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# Unsteady mixed convection visco-elastic flow and heat transfer in a thin film flow over a porous stretching sheet with internal heat generation

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**In this study, a two-dimensional unsteady mixed convection flow of an incompressible visco-elastic fluid over a porous thermal forming thin film stretching sheet with internal heat generation has been**

**studied. Two novel items  $G = g\beta[\frac{(1-at)^2}{b^2}]Ax$  and  $\phi = -\frac{v}{k}ax(1-ct)^{-1}$  are presented for free convection**

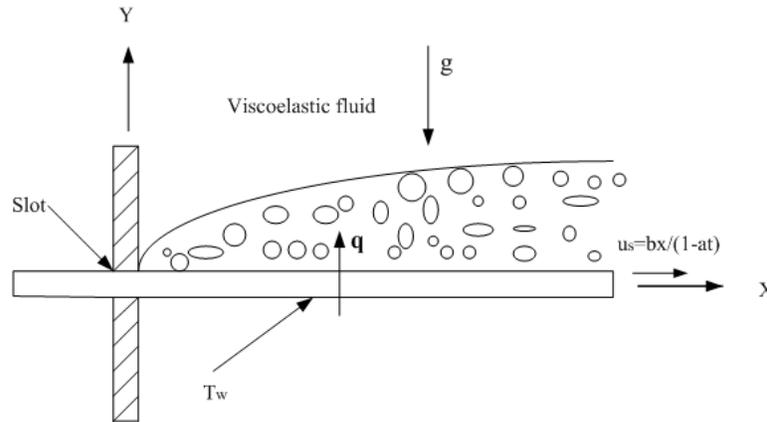
**in porous space important parameters and an internal heat generation phenomena, respectively. The similarity transformation and an implicit finite-difference method have been used to analyze the present problem. The numerical solutions of the flow velocity distributions, temperature profiles, the wall unknown values of  $f''(0)$  and  $\theta'(0)$  for calculating the heat transfer of the similar boundary-layer flow are carried out as functions of the unsteadiness parameter (S), the Prandtl number (Pr), the visco-elastic parameter ( $\alpha$ ), the porous parameter ( $\phi$ ), the space-dependent parameter (A) and temperature-dependent parameter (B) for heat source/sink and the free convection parameter (G). The effects of these parameters have also been discussed. The results show that it will produce greater heat transfer effect with a larger Pr, G and  $\alpha$ , but S, A and B will reduce heat transfer effect.**

**Key words:** Visco-elastic, thin film flow, porous medium, finite-difference method, heat transfer, unsteady stretching sheet, mixed convection, internal heat generation.

## INTRODUCTION

The study of visco-elastic fluids has become of increasing importance in the last few years. Qualitative analyses of these studies have significant bearing on several industrial applications such as polymer sheet extrusion from a dye, drawing of plastic films, etc. The manufacturing process at high temperature needs cooling of the thermal forming thin film stretching sheet. The flows, maybe, need visco-elastic fluids to produce a good effect to reduce the temperature from the thin sheet. And also, the fluids have processed many types of effects (that is, magnetic force, buoyancy and mass diffusion) into the problem, and have become a hybrid system that need to be analyzed by many different ways. It is a well-known fact in the studies of non-Newtonian fluid flows (Hartnett, 1992). Rajagopal et al. (1983) studied a Falkner-Skan flow field of a second-grade visco-elastic fluid. Massoudi and Ramezan (1989) studied a wedge

flow with suction and injection along walls of a wedge by the similarity method and finite-difference calculations. An excellent review of boundary layers in non-linear fluids was recently written (Rajagopal, 1995). These are related studies to the present investigation about second-grade fluids. All the aforementioned are dealing with forced convection problems. Vajravelu and Soewono (1994) had solved the fourth order non-linear systems arising in combined free and forced convection flow of a second order fluid, over a stretching sheet. The stretching sheet flow of a non-Newtonian fluid is also one of the important flow fields in real world; Raptis (1989) had studied heat transfer of a visco-elastic fluid. Recently, Sanjayanand et al. (2006), Cortell (2007) and Seddeek (2007) had studied the heat and mass transfer problems about the visco-elastic boundary layer flow over a stretching sheet with magnetic effect. On the other hand, researches in



**Figure 1.** A sketch of the physical model for unsteady visco-elastic fluid flow with mixed convection effect over a thin film porous boundary layer stretching sheet with internal heat generation.

connection with visco-elastic fluid or second grade non-Newtonian fluids, but they are not the mixed convection flow (Sujit, 2006).

Sakiadis (1961) was the first to study the boundary layer flow generated by a continuous stretching surface moving with a constant velocity. Several authors (Vajravelu and Roper, 1999; Vajravelu, 2001; Liu, 2004; Sajid and Hayat, 2008) investigated the heat transfer problem in a stretching sheet with a linear or non-linear surface velocity and a uniform or different surface temperature condition. Abo-Eldahab and Aziz (2004) extended the problem to involve a space-dependent exponentially decaying with internal heat generation or absorption. Abel et al. (2007) and Bataller (2007) presented the effects of non-uniform heat source on visco-elastic fluid flow and heat transfer over a stretching sheet. Moreover, many authors (Mukhopadhyay et al., 2005; Vajravelu and Roper, 2005; Pantokratoras, 2008; Vajravelu and Roper, 2008; Mukhopadhyay and Layek, 2008) extended to consider the effects of variable fluid properties or specific dimensionless parameters on the flow over a stretching sheet. In all these aforementioned studies, the flow and temperature fields have been considered to be at a steady state. Some authors (Andersson et al., 2000; Dandapat et al., 2003; Ali and Magyari, 2007; Dandapat et al., 2007) studied the problem for unsteady stretching surface condition by using a similar method to transform governing time-dependent boundary layer equations into a set of non-linear ordinary differential equations. Most recently, Noor et al. (2009) had studied the MHD (magnetohydrodynamic) flow and heat transfer in a thin liquid film on an unsteady stretching sheet problem. Kai-Long and Hsu (2009a, 2009b) studied the related conjugate heat transfer problems, but not toward the unsteady problems. Most recently Kumaran et al. (2011) and Abel et al. (2009) had studied the unsteady

problems, a mathematical model was presented for a free convection boundary layer flow of a continuously moving vertical porous plate in a chemically reactive medium in a transverse magnetic field (Ibrahim and Makinde, 2010). Abdullah et al. (2009) studied the enhancement of natural convection heat transfer from a horizontal rectangular fin embedded with rectangular perforations of aspect ratio of two using finite element methods. Makinde (2009) studied the hydromagnetic boundary layer flow with heat and mass transfer over a vertical plate in magnetic field and a convective heat exchange at the surface with the surrounding.

The aforementioned provides the motivation for the present analysis to study the flow and heat transfer in an incompressible porous visco-elastic fluid caused by mixed convection effect on a thermal forming thin film stretching sheet with internal heat generation. It is a point of view in examining the influence of flow and heat transfer characteristics for forced and free convection effect phenomena. The buoyancy force, couple with the internal heat generation is important in the present problem due to the difference among the previous studies. A similar derivation technique has been used and the resulting non-linear similar equations were solved by using the finite-difference method.

## PROBLEM FORMULATION

Let us consider the unsteady, incompressible, two-dimensional visco-elastic fluid flow of a thin liquid porous film of uniform thickness  $h$  ( $t$ ) over the horizontal thermal forming stretching sheet with heat generation by mixed convection. The fluid motion within the thin film is due to stretching of the elastic sheet. The geometry of the problem is shown in Figure 1.

The fluid flow is modeled as an unsteady, two

dimensional, incompressible visco-elastic laminar porous flow on a horizontal thin elastic thermal forming sheet that issues from a narrow slot at the origin and is a continuous thin film stretching with a velocity  $u_s = bx / (1 - at)$  (Andersson et al., 2000) (where  $a$  and  $b$  are positive constant and  $t < 1/a$ ) in the positive  $x$ -direction. An incompressible, homogeneous, non-Newtonian and second-grade fluid having a constitutive equation based on the postulate of gradually fading memory suggested by Rivlin and Ericksen (1955) is being used for the present flow. The model equation is expressed as follows:

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \tag{1}$$

where  $T$  is the stress tensor,  $p$  is the pressure,  $I$  is the unit tensor,  $\mu$  is the dynamic viscosity,  $\alpha_1$  and  $\alpha_2$  are first and second normal stress coefficients that are related to the material modulus and for the present second-grade fluid or named visco-elastic fluid flow.

$$\mu \geq 0, \alpha_1 > 0, \alpha_1 + \alpha_2 = 0 \tag{2}$$

The kinematic tensors  $A_1$  and  $A_2$  are defined as:

$$A_1 = \nabla V + (\nabla V)^T \tag{3}$$

$$A_2 = \frac{dA_1}{dt} + A_1(\nabla V) + (\nabla V)^T A_1 \tag{4}$$

where  $V$  is velocity and  $d/dt$  is the material time derivative. The well-known Boussinesq approximation is used to represent the buoyancy mixed term. Where  $u, v$  are the velocity components in the  $x$  and  $y$  directions, the unsteady boundary-layer equations for this flow, heat transfer, in usual notations, are expressed as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) + \frac{\alpha_1}{\rho} \left[ \frac{\partial^3 u}{\partial t \partial y^2} + u \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial y^2} + v \frac{\partial^3 u}{\partial y^3} \right] \tag{6}$$

$$\rho c_p \left[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = k \frac{\partial^2 T}{\partial y^2} + \left( \frac{ku_s}{xv} \right) [A(T_w - T_\infty)e^{-\eta} + B(T - T_\infty)] \tag{7}$$

The correspondence boundary conditions are as follows:

$$u = u_s(x, t), \quad v = 0, \quad T = T_w(x, t), \quad \text{at } y=0 \tag{8}$$

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial T}{\partial y} = 0 \quad \text{at } y = h(t) \tag{9}$$

where  $u_s$  and  $T_w$  are the velocity and temperature of the stretching sheet at the surface  $y = 0$ , respectively.  $T$  is the temperature,  $g$  is the magnitude of the gravity,  $\nu$  is the

kinematic viscosity,  $\alpha = -\frac{\alpha_1}{\rho}$  is the visco-elastic

parameter,  $\beta$  is the coefficient of thermal expansion,  $T_\infty$  is the temperature of the ambient fluid,  $\rho$  is the density,  $c_p$  is the specific heat at constant pressure and  $k$  is the conductivity, respectively. Where the items  $g\beta(T - T_\infty)$ ,

$\frac{\nu}{k}u$  and  $q = \left(\frac{ku_s}{xv}\right)[A(T_w - T_\infty)e^{-\eta} + B(T - T_\infty)]$  are

the free convection, porous medium and an internal heat generation term, respectively. They are the important items present in this study, and the physical phenomenon is shown in Figure 1. The flow is induced due to stretching at  $y = 0$  which moves in the  $x$ -direction with the velocity as:

$$u_s = \frac{bx}{1 - at}, \tag{10}$$

in which  $a$  and  $b$  are positive constants with dimension  $(\text{time})^{-1}$ . It can be noted from Equation 10 that the effective stretching rate  $b/(1 - at)$  increases with time since  $a > 0$ . The surface temperature  $T_w$  of the sheet is given as:

$$T_w = T_0 - T_{ref} \left[ \frac{bx^2}{2\nu} \right] (1 - at)^{-3/2}, \tag{11}$$

where  $T_0$  and  $T_{ref}$  are the temperature at the slit and reference temperature, respectively. Expression of Equation 11 reflects that the sheet temperature decreases from  $T_0$  at the slot in proportion to  $x^2$  and temperature reduction increases with an increase in  $(1 - at)$ . But it should be noticed that Equations 10 and 11, which are responsible for the whole analysis, are valid only for time  $t < 1/a$ . The following dimensionless parameters are introduced.

$$\eta = \sqrt{\frac{b}{\nu}}(1 - at)^{-1/2}y, \quad \psi = \sqrt{b\nu x}(1 - at)^{-1/2}f(\eta),$$

$$T_w - T_\infty = Ax, \quad \frac{T - T_\infty}{T_w - T_\infty} = \theta \tag{12}$$

and the stream function  $\psi(x, y)$  through

$$u = \frac{\partial \psi}{\partial y} = \frac{bx}{1-at} f'(\eta), \tag{13}$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{\frac{bv}{1-at}} f'(\eta), \tag{14}$$

the continuity Equation 1 is identically satisfied, and dimensionless problems of flow and temperature are given as:

$$f''' - f'^2 + ff'' - S \left( f' + \frac{1}{2} \eta f'' \right) + \alpha \left( 2ff''' + S \left( 2f''' + \frac{1}{2} \eta f'''' \right) - f''^2 - ff'''' \right) + G\theta = 0 \tag{15}$$

$$\theta'' - Pr \left[ \frac{1}{2} S (3\theta + \eta \theta') + 2f\theta - f\theta' \right] + Ae^{-\eta} + B\theta = 0, \tag{16}$$

And then associated with boundary conditions become:

$$f'(0) = 1, f(0) = 0, \theta(0) = 1 \tag{17}$$

$$f'(h) = 0, f''(h) = 0, \theta(h) = 0 \tag{18}$$

Here  $S = a/b$  is the unsteadiness parameter,  $A$  is the space-dependent parameter,  $B$  is temperature-dependent parameter and  $G = g\beta \left[ \frac{(1-at)^2}{b^2} \right] Ax$  is the dimensionless

free convection parameters and  $\phi = -\frac{v}{k} ax(1-ct)^{-1}$ , respectively. Here primes indicate the differentiation with respect to  $\eta$ . The skin-friction coefficient  $C_f$  and the Nusselt number  $Nu$  are defined as:

$$C_f = \frac{\tau_w}{\frac{1}{2} \rho u_s^2} = -2 Re_x^{-1/2} f''(0), \tag{19}$$

$$Nu = \frac{hx}{k} = -Re_x^{1/2} \theta'(0), \tag{20}$$

where  $Re_x$  is the local Reynold number and  $C_f$  is the skin-friction coefficient.

### NUMERICAL ANALYSIS

In the present problem, the set of similar equations from Equations 15 to 18 are solved by a finite difference

method. These ordinary differential equations are discretized by a second-order accurate central difference method (Cebeci and Bradshaw, 1984) and a computer program has been developed to solve these equations. Some authors (Vajravelu, 1994; Vajravelu, 2001; Vajravelu and Rollins, 2004; Kai-Long, 2010a, 2010b, 2011a, 2011b) are also using analytical and numerical solutions to solve the related problems. So, some numerical technique methods will be applied to the same area in the future. The finite difference formulas are divided to forward finite-difference for the boundary layer inner edge  $\eta = 0$ , backward finite-difference formula for the boundary layer outer edge  $\eta = \infty$  and centered finite-difference formula for the internal points as follow:

1. Forward finite-difference formulas for first derivative to fourth derivative

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$$

$$f'''(x_i) = \frac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{h^3}$$

$$f''''(x_i) = \frac{f(x_{i+4}) - 4f(x_{i+3}) + 6f(x_{i+2}) - 4f(x_{i+1}) + f(x_i)}{h^4}$$

2. Backward finite-difference formulas for first derivative to fourth derivative

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$$

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}$$

$$f'''(x_i) = \frac{f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})}{h^3}$$

$$f''''(x_i) = \frac{f(x_i) - 4f(x_{i-1}) + 6f(x_{i-2}) - 4f(x_{i-3}) + f(x_{i-4})}{h^4}$$

3. Centered finite-difference formulas for first derivative to fourth derivative

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1}))}{2h}$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{h^2}$$

$$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2}))}{2h^3}$$

**Table 1.** A comparison of  $-\theta'(0)$  for an unsteady quiescent fluid flow ( $S=0, A=0, \Phi=0, \alpha=0, G=0$ ).

B	Pr	$-\theta'(0)$ (Abel et al., 2007)	Present solution	Errors
-1	1	1.710937	1.710935	0.00002
-2	2	2.486000	2.485991	0.00009
-3	3	3.082179	3.082152	0.00027
-4	4	3.585194	3.585137	0.00057
-5	5	4.028535	4.028511	0.00024

**Table 2.** Mixed convection for an unsteady visco-elastic porous fluid flow over a thermal forming thin film stretching sheet with internal heat generation ( $A=1, B=-1, \Phi=0, \eta=7$ ).

$\alpha$	Pr	S	G	$f''(0)$	$\theta'(0)$
0.1	1	0.1	0.01	-31.2252	-0.9756
0.2	2	0.2	0.02	-9.2130	-1.0932
0.3	3	0.3	0.03	-5.0737	-1.2639
0.4	4	0.4	0.04	-3.5736	-1.3269
0.5	5	0.5	0.05	-2.8398	-1.2990

$$f'''(x_i) = \frac{f(x_{i+2}) - 4f(x_{i+1}) + 6f(x_i) - 4f(x_{i-1}) + f(x_{i-2}))}{h^4}$$

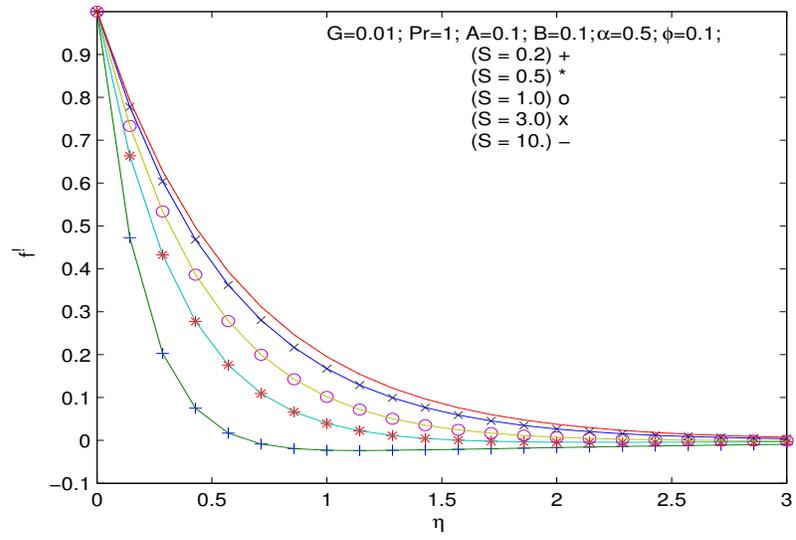
In this study, a program was written to compute finite difference approximations of derivatives for equal spaced discrete data. The code employed centered differences of  $O(h^2)$  for the interior points and forward and backward differences of  $O(h)$  for the first and last points, respectively (Chapra and Canale, 1990). To ensure the convergence of the numerical solution to exact solution, the step sizes  $\Delta\eta$  have been optimized and the results presented here are independent of the step sizes at least up to the fourth decimal place. The convergence criteria based on the relative difference between the current and previous iteration values of the velocity and temperature gradients at wall are employed. When the difference reaches less than  $10^{-6}$  for the flow fields, the solution is assumed to have converged and the iterative process is terminated. The sequence of the aforementioned equations was expressed in difference form using central difference scheme in  $\eta$ -direction. In each iteration step, the equations were then reduced to a system of linear algebraic equations.

## RESULTS AND DISCUSSION

The objective of the present analysis is to study the heat transfer of a visco-elastic porous fluid cooled or heated by a high or low various parameters. An extension of previous works has then been performed to investigate the heat transfer of a visco-elastic porous fluid over a thermal forming thin stretching sheet with mixed

convection effect, which are included. The model for visco-elastic porous fluid has been used in the momentum equations. Effects of dimensionless parameters, the unsteadiness parameter ( $S$ ), the porous parameter ( $\phi$ ), the Prandtl number ( $Pr$ ), the visco-elastic parameter ( $\alpha$ ), the space-dependent parameter ( $A$ ) and temperature-dependent parameter ( $B$ ) for heat source/sink and the free convection parameter ( $G$ ) are mainly of interest in the study. Flow and temperature fields of the visco-elastic porous fluid flow have been analyzed by utilizing the boundary layer concept to obtain a set of coupled momentum equations and energy equations. A similarity transformation has been used to convert the non-linear, coupled partial differential equations to a set of non-linear, coupled ordinary differential equations. A generalized derivation is used to analyze an unsteady flow that has been studied. An accurate finite difference method was used to obtain solutions of these equations. Comparing  $-\theta'(0)$  to results of Vajravelu and Roper (1999) for an unsteady quiescent fluid flow ( $S=0, A=0, \phi=0, \alpha=0, G=0$ ) showed a good agreement and these values are listed in Table 1. Table 2 shows a mixed convection and unsteady visco-elastic fluid flow field results ( $A=1, B=1, \eta=7$ ) for different  $\alpha, Pr, S$  and with different free convection parameters  $G$ .

Figure 2 shows dimensionless velocity gradient  $f'$  versus  $\eta$  as  $G=0.01, S=0.1, A=0.1, B=0.1, \alpha=0.5, \phi=0.1$  and  $S=0.2, 0.5, 1, 3, 10$ . It represents the fluid flow phenomenon toward the flow field. The numerical calculation results are satisfied by the boundary layer conditions at the figure. The momentum was interacting with each other and the figure curves are all having a strong variation with  $\eta$  along the boundary layer for



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