alive at the end of the experiment, constituting the censored data. There are three major types of censored data. The left censored

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data, the interval censored and the most common, the right censored data. The right censored data is more commonly encountered in survival analysis.

In this study, a heterogeneous population is assumed to be composed of two sub-groups mixed together with unknown probabilistic mixing proportions. The random observations taken from this mixed population are supposed to be characterized by a Pareto distribution. So, the two parameter Pareto distribution is considered to model the two components of the mixture.

THE MIXTURE MODEL

A finite two component mixture density with specified parametric form but unknown parameters can be defined as;

$$f(x) = pf_1(x) + (1-p) f_2(x) \quad 0 < p < 1$$

The corresponding mixture distribution function is given by;

$$F(x) = pF_1(x) + (1-p)F_2(x)$$

Consider the mixture of two parameter Pareto distribution given by the pdf;

$$f(x) = p\{\frac{\theta_1\lambda^{\theta_1}}{(\lambda+x)^{\theta_1+1}}\} + (1-p)\{\frac{\theta_2\lambda^{\theta_2}}{(\lambda+x)^{\theta_2+1}}\} \quad x > 0, \quad \theta_1, \theta_2 > 0$$

0 < p < 1 =

$$p\{\theta_1\lambda^{\theta_1}(\lambda+x)^{-(\theta_1+1)}\} + (1-p)\{\theta_2\lambda^{\theta_2}(\lambda+x)^{-(\theta_2+1)}\} \quad (1)$$

And corresponding distribution function is;

$$F(x) = p(1-(\frac{\lambda}{\lambda+x})^{\theta_1}) + q(1-(\frac{\lambda}{\lambda+x})^{\theta_2})$$

$$= 1 - p(\frac{\lambda}{\lambda+x})^{\theta_1} - q(\frac{\lambda}{\lambda+x})^{\theta_2}$$

SAMPLING LIKELIHOOD FUNCTION

Suppose a life testing experiment is conducted for the above described mixture model assumed to have n units. Let the test be performed and it is observed that out of n, r units are failed till the test termination time T is over and the remaining $n-r$ units are still functioning. As described in Mendenhall and Hader (1958), it has become customary that in many real life situations, only failed objects can easily be identified as a member of subpopulation-1 or subpopulation-2. After knowing the cause of failure, we can identify r_1 and r_2 objects as member of first and second subpopulation, respectively. Here, it is clear that $r = r_1 + r_2$ and the remaining $n-r$ units that are still alive provide no information about the population to which they belong. Let the x_{ij} be denoted as the failure time of the *jth* unit belonging to the *ith* subpopulation. Where $j = 1, 2, 3, ...r_i$; $i = 1, 2$; $x_{1j}, x_{2j} > 0$

Now the likelihood function considering the given conditions can be written as:

$$L(\theta_1, \theta_2, p \mid \mathrm{x}) \propto \{\prod_{j=1}^{r_1} pf_1(x_{1j})\}\{\prod_{j=1}^{r_2} qf_2(x_{2j})\}\{1-F(T)\}^{n-r}$$

$$\propto \{\prod_{j=1}^{r_1} p(\theta_1\lambda^{\theta_1}(\lambda+x)^{-(\theta_1+1)})\} + \{\prod_{j=1}^{r_2}(1-p)(\theta_2\lambda^{\theta_2}(\lambda+x)^{-(\theta_2+1)})\} \quad (2)$$

$$\times\{p(\frac{\lambda}{\lambda+T})^{\theta_1} + q(\frac{\lambda}{\lambda+T})^{\theta_2}\}^{n-r}$$

This can further be simplified as:

$$L(\theta_1, \theta_2, p \mid \mathrm{x}) \propto \sum_{k=0}^{n-r} \binom{n-r}{k} p^{n-r_2-k} q^{r_2+k} \theta_1^{r_1} \mathrm{e}^{-\theta_1\{\sum \ln(\lambda+x_{1j}) - r_1\ln(\lambda) + (n-r-k)\ln(1+\frac{T}{\lambda})\}} \theta_2^{r_2} \mathrm{e}^{-\theta_2\{\sum \ln(\lambda+x_{2j}) - r_2\ln(\lambda) + k\ln(1+\frac{T}{\lambda})\}} \quad (3)$$

Maximum likelihood estimators and their variances

Maximum likelihood estimates of θ_1, θ_2 and p can be obtained by solving the system of non linear Equations (4) to (6), obtained by setting the first order derivatives of the log likelihood of Equation (2) to zero. For simplicity, assume $1+\frac{T}{\lambda}$ = w

$$\frac{\partial l}{\partial \theta_1} = \frac{r_1}{\theta_1} + r_1\ln(\lambda) - r_1\ln(\lambda + x_{1j}) + \frac{(n-r)pw^{-\theta_1}\ln(w)}{pw^{-\theta_1} + qw^{-\theta_2}} = 0 \quad (4)$$

$$\frac{\partial l}{\partial \theta_2} = \frac{r_2}{\theta_2} + r_2\ln(\lambda) - r_2\ln(\lambda + x_{2j}) + \frac{(n-r)qw^{-\theta_2}\ln(w)}{pw^{-\theta_1} + qw^{-\theta_2}} = 0 \quad (5)$$

$$\frac{\partial l}{\partial p} = \frac{r_1}{p} - \frac{r_2}{1-p} + \frac{(n-r)(w^{-\theta_1} - w^{-\theta_2})}{pw^{-\theta_1} + qw^{-\theta_2}} = 0 \quad (6)$$

Variances of the maximum likelihood estimates are on the main diagonal of the inverted information matrix. The information matrix is given by the expectation of the negative Hessian as;

$$I(\theta) = -E\begin{pmatrix} \frac{\partial^2 l}{\partial\theta_1^2} & \frac{\partial^2 l}{\partial\theta_1\partial\theta_2} & \frac{\partial^2 l}{\partial\theta_1\partial p} \\ \frac{\partial^2 l}{\partial\theta_2\partial\theta_1} & \frac{\partial^2 l}{\partial\theta_2^2} & \frac{\partial^2 l}{\partial\theta_2\partial p} \\ \frac{\partial^2 l}{\partial p\partial\theta_1} & \frac{\partial^2 l}{\partial p\partial\theta_2} & \frac{\partial^2 l}{\partial p^2} \end{pmatrix} \quad (7)$$

Where, $\frac{\partial^2 l}{\partial\theta_1^2} = -\frac{r_1}{\theta_1^2} + \frac{(n-r)pw^{-\theta_1}\ln(w)^2}{pw^{-\theta_1}+qw^{-\theta_2}} - \frac{(n-r)p^2w^{-2\theta_1}\ln(w)^2}{(pw^{-\theta_1}+qw^{-\theta_2})^2}$

$$\frac{\partial^2 l}{\partial\theta_2^2} = -\frac{r_2}{\theta_2^2} - \frac{(n-r)q^2w^{-2\theta_2}\ln(w)^2}{(pw^{-\theta_1}+qw^{-\theta_2})^2} + \frac{(n-r)qw^{-\theta_2}\ln(w)^2}{pw^{-\theta_1}+qw^{-\theta_2}}$$

$$\frac{\partial^2 l}{\partial p^2} = -\frac{r_1}{p^2} - \frac{r_2}{(1-p)^2} - \frac{(n-r)(w^{-\theta_1}-w^{-\theta_2})^2}{(pw^{-\theta_1}+(1-p)w^{-\theta_2})^2}$$

$$\frac{\partial^2 l}{\partial\theta_1\theta_2} = -\frac{(n-r)pqw^{-\theta_1-\theta_2}\ln(w)^2}{(pw^{-\theta_1}+qw^{-\theta_2})^2}$$

$$\frac{\partial^2 l}{\partial\theta_1\partial p} = \frac{(n-r)pw^{-\theta_1}(w^{-\theta_1}-w^{-\theta_2})\ln(w)}{(pw^{-\theta_1}+qw^{-\theta_2})^2} - \frac{(n-r)w^{-\theta_1}\ln(w)}{(pw^{-\theta_1}+(1-p)w^{-\theta_2})}$$

$$\frac{\partial^2 l}{\partial\theta_2\partial p} = \frac{(n-r)w^{-\theta_2}\ln(w)}{(pw^{-\theta_1}+(1-p)w^{-\theta_2})} + \frac{(n-r)(1-p)w^{-\theta_2}(w^{-\theta_1}-w^{-\theta_2})\ln(w)}{(pw^{-\theta_1}+qw^{-\theta_2})^2}$$

BAYES ESTIMATES ASSUMING UNINFORMATIVE PRIORS

In Bayesian parameter estimation problem, the use of an uninformative prior typically yields results which are not too different from conventional statistical analysis. The Uniform and Jeffreys prior are the most familiar examples of uninformative priors that are used to conduct Bayesian analysis when no prior information is available.

Bayes estimators and their posterior risks using uniform prior under "squared error loss function"

Let us assume that θ_1, θ_2 are uniformly distributed over $(0,\infty)$ and p is uniformly distributed over $(0,1)$. So $h_1(\theta_1) = k_1:\ 0<\theta_1<\infty$ $h_2(\theta_2) = k_2:\ 0<\theta_2<\infty$ and $h_3(p)=1;\ 0<p<1$. Assuming independence, we have an improper joint prior that is proportional to a constant. When this improper prior is incorporated with the likelihood, Equation (3) yields a proper joint posterior distribution

$$g(\theta_1,\theta_2,p\mid \mathrm{x}) = c^{-1}\sum_{k=0}^{n-r} p^{n-r_2-k}q^{r_2+k}\theta_1^{r_1}\,\mathrm{e}^{-\theta_1\beta_1}\,\theta_2^{r_2}\,\mathrm{e}^{-\theta_2\beta_2} \quad (8)$$

Where c $= \sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_1+1)}{\beta_1^{r_1+1}}\frac{\Gamma(r_2+1)}{\beta_2^{r_2+1}}$ and marginal distributions of θ_i can be obtained as under.

$$p(\theta_i) = c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\left\{\prod_{j\neq i}\frac{\Gamma(r_j+1)}{(\beta_j)^{r_j+1}}\right\}\lambda_i^{r_i}e^{-\theta_i\beta_i}$$
$$i=1,2 \quad (9)$$

The marginal distribution for p can be obtained on the same lines. The respective marginal distributions yield the following Bayes estimators of θ_i and p under the squared error loss function.

$$\hat{\theta}_{i(s)}\big|\mathrm{x} = c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i+2)}{(\beta_i)^{r_i+2}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+1)}{(\beta_j)^{r_j+1}}\right\}$$

$$\hat{p}_{(s)}\big|\mathrm{x} = c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+2, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+1)}{(\beta_i)^{r_i+1}}\right\}$$

Where c is as defined earlier. B(.,.) and $\Gamma(.)$ are Beta and Gamma functions, respectively. Notations in the foot of estimator represent loss function used. The expressions for the posterior risk for the parameters of a Pareto mixture are:

$$\pi(\hat{\theta}_{i(s)}\big|\mathrm{x}) = \left[c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i+3)}{(\beta_i)^{r_i+3}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+1)}{(\beta_j)^{r_j+1}}\right\}\right] - \left[\hat{\theta}_{i(s)}\big|\mathrm{x}\right]^2$$

$$\pi(\hat{p}_{(s)}\big|\mathrm{x}) = \left[c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+3, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+1)}{(\beta_i)^{r_i+1}}\right\}\right] - \left[\hat{p}_{(s)}\big|\mathrm{x}\right]^2$$

Where $\beta_1 = \sum\ln(\lambda+x_{1j}) - r_1\ln(\lambda) + (n-r-k)\ln(1+\frac{T}{\lambda})\}$ and $\beta_2 = \sum\ln(\lambda+x_{2j}) - r_2\ln(\lambda) + k\ln(1+\frac{T}{\lambda})\}$

Bayes estimators and their posterior risk using uniform prior under "weighted loss function"

The marginal distribution for θ_i under the weighted loss function can be written as:

$$p(\theta_i) = c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\left\{\prod_{j\neq i}\frac{\Gamma(r_j+1)}{(\beta_j)^{r_j+1}}\right\}\lambda_i^{r_i-1}e^{-\theta_i\beta_i} \quad (10)$$

Marginal distribution for p can similarly be obtained. These marginal distribution leads towards following Bayes estimators:

$$\hat{\theta}_{i(w)}|\mathrm{x} = \left[c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i)}{(\beta_i)^{r_i}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j+1)}{(\beta_j)^{r_j+1}}\right\}\right]^{-1}$$

$$\hat{p}_{(w)}|\mathrm{x} = \left[c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i+1)}{(\beta_i)^{r_i+1}}\right\}\right]^{-1}$$

Posterior risk for the weighted loss function can be obtained from these expressions:

$\pi(\hat{\theta}_i|\mathrm{x}) = \left[\hat{\theta}_{i(s)}|\mathrm{x} - \hat{\theta}_{i(w)}|\mathrm{x}\right]$ and $\pi(\hat{p}_{(W)}|\mathrm{x}) = \left[\hat{p}_{(s)}|\mathrm{x} - \hat{p}_{(w)}|\mathrm{x}\right]$

Bayes estimators and their posterior risk using Jeffrey's prior under "squared error loss function"

Jeffrey's prior is defined as the density of the parameters proportional to the square-root of the determinant of the Fisher information matrix. So, the Jeffrey's prior according to the model as defined in Equation (1) is $h(\theta_i)\propto \theta_i^{-1} = \frac{1}{\theta_i}$ and $p \sim U(0,1)$ $0 < \theta_i < \infty$.
Assuming independence, combining the joint prior with

likelihood Equation (3) we get the joint posterior as follows:

$$g(\theta_1, \theta_2, p \mid \mathrm{x}) = c^{-1}\sum_{k=0}^{n-r} p^{n-r_2-k} q^{r_2+k}\theta_1^{r_1-1}\,\mathrm{e}^{-\theta_1\beta_1}\,\theta_2^{r_2-1}\,\mathrm{e}^{-\theta_2\beta_2} \quad (11)$$

Where c =

$$\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_1)}{\beta_1^{r_1}}\frac{\Gamma(r_2)}{\beta_2^{r_2}}$$

Respective marginal distributions yield the Bayes estimators for $\theta_1, \theta_2 and\ p$ as under.

$$\hat{\theta}_{i(s)}|\mathrm{x} = c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i+1)}{(\beta_i)^{r_i+1}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j)}{(\beta_j)^{r_j}}\right\}$$

i =1,2

$$\hat{p}_{(s)}|\mathrm{x} = c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+2, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i)}{(\beta_i)^{r_i}}\right\}$$

and the expressions for the posterior risk of $\theta_1, \theta_2 and\ p$ are:

$$\pi(\hat{\theta}_{i(s)}|\mathrm{x}) = \left[c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i+2)}{(\beta_i)^{r_i+2}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j)}{(\beta_j)^{r_j}}\right\}\right] - \left[\hat{\theta}_{i(s)}|\mathrm{x}\right]^2$$

$$\pi(\hat{p}_{(s)}|\mathrm{x}) = \left[c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+3, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i)}{(\beta_i)^{r_i}}\right\}\right] - \left[\hat{p}_{(s)}|\mathrm{x}\right]^2$$

Bayes estimators and their posterior risk using Jeffreys prior under "weighted loss function"

Bayes estimators for θ_1, θ_2, p using Jeffreys prior under weighted loss function are:

$$\hat{\theta}_{i(w)}|\mathrm{x} = \left[c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2+1, r_2+k+1)\frac{\Gamma(r_i-1)}{(\beta_i)^{r_i-1}}\left\{\prod_{i\neq j}\frac{\Gamma(r_j)}{(\beta_j)^{r_j}}\right\}\right]^{-1}$$

$$\hat{p}_{(w)}|\mathrm{x} = \left[c^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-k-r_2, r_2+k+1)\left\{\prod_{i=1}^{2}\frac{\Gamma(r_i)}{(\beta_i)^{r_i}}\right\}\right]^{-1}$$

and posterior risk can be evaluated from the following expressions:

$\pi(\hat{\theta}_{i(w)}|\mathrm{x}) = \left[\hat{\theta}_{i(s)}|\mathrm{x} - \hat{\theta}_{i(w)}|\mathrm{x}\right]$ and $\pi(\hat{p}|\mathrm{x}) = \left[\hat{p}_{(s)}|\mathrm{x} - \hat{p}_{(w)}|\mathrm{x}\right]$

BAYES ESTIMATES ASSUMING INFORMATIVE PRIOR

Bayesian analysis rests on the belief that in most practical situations the statistician will possess some prior information concerning the probable value of the parameter. This information is often reasonably summarized and add objectivity to the analysis. Hence, the use of informative prior may lead towards more efficient Bayes estimates accompanied by lower posterior risk.

Bayes estimators and their posterior risk using gamma prior under "squared error loss function"

Let us assume that θ_1, θ_2 are independent and follow Gamma distribution as prior with shape and scale parameters a_1, b_1, a_2, b_2 respectively also p follows uniform distribution. Then joint prior of θ_1, θ_2 and p may be written as:

$$h(\theta_1, \theta_2, p|\mathrm{x}) \propto \theta_1^{a_1-1} e^{-b_1\theta_1} \theta_2^{a_2-1} e^{-b_2\theta_2} \quad (12)$$

So by combining likelihood Equation (3) and joint prior Equation (12), the joint posterior distribution of θ_i and p is obtained as under.

$$g(\theta_1, \theta_2, p \mid \mathrm{x}) = c^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} p^{(n-r_2-k+1)-1} (1-p)^{(r_2+k+1)-1} \theta_1^{(r_1+a_1)-1} \mathrm{e}^{-\theta_1 \beta_1^*} \theta_2^{(r_2+a_2)-1} \mathrm{e}^{-\theta_2 \beta_2^*} \quad (13)$$

Where $\beta_1^* = \beta_1 + b_1$ and $\beta_2^* = \beta_2 + b_2$ and c =

$$\sum_{k=0}^{n-r} \binom{n-r}{k} B(n - r_2 - k + 1, r_2 + k + 1) \frac{\Gamma(r_1 + a_1)}{\beta_1^{*r_1+a_1}} \frac{\Gamma(r_2 + a_2)}{\beta_2^{*r_2+a_2}}$$

The expressions for the Bayes estimators of θ_1, θ_2 and p under the squared error loss function by utilizing the respective marginal densities are specified below.

$$\hat{\theta}_{i(s)}|\mathrm{x} = c^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-k-r_2+1, r_2+k+1) \frac{\Gamma(r_i + a_i + 1)}{(\beta_i^*)^{r_i+a_i+1}} \left\{ \prod_{i \neq j} \frac{\Gamma(r_j + a_j)}{(\beta_j^*)^{r_j+a_j}} \right\}$$

$$\hat{p}_{(s)}|\mathrm{x} = c^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-k-r_2+2, r_2+k+1) \left\{ \prod_{i=1}^{2} \frac{\Gamma(r_i + a_i)}{(\beta_i^*)^{r_i+a_i}} \right\}$$

The expressions for Bayes Posterior Risk for θ_i *and* p are:

$$\pi(\hat{\theta}_{i(s)}|\mathrm{x}) = \left[c^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-k-r_2+1, r_2+k+1) \frac{\Gamma(r_i + a_i + 2)}{(\beta_i^*)^{r_i+a_i+2}} \left\{ \prod_{i \neq j} \frac{\Gamma(r_j + a_j)}{(\beta_j^*)^{r_j+a_j}} \right\} \right] - \left[\hat{\theta}_{i(s)}|\mathrm{x} \right]^2$$

$$\pi(\hat{p}_{(s)}|\mathrm{x}) = c^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-k-r_2+3, r_2+k+1) \left\{ \prod_{i=1}^{2} \frac{\Gamma(r_i + a_i)}{(\beta_i^*)^{r_i+a_i}} \right\} - \left[\hat{p}_{(s)}|\mathrm{x} \right]^2$$

Bayes estimators and their posterior risk using gamma prior under "weighted loss function"

Use of weighted loss function yields the following Bayes estimators.

$$\hat{\theta}_{i(w)}|\mathrm{x} = \left[c^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-k-r_2+1, r_2+k+1) \frac{\Gamma(r_i + a_i - 1)}{(\beta_i^*)^{r_i+a_i-1}} \left\{ \prod_{i \neq j} \frac{\Gamma(r_j + a_j)}{(\beta_j^*)^{r_j+a_j}} \right\} \right]^{-1}$$

$$\hat{p}_{(w)}|\mathrm{x} = \left[c^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-k-r_2, r_2+k+1) \left\{ \prod_{i=1}^{2} \frac{\Gamma(r_i + a_i)}{(\beta_i^*)^{r_i+a_i}} \right\} \right]^{-1}$$

and posterior risk can be obtained from the following expressions:

$$\pi(\hat{\theta}_{i(w)}|\mathrm{x}) = \left[\hat{\theta}_{i(s)}|\mathrm{x} - \hat{\theta}_{i(w)}|\mathrm{x} \right]$$ and

$$\pi(\hat{p}|\mathrm{x}) = \left[\hat{p}_{(s)}|\mathrm{x} - \hat{p}_{(w)}|\mathrm{x} \right]$$

POSTERIOR PREDICTIVE DISTRIBUTION AND THE PREDICTIVE INTERVALS USING "GAMMA PRIOR"

The predictive distribution of the future single observation y given the data x can be derived by combining the data and the posterior distribution.

$$p(y \mid \mathrm{x}) = \int_0^1 \int_0^\infty \int_0^\infty g(\theta_1, \theta_2, p \mid x)\, p(y \mid \theta_1, \theta_2, p)\, d\theta_1, d\theta_2, dp \quad (14)$$

$$p(y \mid x) = \sum_{k=0}^{n-r} \binom{n-r}{k} p^{(n-r_2-k)} (1-p)^{(r_2+k)} \theta_1^{(r_1+a_1)-1} \mathrm{e}^{-\theta_1 \beta_1^*} \theta_2^{(r_2+a_2)-1} \mathrm{e}^{-\theta_2 \beta_2^*} (p\theta_1 y^{-\theta_1-1} + q\theta_2 y^{-\theta_2-1})$$

Where $y = \lambda + x$ is the future single observation. So the predictive density is simplified to:

$$p(y \mid x) = \frac{1}{c} \sum_{k=0}^{n-r} \binom{n-r}{k} \frac{1}{y} \Big[B(n - r_2 - k + 2, r_2 + k + 1) \frac{\Gamma(r_1 + a_1 + 1)}{(\beta_1^{**} + \ln y)^{r_1+a_1+1}} \frac{\Gamma(r_2 + a_2)}{(\beta_2^{**})^{r_2+a_2}} + B(n - r_2 - k + 1, r_2 + k + 2) \frac{\Gamma(r_1 + a_1)}{(\beta_1^{**})^{r_1+a_1}} \frac{\Gamma(r_2 + a_2 + 1)}{(\beta_2^{**} + \ln y)^{r_2+a_2+1}} \Big] \quad (15)$$

Where $\beta_1^{**} = \beta_1^* - \ln\lambda$ and $\beta_2^{**} = \beta_2^* - \ln\lambda$ where $\beta_1^* = \beta_1 + b_1$ and $\beta_2^* = \beta_2 + b_2$ and;

$$c=\sum_{k=0}^{n-r}\binom{n-r}{k}\frac{\Gamma(r_1+a_1)}{(\beta_1^{**})^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{(\beta_2^{**})^{r_2+a_2}}[B(n-r_2-k+2,r_2+k+1)+B(n-r_2-k+1,r_2+k+2)]$$

The $(1-\alpha)$100% Bayesian predictive interval (L,U) is obtained by solving the following equations:

$$\int_0^L p(y\,|\,x)dy=\frac{\alpha}{2}=\int_U^\infty p(y\,|\,x)dy$$

Now lower and upper bound of predictive intervals, respectively are:

$$\frac{1}{c}\sum_{k=0}^{n-r}\binom{n-r}{k}\Gamma(r_1+a_1)\Gamma(r_2+a_2)[\frac{B(n-r_2-k+2,r_2+k+1)}{(\beta_2^{**})^{r_2+a_2}}\{((\beta_1^{**})^{(r_1+a_1)})^{-1}-((\beta_1^{**}+\ln L)^{(r_1+a_1)})^{-1}\}+ \frac{B(n-r_2-k+1,r_2+k+2)}{(\beta_1^{**})^{r_1+a_1}}\{((\beta_2^{**})^{(r_2+a_2)})^{-1}-((\beta_2^{**}+\ln L)^{(r_2+a_2)})^{-1}]-\frac{\alpha}{2}=0 \quad (16)$$

$$\frac{1}{c}\sum_{k=0}^{n-r}\binom{n-r}{k}\Gamma(r_1+a_1)\Gamma(r_2+a_2)[\frac{B(n-r_2-k+2,r_2+k+1)}{(\beta_1^{**}+\ln U)^{(r_1+a_1)}(\beta_2^{**})^{r_2+a_2}}+\frac{B(n-r_2-k+1,r_2+k+2)}{(\beta_1^{**})^{r_1+a_1}(\beta_2^{**}+\ln U)^{(r_2+a_2)}}]-\frac{\alpha}{2}=0 \quad (17)$$

Saleem and Aslam (2008) used predictive intervals for the Rayleigh mixture to obtain hyper parameters. We evaluate these two Equations (16) and (17) for different values of hyper parameters and select the hyper parameters where the difference between these bounds is minimum.

SIMULATION STUDY

In order to investigate the properties of the derived estimators, a large scale simulation study is carried out for the Pareto random variable (mixture) with parameters θ_1,θ_2 *and* p. This simulation is carried out in Minitab by generating pseudo-random uniform numbers by using inverse transform method. We simulate 1000 data sets for size of each n. We use $n=50,100,200,300$ *and* 500 to generate random data from the two parameter Pareto mixture with (θ_1,θ_2) = (0.5,1.5), (1.0,4.0), (2.5,0.5), (4.0,1.0) and $p=0.40$, considering different values for parameter λ at 10% fixed censoring rate. To generate mixture data probabilistic mixing is used. As censored sampling is being used, all the observations greater than T are excluded. Maximum likelihood estimates are not discussed as major focus is on Bayesian approach but they can be found by solving system of three non-linear Equations (4) to (6). Further information matrix can be inverted to have variances of the above three ML estimates on the main diagonal and their co variances on the off diagonal positions. Bayes estimates are calculated in Mathematics using the expressions derived in Section 4 to 5. Numerical results obtained as a result of simulation are presented in Tables 1 to 8. Hyper-parameters to be used in informative prior are elicited by using Equations (16) and (17). So elicited hyper-parameters are $a_1=a_2=2,\ b_1,b_2=1$.

A REAL-DATA APPLICATION

Data Given in Table 9 is the data of Gross fixed investment recorded from the fiscal years 1973 to 2005. Taken from “investments and savings at current prices” for the same period by Federal Bureau of Statistics, Pakistan. These data come from two sectors, public sector and private sector. Considering the public sector as sub-population I and private sector as sub-population-II, data is suitable for fitting a mixture model. The Real data set is fitted to the mixture of two parameter Pareto model which is popular for representing income, investment, etc. Data is truncated at T = 170,000; so the values greater than this T are ignored. The following necessary information needed to fit our mixture model are extracted from Table 9.

$$n=66,\quad n-r=18\quad r_1=25\quad r_2=23\quad r=r_1+r_2=48$$
$$\sum\ln(x_{1j}+\lambda)=265.46854\quad \sum\ln(x_{2j}+\lambda)=234.879$$

Bayes estimates are attained by using uniform prior. The obtained estimates are $\hat{\theta}_1=0.0648$, $\hat{\theta}_2=0.0677$ and $\hat{p}=0.5228$ with standard errors $\pi(\hat{\theta}_1)=0.0002617,\ \pi(\hat{\theta}_2)=0.0002473$ and $\pi(\hat{p})=0.00544$. Obtained estimates reveal that investment and savings for both sectors are almost same.

Table 1. Bayes estimates of Pareto mixture parameters and their posterior risk (in parenthesis) under squared error loss function when $\theta_1 = 0.5, \theta_2 = 1.5, \lambda = (0.5, 1.5, 2.5)\, and\ p = 0.40$.

10% censoring		Uniform prior			Jeffreys prior			Gamma prior		
						$\lambda = 0.5$				
(θ_1, θ_2, p)	n	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$
	50	0.6305(0.0449)	1.5287(0.1382)	0.4026(0.0053)	0.5896(0.0395)	1.4766(0.1298)	0.4027(0.0052)	0.5159(0.0232)	0.9918(0.0506)	0.3847(0.0053)
	100	0.5976(0.0162)	1.5109(0.0620)	0.3967(0.0026)	0.5787(0.0154)	1.4838(0.0607)	0.3966(0.0026)	0.5472(0.0135)	1.1828(0.0386)	0.3857(0.0027)
	200	0.5293(0.0049)	1.5633(0.0276)	0.4045(0.0013)	0.5211(0.0049)	1.5488(0.0272)	0.4045(0.0012)	0.5058(0.0048)	1.3766(0.0013)	0.4008(0.0013)
	300	0.5878(0.0045)	1.4879(0.0191)	0.3947(0.0009)	0.5817(0.0045)	1.4787(0.0189)	0.3946(0.0009)	0.5301(0.0044)	1.3636(0.0166)	0.3910(0.0009)
	500	0.5860(0.0027)	1.5003(0.0114)	0.3969(0.0005)	0.5825(0.0025)	1.4947(0.0112)	0.3969(0.0005)	0.5350(0.0026)	1.4225(0.0104)	0.3949(0.0005)
						$\lambda = 1.5$				
	50	0.4105(0.0115)	1.5800(0.1022)	0.4186(0.0047)	0.3858(0.0108)	1.5223(0.0989)	0.4182(0.0047)	0.3631(0.0101)	1.0268(0.0519)	0.4051(0.0051)
	100	0.4704(0.0078)	1.5199(0.0518)	0.4026(0.0025)	0.4559(0.0076)	1.4916(0.0507)	0.4023(0.0025)	0.4352(0.0070)	1.2031(0.0346)	0.3939(0.0026)
(0.5,1.5,0.4)	200	0.4892(0.0042)	1.4773(0.0251)	0.4006(0.0014)	0.4816(0.0042)	1.4634(0.0247)	0.4005(0.0013)	0.4689(0.0040)	1.3087(0.0203)	0.3963(0.0013)
	300	0.4986(0.0030)	1.4799(0.0169)	0.3956(0.0008)	0.4934(0.0030)	1.4707(0.0166)	0.3955(0.0008)	0.4840(0.0029)	1.3632(0.0145)	0.3925(0.0009)
	500	0.5110(0.0019)	1.4592(0.0098)	0.3954(0.0005)	0.5079(0.0018)	1.4536(0.0100)	0.3953(0.0005)	0.5018(0.0018)	1.3883(0.0091)	0.3935(0.0005)
						$\lambda = 2.5$				
	50	0.5132(0.0194)	1.4029(0.0941)	0.3999(0.0051)	0.4824(0.0182)	1.3513(0.0897)	0.3993(0.0051)	0.4300(0.0131)	0.9565(0.0396)	0.3834(0.0052)
	100	0.4958(0.0089)	1.4513(0.0488)	0.3937(0.0025)	0.4805(0.0086)	1.4243(0.0477)	0.3933(0.0026)	0.4525(0.0075)	1.1644(0.0306)	0.3841(0.0026)
	200	0.4211(0.0030)	1.5138(0.0236)	0.4055(0.0012)	0.4145(0.0029)	1.4997(0.0236)	0.4054(0.0012)	0.4043(0.0004)	1.3464(0.0198)	0.4026(0.0013)
	300	0.4228(0.0019)	1.5124(0.0157)	0.4018(0.0008)	0.4183(0.0020)	1.5030(0.0158)	0.4017(0.0008)	0.4110(0.0020)	1.3976(0.0140)	0.3999(0.0008)
	500	0.4308(0.0013)	1.4963(0.0095)	0.4024(0.0005)	0.4281(0.0013)	1.4908(0.0092)	0.4023(0.0005)	0.4235(0.0012)	1.4263(0.0088)	0.4012(0.0005)

Graphical representaion of marginal densities using real data

Assuming symmetry graphs of the marginal densities confirm the results of Bayes estimates Figure 1.

Testing of hypotheses

In order to conduct testing procedure set of hypotheses are formulated and posterior probabilities are obtained that are given in Table 10 by using the above mentioned real life data. Posterior probabilities are obtained in Mathematics by using marginal densities. In Bayesian statistics, to make decision upon hypotheses is very simple. Posterior probabilities obtained under H_0 and H_1 directly help to make inference. It is found by testing procedure that the parameters θ_1, θ_2 are believed to be greater than 0.05 but they do not exceed 0.08 as when value of a parameter exceeds 0.08, data supports alternative hypothesis. So it can be concluded that $0.05 \le \theta_1, \theta_2 \le 0.08$. Similarly testing for different values we find that $p \ge 0.5$.

Conclusion

Some important features of the Bayes estimates (Table 1 to 8) observed from simulation study are as under. The posterior risk of the estimates θ_1, θ_2 is quite larger for larger values of parameter and is smaller when smaller values of parameters are considered. However posterior risks are reduced in any case as the sample size is increased. We get somewhat more precise and more efficient estimates as we shift the value of λ to a greater point. Comparing the three used priors, it is evident that estimates are bit precise in

Table 2. Bayes estimates of Pareto mixture parameters and their posterior risk (in parenthesis) under squared error loss function when $\theta_1 = 1.0, \theta_2 = 4.0, \lambda = (0.5, 1.5, 2.5)\ and\ p = 0.40$.

10% censoring		Uniform prior			Jeffreys prior			Gamma prior		
					$\lambda = 0.5$					
(θ_1, θ_2, p)	n	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$
	50	1.2441(0.5730)	4.1858(0.9297)	0.3968(0.0051)	1.1145(0.3460)	4.0827(0.8132)	0.3984(0.0049)	0.8193(0.0646)	2.7436(0.1571)	0.3698(0.0053)
	100	1.4934(0.1723)	3.8726(0.4489)	0.3899(0.0027)	1.4361(0.1467)	3.8129(0.4264)	0.3903(0.0076)	1.1834(0.0682)	2.9919(0.1414)	0.3679(0.0027)
	200	1.3098(0.0297)	4.1329(0.1751)	0.4024(0.0012)	1.2889(0.0293)	4.0957(0.1741)	0.4023(0.0012)	1.1694(0.0270)	3.0608(0.1189)	0.3943(0.0013)
	300	1.2421(0.0175)	4.1113(0.1098)	0.4017(0.0008)	1.2287(0.0174)	4.0870(0.1094)	0.4017(0.0008)	1.1419(0.0155)	3.3802(0.0832)	0.3983(0.0009)
	500	1.2779(0.0113)	4.0186(0.0649)	0.3993(0.0005)	1.2696(0.0113)	4.0042(0.0648)	0.3993(0.0005)	1.2125(0.0105)	3.5621(0.0548)	0.3971(0.0005)
					$\lambda = 1.5$					
	50	1.0545(0.0795)	3.9523(0.6657)	0.3871(0.0048)	0.9868(0.0746)	3.8087(0.6397)	0.3863(0.0048)	0.7227(0.0367)	2.7903(0.1186)	0.3543(0.0050)
	100	0.9683(0.0322)	4.0218(0.3279)	0.3948(0.0024)	0.9370(0.0314)	3.9483(0.3225)	0.3946(0.0025)	0.8136(0.0240)	3.0240(0.1260)	0.3762(0.0025)
(1.0,4.0,0.40)	200	0.8505(0.0120)	4.0265(0.1474)	0.4044(0.0012)	0.8368(0.0117)	3.9911(0.1466)	0.4044(0.0012)	0.7761(0.0101)	3.0890(0.0957)	0.4009(0.0012)
	300	0.8674(0.0084)	4.0156(0.0994)	0.4022(0.0008)	0.8580(0.0083)	3.9920(0.0984)	0.4021(0.0008)	0.8136(0.0075)	3.3427(0.0738)	0.3997(0.0008)
	500	0.9049(0.0056)	3.9911(0.0605)	0.3996(0.0005)	0.8991(0.0129)	3.9768(0.0600)	0.3996(0.0005)	0.8695(0.0051)	3.5574(0.0504)	0.3978(0.0005)
					$\lambda = 2.5$					
	50	0.7842(0.0391)	4.2334(0.6443)	0.4015(0.0046)	0.7361(0.0367)	4.0869(0.6262)	0.4012(0.0046)	0.7095(0.0254)	1.8020(0.1387)	0.3743(0.0050)
	100	0.7447(0.0187)	4.0885(0.3028)	0.3999(0.0023)	0.7202(0.0181)	4.0170(0.2988)	0.3997(0.0024)	0.7425(0.0145)	2.5034(0.1319)	0.3890(0.0025)
	200	0.7707(0.0098)	3.9839(0.1463)	0.4039(0.0012)	0.7582(0.0097)	3.9485(0.1454)	0.4038(0.0012)	0.7098(0.0085)	3.0561(0.0939)	0.3997(0.0012)
	300	0.7619(0.0064)	3.9703(0.0955)	0.4022(0.0008)	0.7536(0.0064)	3.9469(0.0954)	0.4022(0.0008)	0.7196(0.0058)	3.3164(0.0707)	0.4000(0.0008)
	500	0.7550(0.0038)	3.9892(0.0578)	0.4009(0.0005)	0.7500(0.0038)	3.9752(0.0575)	0.4009(0.0005)	0.7287(0.0036)	3.5726(0.0476)	0.3998(0.0005)

Table 3. Bayes estimates of Pareto mixture parameters and their posterior risk (in parenthesis) under squared error loss function when $\theta_1 = 2.5, \theta_2 = 0.5, \lambda = (0.5, 1.5, 2.5)\ and\ p = 0.40$.

10% censoring		Uniform prior			Jeffreys prior			Gamma prior		
					$\lambda = 0.5$					
(θ_1, θ_2, p)	n	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$
	50	2.5803(0.3658)	0.6607(0.0178)	0.4067(0.0046)	2.4415(0.3515)	0.6356(0.0172)	0.4073(0.0047)	1.2530(0.0853)	0.5762(0.0145)	0.4277(0.0051)
	100	2.5580(0.1625)	0.5263(0.0056)	0.4021(0.0023)	2.4946(0.1590)	0.5158(0.0054)	0.4021(0.0023)	1.6746(0.0770)	0.4881(0.0048)	0.4035(0.0023)
(2.5,0.5,0.40)	200	2.5342(0.0808)	0.5499(0.0030)	0.4011(0.0012)	2.5023(0.0800)	0.5443(0.0031)	0.4011(0.0012)	2.011(0.0533)	0.5277(0.0027)	0.4016(0.0009)
	300	2.5219(0.0538)	0.5605(0.0021)	0.4008(0.0008)	2.5006(0.0535)	0.5568(0.0021)	0.4008(0.0008)	2.1503(0.0402)	0.5448(0.0020)	0.4011(0.0008)
	500	2.5240(0.0322)	0.5445(0.0012)	0.4005(0.0005)	2.5113(0.0320)	0.5424(0.0011)	0.4005(0.0005)	2.2894(0.0268)	0.5354(0.0011)	0.4006(0.0005)

Table 3. Contd.

				$\lambda = 1.5$					
50	2.6381(0.3350)	0.4365(0.0077)	0.4039(0.0046)	2.5110(0.3197)	0.4191(0.0073)	0.4040(0.0045)	1.2597(0.0932)	0.3893(0.0063)	0.4083(0.0047)
100	2.5890(0.1678)	0.4710(0.0034)	0.4020(0.0024)	2.5251(0.1616)	0.4606(0.0043)	0.4021(0.0023)	1.6945(0.0767)	0.4388(0.0039)	0.4032(0.0023)
200	2.5327(0.0807)	0.4852(0.0023)	0.4011(0.0012)	2.5010(0.0797)	0.4803(0.0024)	0.4011(0.0012)	2.0139(0.0527)	0.4676(0.0022)	0.4015(0.0012)
300	2.5219(0.0538)	0.5010(0.0017)	0.4008(0.0008)	2.5007(0.0532)	0.4977(0.0016)	0.4008(0.0008)	2.1513(0.0402)	0.4884(0.0016)	0.4011(0.0008)
500	2.5168(0.0316)	0.4790(0.0010)	0.4005(0.0004)	2.5041(0.0318)	0.4771(0.0009)	0.4005(0.0004)	2.2846(0.0265)	0.4719(0.0009)	0.4005(0.0005)
				$\lambda = 2.5$					
50	2.6687(0.3460)	0.4450(0.0079)	0.4040(0.0046)	2.5388(0.3322)	0.4273(0.0076)	0.4041(0.0046)	1.2501(0.0941)	0.3974(0.0066)	0.4105(0.0047)
100	2.5568(0.1616)	0.4467(0.0040)	0.4020(0.0024)	2.4937(0.0039)	0.4378(0.0039)	0.4021(0.0023)	1.6821(0.0754)	0.4183(0.0036)	0.4031(0.0023)
200	2.5092(0.0781)	0.4145(0.0017)	0.401(0.0012)	2.4782(0.0768)	0.4103(0.0017)	0.4010(0.0012)	2.0083(0.0504)	0.4013(0.0016)	0.4011(0.0012)
300	2.5238(0.0531)	0.4632(0.0014)	0.4007(0.0008)	2.5026(0.0531)	0.4601(0.0014)	0.4008(0.0008)	2.1561(0.0396)	0.4523(0.0013)	0.4009(0.0008)
500	2.5237(0.0323)	0.4803(0.0009)	0.4006(0.0004)	2.5109(0.0322)	0.4783(0.0010)	0.4006(0.0004)	2.2876(0.0269)	0.4732(0.0009)	0.4007(0.0005)

Table 4. Bayes estimates of Pareto mixture parameters and their posterior risk (in parenthesis) under squared error loss function when $\theta_1 = 4.0, \theta_2 = 1.0, \lambda = (0.5, 1.5, 2.5)\ and\ p = 0.40$.

10% censoring		Uniform prior			Jeffreys prior			Gamma prior		
					$\lambda = 0.5$					
(θ_1, θ_2, p)	n	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$
	50	4.2649(1.7583)	1.1094(0.0508)	0.4047(0.0048)	4.0461(0.9100)	1.067(0.0496)	0.4050(0.0046)	2.3041(0.1537)	0.9459(0.0520)	0.4404(0.0056)
	100	3.9797(0.4544)	1.2887(0.0343)	0.4042(0.0023)	3.8695(0.4474)	1.2645(0.0335)	0.4044(0.0024)	2.6417(0.1466)	1.14821(0.0325)	0.4245(0.0027)
	200	4.010(0.2186)	1.2398(0.0155)	0.4022(0.0012)	3.9556(0.2199)	1.2278(0.0153)	0.4023(0.0012)	2.7089(0.1256)	1.1522(0.0142)	0.4080(0.0013)
	300	4.0297(0.1479)	1.2524(0.0105)	0.4017(0.0008)	3.9939(0.1464)	1.2443(0.0104)	0.4017(0.0008)	3.0835(0.0981)	1.1856(0.0097)	0.4044(0.0008)
	500	4.0028(0.0857)	1.1871(0.0056)	0.4010(0.0022)	3.9817(0.0856)	1.1824(0.0056)	0.4011(0.0004)	3.4063(0.0663)	1.1479(0.0054)	0.4020(0.0005)
					$\lambda = 1.5$					
	50	4.0407(0.8772)	0.9787(0.0389)	0.4061(0.0046)	3.8264(0.8435)	0.9412(0.0375)	0.4066(0.0046)	2.3840(0.1174)	0.7988(0.0280)	0.4348(0.0052)
	100	4.0997(0.4496)	0.9878(0.0196)	0.4033(0.0024)	3.9915(0.4404)	0.9685(0.0192)	0.4035(0.0023)	2.3823(0.1411)	0.8814(0.0165)	0.4159(0.0025)
(4.0,1.0,0.40)	200	4.0528(0.2088)	0.8968(0.0081)	0.4012(0.0012)	4.0014(0.2066)	0.8879(0.0080)	0.4012(0.0012)	3.0687(0.1245)	0.8362(0.0069)	0.4010(0.0012)
	300	4.0281(0.1420)	0.9536(0.0061)	0.4013(0.0008)	3.9931(0.1414)	0.9473(0.0061)	0.4013(0.0008)	3.1182(0.0924)	0.9119(0.0056)	0.4029(0.0008)
	500	4.0227(0.0825)	0.8809(0.0030)	0.4005(0.0005)	4.0024(0.0816)	0.8774(0.0030)	0.4005(0.0005)	3.4525(0.0622)	0.8575(0.0029)	0.4008(0.0005)
					$\lambda = 2.5$					
	50	4.2651(0.8981)	0.7825(0.0296)	0.4042(0.0046)	4.0545(0.8603)	0.7515(0.0238)	0.4044(0.0045)	2.4391(0.1353)	0.6599(0.0191)	0.4235(0.0051)
	100	4.0401(0.4200)	0.8527(0.0146)	0.4023(0.0026)	3.9367(0.4111)	0.8359(0.0142)	0.4027(0.0023)	2.5098(0.1403)	0.7672(0.0123)	0.4109(0.0025)
	200	4.133(0.2184)	0.8404(0.0070)	0.4013(0.0012)	4.0804(0.2159)	0.8320(0.0070)	0.4013(0.0012)	2.8695(0.1157)	0.7917(0.0064)	0.4032(0.0012)
	300	3.9558(0.1332)	0.8266(0.0045)	0.4009(0.0008)	3.9223(0.1320)	0.8211(0.0045)	0.4009(0.0008)	3.1017(0.0866)	0.7936(0.0041)	0.4017(0.0008)
	500	4.0396(0.0846)	0.8610(0.0030)	0.4008(0.0005)	4.0188(0.0840)	0.8576(0.0030)	0.4008(0.0005)	3.4518(0.0639)	0.8393(0.0029)	0.4014(0.0005)

Table 5. Bayes estimates of Pareto mixture parameters and their posterior risk (in parenthesis) under weighted loss function when $\theta_1 = 0.5, \theta_2 = 1.5, \lambda = (0.5, 1.5, 2.5)$ *and* $p = 0.40$.

10% censoring		Uniform prior			Jeffreys prior			Gamma prior		
		$\lambda = 0.5$								
(θ_1, θ_2, p)	n	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$
	50	0.5722(0.0583)	1.4328(0.0959)	0.3886(0.0140)	0.5338(0.0558)	1.384(0.0926)	0.3888(0.0139)	0.4739(0.0420)	0.9413(0.0505)	0.3700(0.0147)
	100	0.5726(0.025)	1.4681(0.0428)	0.3900(0.0067)	0.5540(0.0247)	1.4415(0.0423)	0.3898(0.0068)	0.5238(0.0234)	1.1499(0.0329)	0.3785(0.0072)
	200	0.5200(0.0093)	1.5455(0.0178)	0.4015(0.0030)	0.51176(0.0093)	1.5310(0.0178)	0.4014(0.0031)	0.4965(0.0093)	1.3595(0.0171)	0.3976(0.0032)
	300	0.5802(0.0076)	1.4750(0.0129)	0.3925(0.0022)	0.5741(0.0076)	1.4658(0.0129)	0.3924(0.0022)	0.5625(0.0076)	1.3513(0.0123)	0.3888(0.0022)
	500	0.5817(0.0043)	1.4927(0.0076)	0.3956(0.0013)	0.5780(0.0045)	1.4871(0.0076)	0.3956(0.0013)	0.5706(0.0044)	1.4150(0.0075)	0.3936(0.0013)
		$\lambda = 1.5$								
	50	0.3833(0.0272)	1.5126(0.0674)	0.4068(0.0118)	0.3585(0.0273)	1.4545(0.0678)	0.4063(0.0119)	0.3366(0.0265)	0.9750(0.0518)	0.3920(0.0131)
	100	0.4540(0.0164)	1.4853(0.0346)	0.3963(0.0063)	0.4394(0.0165)	1.4569(0.0347)	0.3960(0.0063)	0.4193(0.0159)	1.1740(0.0291)	0.3873(0.0066)
(0.5,1.5,0.40)	200	0.4806(0.0086)	1.4603(0.0170)	0.3975(0.0031)	0.4731(0.0085)	1.4463(0.0171)	0.3973(0.0032)	0.4605(0.0084)	1.2930(0.0157)	0.3930(0.0033)
	300	0.4927(0.0059)	1.4685(0.0114)	0.3935(0.0021)	0.4875(0.0059)	1.4593(0.0114)	0.3933(0.0022)	0.4782(0.0058)	1.3524(0.0108)	0.3903(0.0022)
	500	0.5074(0.0036)	1.4523(0.0069)	0.394(0.0013)	0.5042(0.0037)	1.4468(0.0068)	0.3940(0.0013)	0.4982(0.0036)	1.3817(0.0066)	0.3922(0.0013)
		$\lambda = 2.5$								
	50	0.4768(0.0364)	1.3354(0.0675)	0.3865(0.0134)	0.4461(0.0363)	1.2846(0.0667)	0.3858(0.0135)	0.4000(0.0300)	1.0455(0.0789)	0.3692(0.0142)
	100	0.4782(0.0176)	1.4176(0.0337)	0.3871(0.0066)	0.4629(0.0176)	1.3907(0.0336)	0.3867(0.0066)	0.4379(0.0146)	1.1381(0.0263)	0.3772(0.0069)
	200	0.4140(0.0071)	1.4980(0.0158)	0.4025(0.0030)	0.4075(0.0070)	1.4839(0.0158)	0.4024(0.0030)	0.3974(0.0069)	1.3312(0.0152)	0.3995(0.0031)
	300	0.4180(0.0048)	1.5019(0.0105)	0.3997(0.0021)	0.4135(0.0048)	1.4925(0.0105)	0.3996(0.0114)	0.4063(0.0047)	1.3876(0.0100)	0.3978(0.0021)
	500	0.4279(0.0029)	1.4900(0.0063)	0.4011(0.0013)	0.4252(0.0029)	1.4844(0.0064)	0.4011(0.0012)	0.4206(0.0029)	1.4202(0.0061)	0.4000(0.0012)

Table 6. Bayes estimates of Pareto mixture parameters and their posterior risk (in parenthesis) under weighted loss function when $\theta_1 = 1.0, \theta_2 = 4.0, \lambda = (0.5, 1.5, 2.5)$ *and* $p = 0.40$.

10% censoring		Uniform prior			Jeffreys prior			Gamma prior		
		$\lambda = 0.5$								
(θ_1, θ_2, p)	n	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$
	50	1.0624(0.1817)	3.8916(0.2942)	0.3828(0.0140)	0.9757(0.1388)	3.8345(0.2482)	0.3851(0.0133)	0.7468(0.0725)	1.6537(0.0899)	0.3548(0.0150)
	100	1.4099(0.0835)	3.7433(0.1239)	0.3828(0.0071)	1.3587(0.0774)	3.6900(0.1229)	0.3833(0.0070)	1.1286(0.0548)	2.2302(0.0617)	0.3602(0.0077)
(1.0,4.0,0.40)	200	1.2871(0.0227)	4.0900(0.0429)	0.3994(0.0030)	1.2662(0.0227)	4.0528(0.0429)	0.3993(0.0030)	1.1467(0.0227)	3.0213(0.0395)	0.3910(0.0033)
	300	1.2280(0.0141)	4.0846(0.0267)	0.3997(0.0020)	1.2146(0.0141)	4.0603(0.0267)	0.3996(0.0021)	1.1283(0.0136)	3.3553(0.0249)	0.3963(0.0020)
	500	1.2691(0.0088)	4.0024(0.0162)	0.3981(0.0012)	1.2608(0.0088)	3.9880(0.0162)	0.3981(0.0012)	1.2039(0.0086)	3.5465(0.0156)	0.3959(0.0012)

Table 6. Contd.

					$\lambda = 1.5$					
	50	0.9800(0.0745)	3.7825(0.1698)	0.3739(0.0132)	0.9123(0.0745)	3.6397(0.1690)	0.3730(0.0133)	0.6724(0.0503)	1.7247(0.0656)	0.3394(0.0149)
	100	0.9349(0.0334)	3.9395(0.0823)	0.3886(0.0062)	0.9036(0.0334)	3.8659(0.0824)	0.3883(0.0063)	0.7845(0.0291)	2.9732(0.0508)	0.3692(0.0070)
	200	0.8365(0.0140)	3.9897(0.0368)	0.4015(0.0029)	0.8227(0.0141)	3.9543(0.0368)	0.4014(0.0030)	0.7630(0.0131)	3.0577(0.0313)	0.3978(0.0031)
	300	0.8578(0.0096)	3.9909(0.0247)	0.4002(0.0020)	0.8484(0.0096)	3.9672(0.0248)	0.4001(0.0020)	0.8045(0.0091)	3.3207(0.0220)	0.3977(0.0020)
	500	0.8989(0.0060)	3.9759(0.0152)	0.3984(0.0012)	0.8930(0.0061)	3.9616(0.0152)	0.3984(0.0012)	0.8636(0.0059)	3.5433(0.0141)	0.3966(0.0012)
					$\lambda = 2.5$					
	50	0.7346(0.0496)	4.0787(0.1547)	0.3896(0.0119)	0.6864(0.0497)	3.9310(0.1559)	0.3893(0.0119)	0.5685(0.0410)	1.7254(0.0766)	0.3602(0.0141)
	100	0.7197(0.0250)	4.0139(0.0746)	0.3939(0.0060)	0.6952(0.0250)	3.9422(0.0748)	0.3937(0.0060)	0.6202(0.0223)	2.4504(0.0530)	0.3825(0.0065)
	200	0.7580(0.0127)	3.9471(0.0368)	0.4009(0.0030)	0.7455(0.0127)	3.9116(0.0369)	0.4008(0.0030)	0.6979(0.0119)	3.0251(0.0310)	0.3966(0.0031)
	300	0.7535(0.0084)	3.9460(0.0243)	0.4002(0.0020)	0.7452(0.0084)	3.9227(0.0242)	0.4002(0.0020)	0.7116(0.0080)	3.2949(0.0215)	0.3980(0.0020)
	500	0.7500(0.0050)	3.9748(0.0144)	0.3997(0.0012)	0.7450(0.0050)	3.9607(0.0145)	0.3997(0.0012)	0.7238(0.0049)	3.5592(0.0134)	0.3986(0.0012)

Table 7. Bayes estimates of Pareto mixture parameters and their posterior risk (in parenthesis) under weighted loss function when $\theta_1 = 2.5, \theta_2 = 0.5, \lambda = (0.5, 1.5, 2.5)$ *and* $p = 0.40$.

10% censoring		Uniform prior			Jeffreys prior			Gamma prior		
					$\lambda = 0.5$					
(θ_1, θ_2, p)	n	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$
	50	2.4349(0.1454)	0.6339(0.0268)	0.3949(0.0118)	2.2933(0.1482)	0.6086(0.0270)	0.3955(0.0118)	1.0794(0.0736)	0.5511(0.0251)	0.4152(0.0125)
	100	2.4944(0.0636)	0.5158(0.0105)	0.3961(0.0060)	2.4308(0.0638)	0.5053(0.0105)	0.3962(0.0059)	1.6280(0.0466)	0.4782(0.0099)	0.3975(0.0060)
	200	2.5022(0.0320)	0.5443(0.0056)	0.3981(0.0030)	2.4703(0.0320)	0.5389(0.0054)	0.3982(0.0029)	1.9844(0.0266)	0.5224(0.0053)	0.3986(0.0030)
	300	2.5005(0.0214)	0.5568(0.0037)	0.3988(0.0020)	2.4792(0.0214)	0.5530(0.0038)	0.3988(0.0020)	2.1315(0.0188)	0.5412(0.0036)	0.3991(0.0020)
	500	2.5113(0.0127)	0.5424(0.0021)	0.3993(0.0012)	2.4985(0.0155)	0.5402(0.0022)	0.3993(0.0012)	2.4349(0.0154)	0.6339(0.0027)	0.3949(0.0012)
					$\lambda = 1.5$					
	50	2.5105(0.1276)	0.4191(0.0174)	0.3922(0.0117)	2.3827(0.1283)	0.4016(0.0175)	0.3923(0.0117)	1.1825(0.0772)	0.3731(0.0162)	0.3964(0.0119)
	100	2.5249(0.0641)	0.4606(0.0104)	0.3961(0.0059)	2.4608(0.0643)	0.4512(0.0094)	0.3962(0.0059)	1.6486(0.0459)	0.4299(0.0089)	0.3972(0.0060)
(2.5,0.5,0.40)	200	2.5010(0.0317)	0.4803(0.0049)	0.3981(0.0030)	2.4692(0.0318)	0.4755(0.0048)	0.3981(0.0030)	1.9877(0.0262)	0.4629(0.0047)	0.3985(0.0030)
	300	2.5006(0.0213)	0.4977(0.0033)	0.3988(0.0020)	2.4793(0.0214)	0.4943(0.0034)	0.3988(0.0020)	2.1326(0.0187)	0.4851(0.0033)	0.3992(0.0019)
	500	2.5041(0.0127)	0.4771(0.0019)	0.3993(0.0012)	2.4915(0.0126)	0.4752(0.0019)	0.3993(0.0012)	2.2730(0.0116)	0.4700(0.0019)	0.3993(0.0012)
					$\lambda = 2.5$					
	50	2.5381(0.1306)	0.4272(0.0178)	0.3923(0.0117)	2.4073(0.1315)	0.4095(0.0178)	0.3924(0.0117)	1.1727(0.0774)	0.3809(0.0165)	0.3985(0.0120)
	100	2.4936(0.0632)	0.4378(0.0089)	0.3961(0.0059)	2.4303(0.0634)	0.4289(0.0089)	0.3961(0.0060)	1.6370(0.0451)	0.4099(0.0084)	0.3971(0.0060)
	200	2.4782(0.0310)	0.4103(0.0042)	0.3980(0.0030)	2.4471(0.0311)	0.4062(0.0041)	0.3980(0.0030)	1.9831(0.0252)	0.3973(0.0040)	0.3981(0.0030)
	300	2.5026(0.0212)	0.4601(0.0031)	0.3988(0.0019)	2.4815(0.0211)	0.4570(0.0031)	0.3988(0.0020)	2.1376(0.0185)	0.4492(0.0031)	0.3989(0.0020)
	500	2.5108(0.0129)	0.4783(0.0020)	0.3994(0.0012)	2.4980(0.0129)	0.4764(0.0019)	0.3994(0.0012)	2.2758(0.0118)	0.4713(0.0019)	0.3995(0.0012)

Table 8. Bayes estimates of Pareto mixture parameters and their posterior risk (in parenthesis) under weighted loss function when $\theta_1 = 4.0, \theta_2 = 1.0, \lambda = (0.5, 1.5, 2.5)$ *and* $p = 0.40$ ·

10% censoring		Uniform prior			Jeffreys prior			Gamma prior		
					$\lambda = 0.5$					
(θ_1, θ_2, p)	n	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{p}$
	50	4.0355(0.2294)	1.0642(0.0452)	0.3930(0.0117)	3.8089(0.2372)	1.0213(0.0457)	0.3932(0.0118)	1.2002(1.1039)	0.8941(0.0518)	0.4272(0.0132)
	100	3.8633(0.1164)	1.2624(0.0263)	0.3982(0.0060)	3.7514(0.1181)	1.2380(0.0265)	0.3985(0.0059)	1.8658(0.7759)	1.1205(0.0277)	0.4180(0.0065)
	200	3.9539(0.0561)	1.2272(0.0126)	0.3992(0.0030)	3.8997(0.0559)	1.2152(0.0126)	0.3993(0.0030)	2.6619(0.0470)	1.1399(0.0123)	0.4050(0.0030)
	300	3.9930(0.0367)	1.2440(0.0084)	0.3997(0.0020)	3.9570(0.0369)	1.2358(0.0085)	0.3997(0.0020)	3.0514(0.0321)	1.1774(0.0082)	0.4023(0.0021)
	500	3.9814(0.0214)	1.1823(0.0048)	0.3998(0.0012)	3.9603(0.0214)	1.1776(0.0048)	0.3999(0.0012)	3.3868(0.0195)	1.1433(0.0046)	0.4008(0.0012)
					$\lambda = 1.5$					
	50	3.8183(0.2224)	0.9391(0.0396)	0.3943(0.0118)	3.6001(0.2263)	0.9014(0.0398)	0.3948(0.0118)	1.3002(1.0838)	0.7641(0.0347)	0.4223(0.0125)
	100	3.9890(4.0997)	0.9679(0.0199)	0.3974(0.0059)	3.8799(0.1116)	0.9486(0.0199)	0.3975(0.0060)	2.0142(0.3681)	0.8627(0.0187)	0.4097(0.0062)
(4.0,1.0,0.40)	200	4.0010(0.0518)	0.8878(0.0090)	0.3982(0.0030)	3.9496(0.0518)	0.8789(0.0090)	0.3983(0.0029)	2.7946(0.2741)	0.8328(0.0034)	0.3999(0.0011)
	300	3.9927(0.0354)	0.9472(0.0064)	0.3993(0.0020)	3.9577(0.0354)	0.9410(0.0063)	0.3993(0.0020)	3.0885(0.0297)	0.9058(0.0061)	0.4009(0.0020)
	500	4.0023(0.0204)	0.8773(0.0036)	0.3993(0.0012)	3.9819(0.0205)	0.8738(0.0036)	0.3993(0.0012)	3.4345(0.0180)	0.8541(0.0034)	0.3996(0.0012)
					$\lambda = 2.5$					
	50	4.0523(0.2128)	0.7511(0.0314)	0.3925(0.0117)	3.8394(0.2151)	0.7201(0.0314)	0.3927(0.0117)	1.3451(1.0940)	0.6313(0.0286)	0.4111(0.0124)
	100	3.9354(0.1047)	0.8356(0.0171)	0.3966(0.0057)	3.8313(0.1054)	0.8187(0.0172)	0.3967(0.0060)	2.0426(0.4672)	0.7513(0.0159)	0.4048(0.0061)
	200	4.0800(0.0530)	0.8320(0.0084)	0.3983(0.0030)	4.0273(0.0531)	0.8236(0.0084)	0.3983(0.0030)	2.8289(0.0406)	0.7837(0.0080)	0.4002(0.0030)
	300	3.9221(0.0337)	0.8211(0.0055)	0.3989(0.0020)	3.8885(0.0338)	0.8156(0.0055)	0.3989(0.0020)	3.0738(0.0279)	0.7882(0.0054)	0.3997(0.0020)
	500	4.0186(0.0210)	0.8576(0.0034)	0.3996(0.0012)	3.9978(0.0210)	0.8542(0.0034)	0.3996(0.0012)	3.4332(0.0186)	0.8359(0.0034)	0.4002(0.0012)

Table 9. Investments and savings at current prices for the year 1973-2005 (million rupees).

				Public sector				
3,921	6,774	11,010	16,287	18,643	20,251	21,857	26,421	26,099
31,258	35,003	37,793	42,085	47,586	55,691	59,497	69,008	71,513
86,420	106,482	121,876	130,508	155,089	175,298	166,036	141,380	177,813
212,661	236,228	183,909	191,329	267,110	286,149			
				Private sector				
3,726	3,840	5,208	6,483	7,779	8,709	9,749	12,955	21,609
23,331	26,758	31,419	35,840	39,959	44,349	51,769	64,162	76,563
91,226	118,878	134,768	150,369	163,219	193,126	230,823	261,465	231,544
394,749	423,097	496,464	545,104	597,591	713,157			

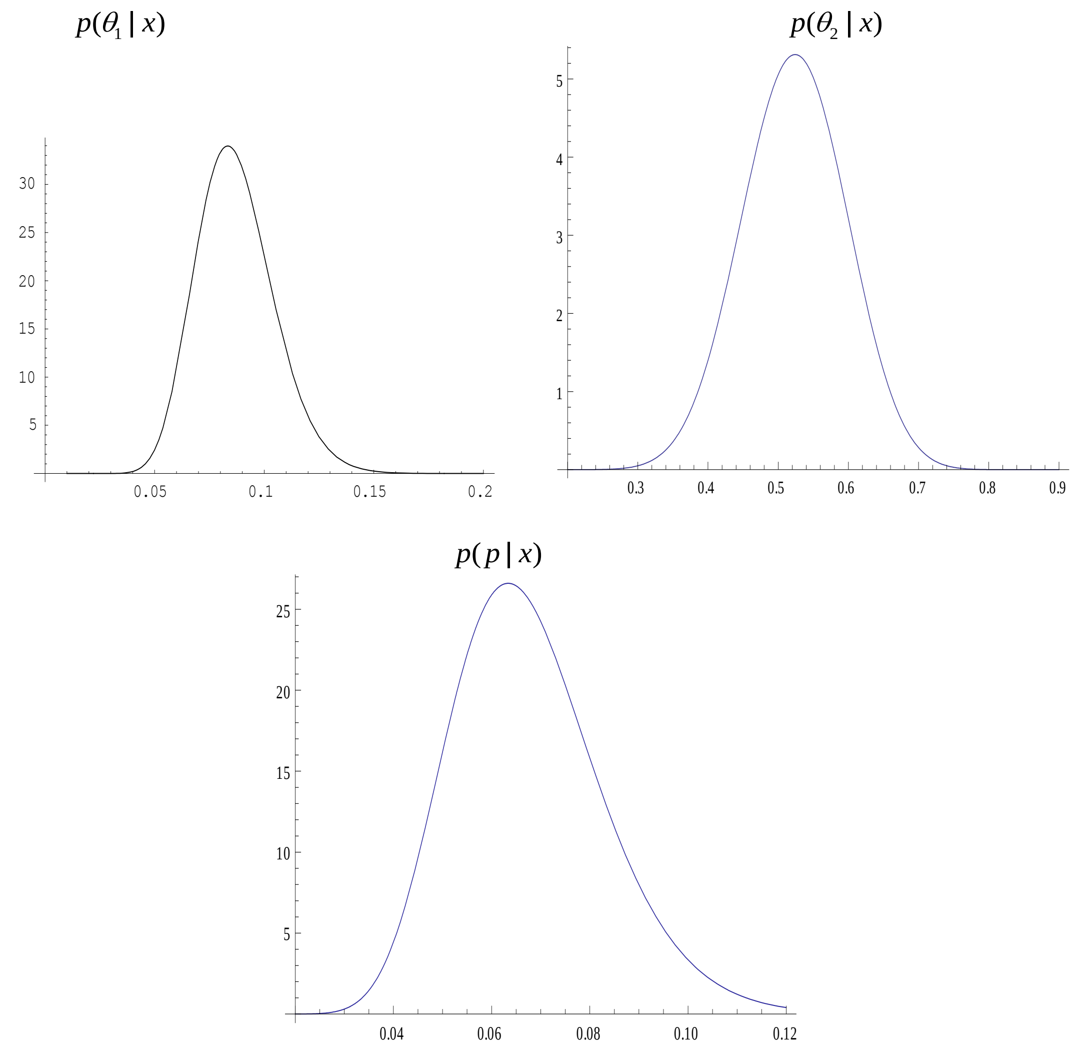

Figure 1. Symmetry graphs of the marginal densities.

Table 10. Testing of hypotheses for different values of parameter.

Hypotheses	Posterior probabilities under H_0	Posterior probabilities under H_1	Conclusion
$H_0:\theta_1 \geq 0.05, H_1:\theta_1 < 0.05$	0.8598	0.1402	Decisive evidence in favor of H_0
$H_0:\theta_2 \geq 0.05, H_1:\theta_2 < 0.05$	0.8829	0.1171	Decisive evidence in favor of H_0
$H_0: p \geq 0.5, H_1: p < 0.5$	0.6017	0.3983	Supports H_0
$H_0:\theta_1 \geq 0.08, H_1:\theta_1 < 0.08$	0.1414	0.8586	Decisive evidence in favor of H_1
$H_0:\theta_2 \geq 0.08, H_1:\theta_2 < 0.08$	0. 2027	0.7973	Decisive evidence in favor of H_1
$H_0: 0.55 < p < 1, H_1: p < 0.55$	0.4142	0.5858	Supports H_1 but not worth
$H_0:\theta_1 \geq 0.1, H_1:\theta_1 < 0.1$	0.0154	0.9846	Decisive evidence in favor of H_1
$H_0:\theta_2 \geq 0.1, H_1:\theta_2 < 0.1$	0.0320	0.9668	Decisive evidence in favor of H_1
$H_0: 0.3 < p < 1, H_1: p < 0.3$	0.9991	0.0009	Decisive evidence in favor of H_0

case of Jeffreys prior as compare to uniform prior, and posterior risk are also reduced by the change of prior from uniform to Jeffreys. But in case of informative prior under estimation of the parameter estimates is observed. It is noticed that under-estimation resulted by the use of informative prior occurred only for larger parametric points. This under estimation is serious for smaller sample size and is reduced with increase of sample size. However, this is tolerable because still the feature in which we are most interested is attained, that is the reduction of posterior risk.

Also, problem of under-estimation can be settled by some more suitable choice of hyper-parameters. We also conclude that under/over estimation wherever found is due to censoring. We have opted for 10% censoring rate. Effect of censoring can be studied in more depth by increasing or decreasing censoring rate. Posterior risk is found to be lower for weighted loss function as compare to squared error loss function. Risk is significantly lower for larger values of parameters under weighted loss function. We have considered 40% mixing proportion. Bayes estimates are found to be equally good and efficient for all priors under both loss functions. Posterior risk for mixing proportion is found almost equal for three priors. This stands true for all sample sizes and for both loss functions except exceptions. We observe from the results (Bayes estimates) of simulation that: posterior risk (Gamma prior) < posterior risk (Jeffreys prior) < posterior risk (Uniform prior). In real data example obtained estimates are verified through trends of graphs of marginal densities and by testing of hypotheses procedure. Both approve the authenticity of the obtained estimates.

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