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Full Length Research Paper

A block orthogonal matching pursuit algorithm based on sensing dictionary

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A block version of the orthogonal matching pursuit (OMP) algorithm, termed BOMP, can yield better reconstruction performance for block-sparse signals than conventional algorithms. However, the redundant dictionaries for block sparse signals are block coherent in the particle applications. In this paper, we consider a modified version of BOMP by introducing the concept of sensing dictionary. Exploiting the alternating projection (AP), we propose a method to design sensing dictionary for this modified BOMP. Simulation results show that the modified BOMP with sensing dictionary provides significant improvement for the recovery performance in the case of block coherent dictionary.

Key words: Block-sparsity, sensing dictionary, orthogonal matching pursuit, sparse reconstruction.

INTRODUCTION

The problem of sparse reconstruction has received tremendous attention in recent years. The main goal of this problem is to find the unknown vector $\mathbf{x}$ in the underdetermined inverse problem $\mathbf{y} = \mathbf{\Phi x}$, only a few non-zero entries in $\mathbf{x}$ are assume. From sparse reconstruction's point of view, the matrix $\mathbf{\Phi}$ is termed as redundant dictionary. In general, columns of the dictionary $\mathbf{\Phi}$ are normalized and termed as atoms. The most suitable algorithms for sparse reconstruction are basis pursuit (BP) (Chen et al., 1999) and orthogonal matching pursuit (OMP) (Pati et al., 1993). Recent results have shown that the recovery performance of these algorithms depend on the characteristics of redundant dictionaries. Sparse signals can be exactly reconstructed when the redundant dictionary is incoherent or equi-coherent and the vector $\mathbf{x}$ is sparse enough (Donoho and Huo, 2001; Tropp, 2004).

We consider a new class of sparse signals which have non-zero coefficients in clusters. Such signals are termed as block-sparse (Eldar and Mishali, 2009; Stojnic et al., 2010; Eldar et al., 2010). This sparse model can be used to deal with the problem of sampling signals that lie in a union of subspaces (Eldar and Mishali, 2009; Stojnic et al., 2010). Peotta and Vandergheynst (2007) proved that matching pursuit (MP) correctly identifies an atom at each step when the redundant dictionary can be represented as an incoherent union of coherent blocks (Peotta and Vandergheynst, 2007). Eldar proposed a mixed ℓ_2/ℓ_1 norm optimization for the block-sparse model by using the spectral structure of block-sparsity and proved that this mixed norm method is guaranteed to recover block-sparse signal if the redundant dictionary has small block-restricted isometry constants (Eldar and Mishali, 2009).

Recently, a block-version of OMP, called BOMP, was proposed and a sufficient condition for BOMP was developed (Eldar et al., 2010). However, BOMP can not guarantee to identify correct blocks if different blocks in redundant dictionary are highly coherent.

In practical applications, the redundant dictionaries for

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block-sparse signals, example, direction-of-arrival (DOA) estimation in the presence of mutual coupling, are block coherent, and BOMP may fail to reconstruct the sparse signal in this case. In this paper, we extend BOMP to this block coherent dictionary and propose a modified BOMP for sparse reconstruction by introducing sensing dictionary. A sufficient condition for this modified BOMP to exactly recover block-sparse signal was developed. The problem of designing sensing dictionary is cast as the subspace packing problem in Grassmannian manifolds.

Based on alternating projection (AP) (Tropp et al., 2005; Dhillon et al., 2007), a method for constructing sensing dictionary for block dictionary was proposed in this paper.

BLOCK-SPARSE MODEL AND BOMP

The standard sparsity model treated in the conventional sense assumes that non-zero elements can appear anywhere in the vector $\mathbf{x}$ (Donoho and Huo, 2001; Tropp, 2004). As discussed in block-sparse model (Eldar and Mishali, 2009; Stojnic et al., 2010; Eldar et al., 2010), the non-zero entries of $\mathbf{x}$ appear in blocks or clusters rather than being arbitrarily spread over the vector. We assume that the vector $\mathbf{x} \in \mathbb{C}^{pN\times 1}$ is a concatenation of N blocks and each block has p elements. The vector $\mathbf{x}$ is described as:

$$\mathbf{x}=[\mathbf{x}^T[1]\ \mathbf{x}^T[2]\ \cdots\ \mathbf{x}^T[N]]^T \tag{1}$$

where $\mathbf{x}[i] \in C^{p\times 1}$ for $i=1,...,N$ and T denotes transpose. The vector $\mathbf{x}$ is called block k-sparse if $\mathbf{x}[i]$ has non-zero Euclidean norm for at most k indices (Eldar et al., 2010).

Accordingly, the redundant dictionary $\mathbf{\Phi}$ can be represented as a concatenation of N matrices of size $d\times p$.

$$\mathbf{\Phi}=[\mathbf{\Phi}[1]\ \mathbf{\Phi}[2]\ \cdots\ \mathbf{\Phi}[N]] \tag{2}$$

where $\mathbf{\Phi}[i] \in C^{d\times p}$ for $i=1,...,N$ are termed as blocks.

The redundant dictionary $\mathbf{\Phi}$ is referred as block dictionary. Each matrices $\mathbf{\Phi}[i]$ can be view as a p-dimensional subspaces of C^d. Let x_i denotes the i-th element of $\mathbf{x}$ and ϕ_i denotes the i-th column of the dictionary $\mathbf{\Phi}$, in contrast to $\mathbf{x}[i]$ and $\mathbf{\Phi}[i]$.

A block version of OMP has been proposed for this block sparse case (Eldar et al., 2010). At k-th step, BOMP selects the block that is the best match to the current residual according to:

$$i_k = \arg\max_i \left\| \mathbf{\Phi}^H[i]\mathbf{r}_{k-1} \right\|_2 \tag{3}$$

where H denotes conjugate transpose, $\|\mathbf{a}\|_2$ denotes the Euclidean norm of the vector $\|\mathbf{a}\|_2=\sqrt{\mathbf{a}^H\mathbf{a}}$ and $\mathbf{r}_{k-1}$ is the residual. BOMP can be summarized as:

(1) Initialization: Let the residual $\mathbf{r}_0=\mathbf{y}$, the iteration counter $k=1$ and the index set $\mathbf{E}_0$ be the empty set.
(2) Sensing step: Find the block index i_k by solving the optimization (3). Then, $\mathbf{E}_k=\mathbf{E}_{k-1}\bigcup\{i_k\}$.
(3) Update of the residual: $\mathbf{r}_k=(\mathbf{I}_{kp}-\mathbf{\Phi}_{\mathbf{E}_k}\mathbf{\Phi}_{\mathbf{E}_k}^{\dagger})\mathbf{y}$, where $\mathbf{I}_{kp}$ is the identity matrix of size $Kp\times Kp$, $\mathbf{\Phi}_{\mathbf{E}_k}$ is a set of blocks $\mathbf{\Phi}_{\mathbf{E}_k}=[\mathbf{\Phi}[i_1]\ \cdots\ \mathbf{\Phi}[i_k]]$ and $\dagger$ denotes the pseudo-inverse.
(4) Increment: Set $k=k+1$, and return to step (2) if $k\le K$.

A sufficient condition for BOMP to recover $\mathbf{x}$ is that

$$\rho_c(\mathbf{\Phi}_0^{\dagger}\overline{\mathbf{\Phi}}_0)<1, \tag{4}$$

where

$$\rho_c(\mathbf{A})=\max_j \sum_i \rho(\mathbf{A}[i,j]), \tag{5}$$

where $\rho(\mathbf{A})$ denotes the spectral norm of matrix $\mathbf{A}$ and returns the largest singular value of $\mathbf{A}$, $\mathbf{\Phi}_0$ denote the matrix whose blocks correspond to the nonzero blocks of $\mathbf{x}$, $\overline{\mathbf{\Phi}}_0$ denotes the complementary of $\mathbf{\Phi}_0$ in $\mathbf{\Phi}$, and $\mathbf{A}[i,j]$ is the (i,j)-th block of $\mathbf{A}$.

Eldar et al. (2010) define the block-coherence of the dictionary as:

$$\mu_B=\max_{i,j\ne i}\frac{1}{p}\rho(\mathbf{\Phi}^H[i]\mathbf{\Phi}[j]), \tag{6}$$

if $p=1$, this block-sparse model reduces to the conventional sparse formulation and block-coherence is equal to the coherence μ, which is defined as:

$$\mu=\max_{i,j\ne i}\left|\phi_i^H\phi_j\right|. \tag{7}$$

Eldar et al. (20100 proved that any block K-sparse vector $\mathbf{x}$ can be recovered from $\mathbf{y}=\mathbf{\Phi x}$ using BOMP if:

$$Kp<(\mu_B^{-1}+p-(p-1)\upsilon\mu_B^{-1})/2, \tag{8}$$

where

$$\upsilon=\max_l \max_{i,j\ne i}\left|\phi_i^H\phi_j\right|, \quad \phi_i,\ \phi_j\ \in\ \mathbf{\Phi}[l], \tag{9}$$

is termed as sub-coherence (Eldar et al., 2010).

THE MODIFIED BOMP

A modified version of BOMP

Recent results show that BOMP can recover a block-sparse signal if the block-coherence μ_B is small enough. However, redundant dictionaries for practical signals are usually coherent between different blocks. Here, we modify BOMP for block coherent dictionary by using a sensing dictionary. The block-coherence affects the recovery performance at the sensing step significantly. Instead of sensing with the ordinary dictionary, our idea is to change this sensing step by using a different dictionary, which is called sensing dictionary. Schnass first introduced the concept of sensing dictionary and modified OMP for sparse reconstruction (Schnass and Vandergheynst, 2008). We extend the BOMP algorithm to the case with block coherent dictionary by using a sensing dictionary. The sensing dictionary we used has the same size as the ordinary dictionary, that is, $\mathbf{\Psi} \in \mathbb{C}^{d\times Np}$. Therefore, the modified BOMP selects a block at sensing step according to:

$$i_k = \arg\max_i \left\| \mathbf{\Psi}^H[i]\mathbf{r}_{k-1} \right\|_2. \tag{10}$$

In order to extend the recovery condition (8) to a generalized case, we define the cross block-coherence between $\mathbf{\Psi}$ and $\mathbf{\Phi}$ as:

$$\tilde{\mu}_B = \max_{i,j\neq i} \frac{1}{p}\rho(\mathbf{\Psi}^H[i]\mathbf{\Phi}[j]). \tag{11}$$

Similarly, the cross sub-coherence between $\mathbf{\Psi}$ and $\mathbf{\Phi}$ is defined as:

$$\tilde{\upsilon} = \max_l \max_{i,j\neq i} \left|\psi_i^H \phi_j\right|, \quad \psi_i \in \mathbf{\Psi}[l],\ \phi_j \in \mathbf{\Phi}[l]. \tag{12}$$

The cross sub-coherence considers similarities between atoms from $\mathbf{\Psi}$ and $\mathbf{\Phi}$. While the cross block-coherence measures coherence between blocks from $\mathbf{\Psi}$ and $\mathbf{\Phi}$.

Exact recovery condition

We develop the exact recovery condition for the modified BOMP for block-sparse reconstruction with block coherent dictionary. Similar to the definition of $\mathbf{\Phi}_0$, the sub-dictionary $\mathbf{\Psi}_0$ includes the blocks corresponding to the nonzero blocks of $\mathbf{x}$ and $\overline{\mathbf{\Psi}}_0$ denotes the complementary of $\mathbf{\Psi}_0$.

At first, we introduce some definitions and basic results that will be used in the following discussion. The general mixed ℓ_2/ℓ_q norm of a vector $\mathbf{x}$ is defined as:

$$\|\mathbf{x}\|_{2,q} = \|\mathbf{z}\|_q, \tag{13}$$

where

$$\mathbf{z} = [\|\mathbf{x}[1]\|_2 \quad \|\mathbf{x}[2]\|_2 \quad \cdots \quad \|\mathbf{x}[N]\|_2]^T. \tag{14}$$

The mixed matrix norm is given by

$$\|\mathbf{A}\|_{2,q} = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{Ax}\|_{2,q}}{\|\mathbf{x}\|_{2,q}}. \tag{15}$$

Lemma 1 (Eldar et al., 2010): Let $\mathbf{A}$ be a matrix with size $Mp\times Np$, Denote $\mathbf{A}[i,j]$ by the (i,j)-th $p\times p$ block of $\mathbf{A}$. Then:

$$\|\mathbf{A}\|_{2,\infty} \leq \max_i \sum_j \rho(\mathbf{A}[i,j]) \triangleq \rho_r(\mathbf{A}). \tag{16}$$

In particular, $\rho_r(\mathbf{A}) = \rho_c(\mathbf{A}^H)$.

Lemma 2 (Eldar et al., 2010): The matrix norm $\rho_c(\mathbf{A})$ defined as (5) satisfies the following properties:

Non-negative: $\rho_c(\mathbf{A}) \geq 0$

Positive: $\rho_c(\mathbf{A}) = 0$ if and only if $\mathbf{A} = 0$

Homogeneous: $\rho_c(\alpha\mathbf{A}) = |\alpha|\rho_c(\mathbf{A})$ for all $\alpha \in \mathbb{C}$.

Triangle inequality: $\rho_c(\mathbf{A}+\mathbf{B}) \leq \rho_c(\mathbf{A}) + \rho_c(\mathbf{B})$.

Submultiplicative: $\rho_c(\mathbf{AB}) \leq \rho_c(\mathbf{A})\rho_c(\mathbf{B})$.

Lemma 3 (Eldar et al., 2010): Suppose that $\rho_c(\mathbf{A}) < 1$. Then

$$(\mathbf{I}+\mathbf{A})^{-1} = \sum_{k=0}^{\infty} (-\mathbf{A})^k.$$

Theorem 1: Let $\mathbf{y}$ is a block K-sparse signal in $\mathbf{\Phi}$, that is, $\mathbf{y} = \mathbf{\Phi x}$, where $\mathbf{x}$ has K non-zero blocks. The modified BOMP using sensing dictionary $\mathbf{\Psi}$ can recover the block sparse vector $\mathbf{x}$ if:

$$\rho_c((\mathbf{\Psi}_0^H\mathbf{\Phi}_0)^{-1}\mathbf{\Phi}_0^H\overline{\mathbf{\Psi}}_0) < 1. \tag{17}$$

Proof: We show that if $\mathbf{r}_{k-1}$ is the linear combination of atoms in $\mathbf{\Phi}_0$, that is, $\mathbf{r}_{k-1} = \mathbf{\Phi}_0\mathbf{h}$, then the modified BOMP will select correct block at next step. BOMP using a sensing dictionary $\mathbf{\Psi}$ can select a correct block if:

$$\frac{\left\|\overline{\mathbf{\Psi}}_0^H \mathbf{r}_{k-1}\right\|_{2,\infty}}{\left\|\mathbf{\Psi}_0^H \mathbf{r}_{k-1}\right\|_{2,\infty}} < 1. \tag{18}$$

Substituting $\mathbf{r}_{k-1} = \mathbf{\Phi}_0\mathbf{h}$ into (18) and assuming $\mathbf{\Psi}_0^H\mathbf{\Phi}_0\mathbf{h} = \mathbf{t}$, we can get

$$\frac{\left\|\overline{\mathbf{\Psi}}_0^H \mathbf{r}_{k-1}\right\|_{2,\infty}}{\left\|\mathbf{\Psi}_0^H \mathbf{r}_{k-1}\right\|_{2,\infty}} = \frac{\left\|\overline{\mathbf{\Psi}}_0^H \mathbf{\Phi}_0\mathbf{h}\right\|_{2,\infty}}{\left\|\mathbf{\Psi}_0^H \mathbf{\Phi}_0\mathbf{h}\right\|_{2,\infty}} = \frac{\left\|\overline{\mathbf{\Psi}}_0^H \mathbf{\Phi}_0(\mathbf{\Psi}_0^H\mathbf{\Phi}_0)^{-1}\mathbf{t}\right\|_{2,\infty}}{\|\mathbf{t}\|_{2,\infty}}. \tag{19}$$

With Lemma 1, we have:

$$\frac{\left\|\overline{\boldsymbol{\Psi}}_0^H \mathbf{r}_{k-1}\right\|_{2,\infty}}{\left\|\boldsymbol{\Psi}_0^H \mathbf{r}_{k-1}\right\|_{2,\infty}} \le \rho_r(\overline{\boldsymbol{\Psi}}_0^H \boldsymbol{\Phi}_0 (\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0)^{-1}) = \rho_c((\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0)^{-1} \boldsymbol{\Phi}_0^H \overline{\boldsymbol{\Psi}}_0). \quad (20)$$

Note that the residual $\mathbf{r}_0$ belongs to the range expanded by $\boldsymbol{\Phi}_0$ due to $\mathbf{r}_0 = \mathbf{y}$. Therefore, the modified OMP can select correct block at each step if (17) holds.

This theorem extends (4) to a generalized case, in which the blocks in $\boldsymbol{\Phi}$ can be coherent. Considering the optimal blocks $\boldsymbol{\Phi}_0$ is unknown in advance, we develop a recovery condition using $\tilde{\mu}_B$ and $\tilde{\upsilon}$.

Theorem 2: Let $\tilde{\mu}_B$ be the cross block-coherence and $\tilde{\upsilon}$ the sub-coherence between $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$. If:

$$Kp < (\tilde{\mu}_B^{-1} + P - (P-1)\tilde{\upsilon}\tilde{\mu}_B^{-1})/2, \quad (21)$$

condition (18) is satisfied.

Proof: Using Lemma 2, we have:

$$\rho_c((\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0)^{-1} \boldsymbol{\Phi}_0^H \overline{\boldsymbol{\Psi}}_0) \le \rho_c((\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0)^{-1}) \rho_c(\boldsymbol{\Phi}_0^H \overline{\boldsymbol{\Psi}}_0). \quad (22)$$

Based on the definition of $\rho_c(\mathbf{A})$ and $\tilde{\mu}_B$, the last term in (22) can be bound as

$$\rho_c(\boldsymbol{\Phi}_0^H \overline{\boldsymbol{\Psi}}_0) = \max_{j \notin \Lambda_0} \sum_{i \in \Lambda_0} \rho(\boldsymbol{\Phi}^H[i]\boldsymbol{\Psi}[j]) \le Kp\tilde{\mu}_B, \quad (23)$$

where Λ_0 denotes the index set of blocks in $\boldsymbol{\Phi}_0$. Let

$$\mathbf{A} = \boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0 - \mathbf{I},$$

Using Lemma 3, if $\rho_c(\mathbf{A}) < 1$,

$$(\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0)^{-1} = (\mathbf{I} + \mathbf{A})^{-1} = \sum_{k=0}^{\infty} (-\mathbf{A})^k. \quad (24)$$

Then, we can get:

$$\begin{aligned} \rho_c((\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0)^{-1}) &= \rho_c(\sum_{k=0}^{\infty} (-(\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0 - \mathbf{I}))^k) \\ &\le \sum_{k=0}^{\infty} (\rho_c(\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0 - \mathbf{I}))^k \\ &= \frac{1}{1 - \rho_c(\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0 - \mathbf{I})} \end{aligned} \quad (25)$$

On the other hand, we have:

$$\begin{aligned} \rho_c(\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0 - \mathbf{I}) &= \max_j \sum_i \rho(\boldsymbol{\Psi}_0^H[i]\boldsymbol{\Phi}_0[j] - \mathbf{I}) \\ &\le \max_j \rho(\boldsymbol{\Psi}_0^H[j]\boldsymbol{\Phi}_0[j] - \mathbf{I}) + \max_j \sum_{i \ne j} \rho(\boldsymbol{\Psi}_0^H[i]\boldsymbol{\Phi}_0[j]) \\ &\le (p-1)\tilde{\upsilon} + p(K-1)\tilde{\mu}_B. \end{aligned} \quad (26)$$

Combining (25) with (26) leads to:

$$\rho_c((\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0)^{-1}) \le \frac{1}{1-(p-1)\tilde{\upsilon} - p(K-1)\tilde{\mu}_B}. \quad (27)$$

From (22), (23) and (27), we can obtain:

$$\rho_c((\boldsymbol{\Psi}_0^H \boldsymbol{\Phi}_0)^{-1} \boldsymbol{\Phi}_0^H \overline{\boldsymbol{\Psi}}_0) \le \frac{Kp\tilde{\mu}_B}{1-(p-1)\tilde{\upsilon} - p(K-1)\tilde{\mu}_B} < 1. \quad (28)$$

Therefore, condition (18) is satisfied if (21) holds.

Peotta and Vandergheynst (2007) derived the exact recovery condition for MP with block incoherent dictionary in the conventional sparse case. Schnass and Vandergheynst (2008) developed the sufficient condition for greed algorithm for sparse reconstruction in the case of coherent dictionary.

In a recent literature, Eldar et al. (2010) proved the exact recovery condition for BOMP for block-sparse reconstruction. The theorems above extend these research results to a generalized case with block coherent dictionary. If $p = 1$, (17) and (21) reduce to the results in Schnass and Vandergheynst (2008). The inequalities (17) and (21) becomes (4) and (8) in Eldar et al. (2010) when the sensing dictionary is selected as $\boldsymbol{\Psi} = \boldsymbol{\Phi}$ for block incoherent dictionary. Through constructing an appropriate sensing dictionary with $\tilde{\mu}_B < \mu_B$ for block coherent dictionary, the recovery condition (21) is looser than (8).

ALGORITHM FOR CONSTRUCTING SENSING DICTIONARY

The blocks $\boldsymbol{\Psi}[i]$ and $\boldsymbol{\Phi}[i]$ can be view as p-dimensional subspaces of C^d. Therefore, the ordinary dictionary $\boldsymbol{\Phi}$ and sensing dictionary $\boldsymbol{\Psi}$ are the collection of N K-dimensional subspaces. Constructing a sensing dictionary can be cast as the subspace packing problem. We use the chordal distance to describe the distance between two p-dimensional subspaces.

The chordal distance between $\boldsymbol{\Psi}[i]$ and $\boldsymbol{\Phi}[j]$ is given by Dhillon et al. (2007).

$$dist_{chord}(\boldsymbol{\Psi}[i], \boldsymbol{\Phi}[j]) \overset{def}{=} \sqrt{\sin^2(\theta_1) + ... + \sin^2(\theta_p)} = [p - \left\|\boldsymbol{\Psi}^H[i]\boldsymbol{\Phi}[j]\right\|_F^2]^{1/2}, \quad (29)$$

where θ_i $(i = 1,.., p)$ denote p principal angles formed by $\boldsymbol{\Psi}[i]$ and $\boldsymbol{\Phi}[j]$.

In fact, constructing a sensing dictionary is to find a matrix $\boldsymbol{\Psi}$ such that the Gram type matrix $\mathbf{G} = \boldsymbol{\Psi}^H \boldsymbol{\Phi}$ have the following properties:

(i) $\mathbf{G}$ is Hermitian.
(ii) Each diagonal block $\mathbf{G}[i,i]$ of $\mathbf{G}$ is an identify matrix.
(iii) Off-diagonal block $\mathbf{G}[i, j]$: $\left\|\mathbf{G}[i, j]\right\|_F \le \mu$ for each $i \ne j$, where μ is a positive constant discussed as follows.

A good sensing dictionary should guarantee that the cross block-coherence $\tilde{\mu}_B$ is small enough. This problem is equal to Minimize the chordal distance between $\mathbf{\Psi}[i]$ and $\mathbf{\Phi}[j]$

$$\min_{i \neq j} dist_{chord}(\mathbf{\Psi}[i], \mathbf{\Phi}[j]) = \min_{i \neq j} [p - \left\| \mathbf{\Psi}^H [i] \mathbf{\Phi}[j] \right\|_F^2]^{1/2}. \tag{30}$$

The sensing dictionary satisfies:

$$\min_{i \neq j} [p - \left\| \mathbf{\Psi}^H [i] \mathbf{\Phi}[j] \right\|_F^2]^{1/2} \geq \eta, \tag{31}$$

where η is a given parameter. Rearranging the inequality (31), we have:

$$\max_{i \neq j} \left\| \mathbf{\Psi}^H [i] \mathbf{\Phi}[j] \right\|_F \leq \mu, \tag{32}$$

where $\mu = \sqrt{p - \eta^2}$. Every collection of N p-dimensional subspaces of $\mathbb{C}^d$ satisfies the inequality (Dhillon et al., 2007)

$$\min_{i \neq j} dist_{chord}(\mathbf{A}[i], \mathbf{A}[j]) \leq \eta_0, \tag{33}$$

where

$$\eta_0 = \frac{KN(d-K)}{d(N-1)}. \tag{34}$$

Therefore, the parameter η can be selected as $\eta = \varepsilon \eta_0$, where $\varepsilon \in (0,1]$.

The problem of constructing a sensing dictionary can be formulated as a matrix nearness problem. That is, a good sensing dictionary $\mathbf{\Psi}$ should satisfy that $\mathbf{\Psi}^H \mathbf{\Phi}$ is closest to a structural constraint Hermitian matrix. Here, we construct sensing dictionary as the solution to the optimization.

$$\begin{gathered} \arg\min_{\mathbf{\Psi} \in \mathbb{C}^{d \times Np}} \left\| \mathbf{H} - \mathbf{\Psi}^* \mathbf{\Phi} \right\|_F, \text{ s.t.} \\ \mathbf{H} \in \chi \triangleq \{ \mathbb{C}^{Np \times Np} : \mathbf{H} = \mathbf{H}^*, \mathbf{H}[i,i] = \mathbf{I}_p \text{ and } \left\| \mathbf{H}[i,j] \right\|_F \leq \mu \, for \; i \neq j \}, \end{gathered} \tag{35}$$

where $\mathbf{H}[i, j]$ denotes (i, j)th $p \times p$ block, and $\|\cdot\|_F$ denotes the Frobenius norm. Applying alternating projection (AP) algorithm, we obtain a sensing dictionary to (35) by alternately solving two basic matrices nearness problems.

(1) Given matrix $\tilde{\mathbf{\Psi}}$, we can obtain the Gram type matrix $\tilde{\mathbf{G}} = \tilde{\mathbf{\Psi}}^H \mathbf{\Phi}$. Find a matrix $\tilde{\mathbf{H}}$ that solves $\min_{\mathbf{H} \in \chi} \left\| \mathbf{H} - \tilde{\mathbf{G}} \right\|_F$.

(2) Given Hermitian matrix $\tilde{\mathbf{H}}$, find a matrix $\tilde{\mathbf{\Psi}}$ that solves $\min_{\mathbf{\Psi} \in \mathbb{C}^{d \times Np}} \left\| \mathbf{\Psi}^H \mathbf{\Phi} - \tilde{\mathbf{H}} \right\|_F$.

By alternately finding structural constraint Hermitian matrix and sensing dictionary, this algorithm can construct a sensing dictionary conveniently.

For the matrices nearness problem (1), the constraint set χ is convex. The unique matrix $\tilde{\mathbf{H}}$ in χ nearest to $\tilde{\mathbf{G}}$ has diagonal blocks equal to the identify and off-diagonal blocks that satisfy (Dhillon et al., 2007).

$$\tilde{\mathbf{H}}[i, j] = \begin{cases} \tilde{\mathbf{G}}[i, j] & \left\| \tilde{\mathbf{G}}[i, j] \right\|_F \leq \mu \\ \mu \tilde{\mathbf{G}}[i, j] / \left\| \tilde{\mathbf{G}}[i, j] \right\|_F & otherwise \end{cases}. \tag{36}$$

For the matrices nearness Problem (2), we have:

$$\begin{aligned} \min_{\mathbf{\Psi} \in \mathbb{C}^{d \times Np}} \left\| \mathbf{\Psi}^H \mathbf{\Phi} - \tilde{\mathbf{H}} \right\|_F &= \min_{\mathbf{\Psi} \in \mathbb{C}^{d \times Np}} \left\| \tilde{\mathbf{H}} - \mathbf{\Phi}^H \mathbf{\Psi} \right\|_F \\ &= \min_{\mathbf{\Psi} \in \mathbb{C}^{d \times Np}} \sum_j \left\| \tilde{\mathbf{H}}[j] - \mathbf{\Phi}^H \mathbf{\Psi}[j] \right\|_F. \end{aligned} \tag{37}$$

Solving the optimization $\min_{\mathbf{\Psi}[j] \in \mathbb{C}^{d \times p}} \left\| \tilde{\mathbf{H}}[j] - \mathbf{\Phi}^H \mathbf{\Psi}[j] \right\|_F$, we can get:

$$\mathbf{\Psi}[j] = (\mathbf{\Phi} \mathbf{\Phi}^H)^{-1} \mathbf{\Phi} \tilde{\mathbf{H}}[j]. \tag{38}$$

SIMULATION RESULTS

To illustrate the performance of the modified BOMP, we consider the direction-of-arrival (DOA) estimation in the presence of mutual coupling since the redundant dictionary composed of steering vector are highly coherent. By exploiting the particular construction of the matrix of mutual coupling, this parameters estimation problem can be cast as block-sparse model. Assume K far-field narrowband signals $s_k(t)(k = 1, ..., K)$ impinge on an M-element uniform linear array from the directions $\theta_k \ (k = 1, ..., K)$. In order to illustrate the exact reconstruction performance for block-sparse signal, we consider the DOA estimation in noise-free case. The discrete time samples of the signals $\mathbf{y}(t)$ received at array sensors can be written as:

$$\mathbf{y}(t) = \mathbf{QA}(\boldsymbol{\theta})\mathbf{s}(t), \tag{39}$$

where $\mathbf{y}(t) = [\mathbf{y}_1(t), ..., \mathbf{y}_M(t)]^T$, $\mathbf{Q} \in C^{M \times M}$ is the mutual coupling matrix (MCM) of ULA, $\mathbf{A}(\boldsymbol{\theta}) = [A(\theta_1), ..., A(\theta_K)]$ is the array manifold matrix composed of steering vectors $A(\theta_k) \ (k = 1, ..., K)$, $\mathbf{s}(t) = [s_1(t), ..., s_K(t)]^T$ is the vector composed of the complex amplitude of K signals. Given the observed data $\mathbf{y}(t)$, the goal of DOA estimation is to find the unknown angles $\theta_k \ (k = 1, ..., K)$.

Considering that the MCM $\mathbf{Q}$ for ULA has the especial structure of banded complex symmetric Toeplitz, we can

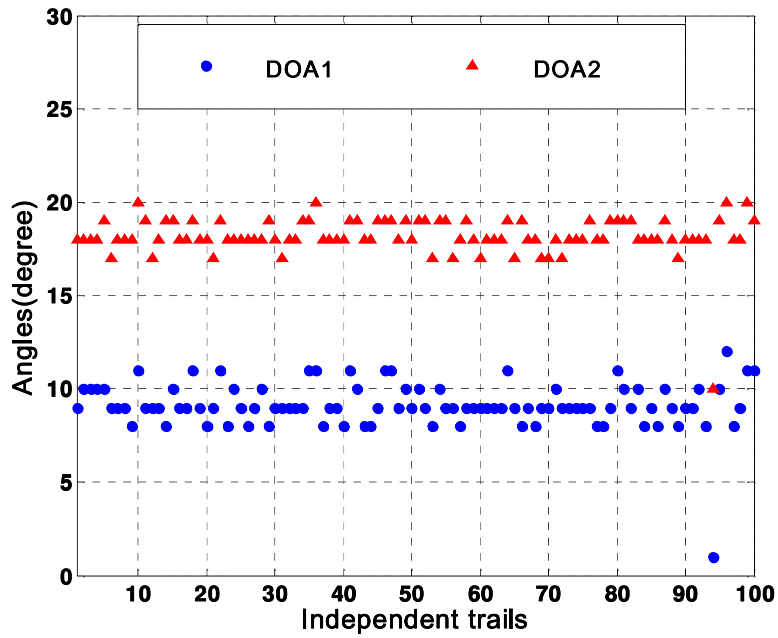

Figure 1. The BOMP for two spatial signals.

obtain:

$$\mathbf{Q}A(\theta_k) = \mathbf{U}(A(\theta_k))\mathbf{c}, \tag{40}$$

where $\mathbf{c} = [c_0, c_1, \cdots, c_{p-1}]^T$, p is the number of non-zero coefficient in $\mathbf{Q}$, $\mathbf{U}(A(\theta_k)) \in C^{M\times p}$ is given by the sum of the two following $M \times P$ matrices:

$$[\mathbf{U}_1(A(\theta_k))]_{ij} = \begin{cases} [A(\theta_k)]_{i+j-1}, & i+j \le N+1 \\ 0, & otherwise \end{cases}, \tag{41}$$

$$[\mathbf{U}_2(A(\theta_k))]_{ij} = \begin{cases} [A(\theta_k)]_{i-j+1}, & i \ge j \ge 2 \\ 0, & otherwise \end{cases}, \tag{42}$$

With (40), (39) can be represented as:

$$\mathbf{y}(t) = \mathbf{U}(\mathbf{I}_K \otimes \mathbf{c})\mathbf{s}(t) = \mathbf{U}(\mathbf{s}(t) \otimes \mathbf{c}), \tag{43}$$

where $\mathbf{U} = [\mathbf{U}(A(\theta_1))\ \mathbf{U}(A(\theta_2))\ \cdots \mathbf{U}(A(\theta_K))]$ and $\otimes$ denotes Kronecker product.

Let $\theta_n (n = 1,...,N)$, in general, $N \square\ M$, be the sampling grids of all directions of interest. We can construct a redundant dictionary $\mathbf{U} \in C^{M\times Np}$ composed of matrices $\mathbf{U}(A(\theta_n))\ (n = 1,...,N)$, that is:

$$\mathbf{U} = [\mathbf{U}(A(\theta_1))\ \mathbf{U}(A(\theta_2))\ \cdots\ \mathbf{U}(A(\theta_N))]. \tag{44}$$

For a single snapshot, this problem of DOA estimation is to find the unknown vector $\mathbf{s}$ by solving:

$$\mathbf{y} = \mathbf{U}\,\mathbf{s}. \tag{45}$$

The block $\mathbf{s}[k]$, which is corresponding to $\mathbf{U}(A(\theta_k))$, is non-zero if source comes from θ_k for some k and zeros otherwise. Therefore, DOA estimation in the presence of mutual coupling is cast as the problem of block-sparse reconstruction.

In our experiments, we consider two narrowband far-field signal sources impinge on 15-elements ULA from $\text{DOA1} = 9°$ and $\text{DOA2} = 18°$, respectively. The complex amplitudes of signals are produced randomly. The array sensors are separated by half a wavelength. We take three non-zero coefficients of mutual coupling into account. Let $c_0 = 1$, $c_1 = 0.6 + 0.4j$ and $c_2 = 0.1 - 0.2j$. The potential directions $\theta_n (n = 1,...,N)$ are obtained from the interval $[0^\circ, 60^\circ]$ with uniform grid $\Delta\theta = 1^\circ$, i.e. $N = 61$. The number of iteration for designing sensing dictionary is 30 and the parameter μ is calculated as $\mu = \sqrt{p - \eta^2}$, where η is selected as $\eta = 0.6\eta_0$.

Simulation results are obtained over 100 independent Monte-Carlo trails to compare the recovery performance of the modified BOMP to that of BOMP. (Figure 1) presents angles estimated via BOMP for the two signals. It is shown in this figure that BOMP can not identify

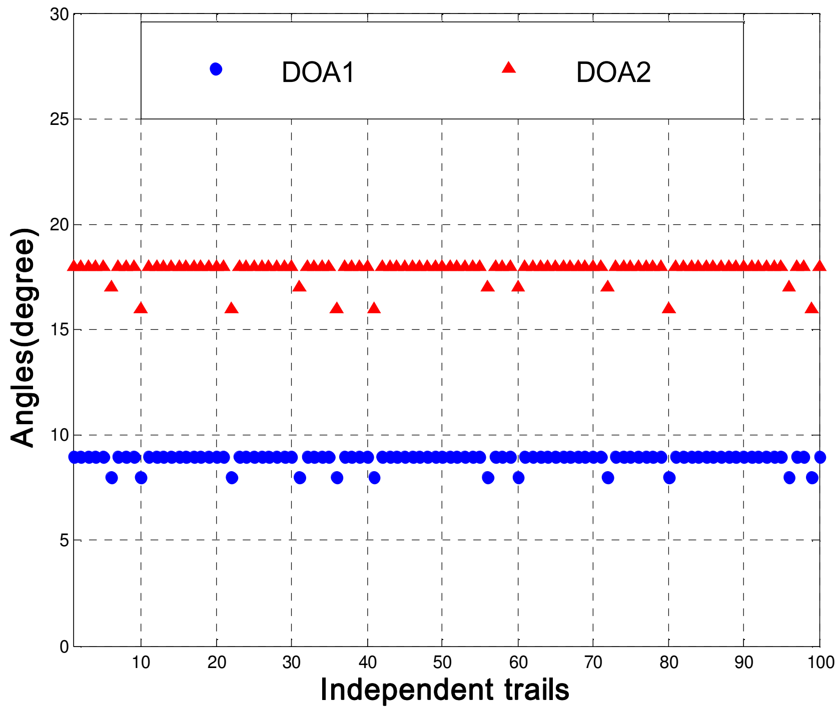

Figure 2. The modified BOMP for two spatial signals.

non zero blocks in $\mathbf{s}$ since the redundant dictionary $\mathbf{U}$ for DOA estimation is block coherent. (Figure 2) shows DOA estimation via modified BOMP. Applying sensing dictionary constructed by AP algorithm, the modified BOMP can identify correct blocks at large probability. Because the complex amplitude is produced randomly, the amplitudes of signals is too small in some trails and the proposed algorithm fails in this case.

Conclusions

In this paper, we extend BOMP to a generalized case through constructing sensing dictionary. A recovery condition is derived for this modified BOMP by introducing the cross block-coherence measure. The modified BOMP using sensing dictionary show superior performance for block-sparse construction, especially in the case of block coherent dictionaries. Based on the AP method, we develop an algorithm to design sensing dictionary for the redundant dictionary with block structure.

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