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Full Length Research Paper

Dirac equation for a spherically pseudoharmonic oscillatory ring-shaped potential

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In this paper, we investigate exact solutions of Dirac equation with spherically pseudoharmonic oscillatory ring-Shaped potential. The generalized parametric Nikiforov-Uvarov method is used obtaining analytical solutions of radial and polar angle parts of Dirac Equation. Energy eigenvalues and corresponding eigenfunctions are obtained in closed forms.

Key words: Dirac equation, ring-shaped non-central pseudoharmonic oscillator potential, Nikiforov-Uvarov method.

INTRODUCTION

The study exact solution of the Schrödinger, Klein-Gordon and Dirac equations with non-central potentials is of considerable interest (Gönül and Zorba, 2000; Taşeli et al., 2002; Kocak et al., 2002; Chen et al. 2002; Chen et al. 2004; Kocak and Gönül, 2005; Chen and Dong, 2005; Chen, 2005; Guo et al., 2006; Souza Dutra and Hott, 2006; Ikhdair and Sever, 2007; Cheng and Dai, 2008; Ikhdair and Sever, 2008; Zhang, 2009; Sadeghi et al., 2009; Berkdemir and Sever, 2009; Zhang et al., 2010; Chen et al., 2010; Gu et al., 2010; Zhou et al., 2009; Xiao and long, 2010; Hamzavi et al., 2010). Quesne (1998) obtained a new ring-shaped potential by replacing the Coulomb part of the Hartmann potential by a harmonic oscillator term. Gang (2004) exactly obtained the energy spectrum of some non-central separable potentials in r and θ dimensions using the method of supersymmetric WKB approximation. Zi-Dong and Gang (2005) obtained exact bound state solutions of the radial Schrödinger equation with the Hartmann potential, as well as two kinds of recursion relations of radial wave functions by using Laplace transformation. Yaşuk et al. (2005) obtained the general solutions of the Schrödinger equation for a non-central potential by using the Nikiforov–Uvarov method. Chang-Yuan et al. (2006) studied exact solutions of scattering states of the Klein–Gordon equation with Coulomb potential plus a new ring-shaped potential under the condition that the scalar potential is equal to the vector potential. Min-Cang and Zhen-Bang (2007) studied the Klein–Gordon equation with equal scalar and vector Makarov potentials by the factorization method. Yan-Fu and Tong-Qing (2007) proposed a new exactly solvable potential which is formed by modified Kratzer potential plus a new ring-shaped potential. Kerimov (2007) studied the nonrelativistic quantum scattering problem for a non-central potential which belongs to a class of potentials exhibiting an 'accidental' degeneracy. Berkdemir and Sever (2008) investigated the pseudospin symmetry solution of the Dirac equation for spin $1/2$ particles moving within the Kratzer potential connected with an angle-dependent potential, systematically. Yeşiltaş (2008) showed that a wide class of non-central potentials can be analysed via the improved picture of the Nikiforov–Uvarov method. Fang et al. (2009) investigated the pseudospin symmetry in the Makarov potential systematically by solving the Dirac equation. Berkdemir and Cheng (2009) studied the problem of the relativistic motion of a $1/2$-spin particle in an exactly solvable potential, which consists of the harmonic oscillator potential plus a novel angle-dependent potential. Zhang et al. (2009) proposed a new ring-shaped non-spherical harmonic oscillator potential which consists of a generalized non-harmonic oscillator potential plus an angle-dependent potential. Yong-Jun and Zheng-Wen

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(2010) presented under the condition of an equal mixing of vector and scalar potentials, exact solutions of hound states of the Klein–Gordon equation with pseudo-Coulomb potential plus a new ring-shaped potential. The ring-shaped non-central pseudoharmonic oscillator (Zhanget et al., 2010) is a kind of physical potential which in spherical coordinates (r,θ,ϕ) is defined by:

$$V(r,\theta)=\frac{A}{2}r^2+\frac{B}{r^2}+\frac{\eta+\beta\cos^2\theta+\gamma\cos^4\theta}{r^2\sin^2\theta\cos^2\theta} \quad (1)$$

Generalized parametric Nikiforov-Uvarov method

To solve second order differential equations, the Nikiforov-Uvarov method (Nikiforov and Uvarov, 1988) can be used with an appropriate coordinate transformation $s=s(r)$:

$$\psi_n''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0 \quad (2)$$

Where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. The following equation is a general form of the Schrödinger-like equation written for any potential (Tezcan and Sever, 2009):

$$\left[\frac{d^2}{ds^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d}{ds}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{\left[s(1-\alpha_3 s)\right]^2}\right]\psi_n(s)=0 \quad (3)$$

According to the Nikiforov-Uvarov method, the eigenfunctions and eigenenergy function become, Equations 4 and 5, respectively:

$$\psi(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s) \quad (4)$$

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)+n(n+1)\alpha_3 +\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0 \quad (5)$$

Where,

$$\alpha_4=\frac{1}{2}(1-\alpha_1),\qquad \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),$$
$$\alpha_6=\alpha_5^2+\xi_1,\qquad \alpha_7=2\alpha_4\alpha_5-\xi_2, \quad (6)$$
$$\alpha_8=\alpha_4^2+\xi_3\qquad \alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6$$

and

$$\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},\qquad \alpha_{11}=\alpha_2-2\alpha_5+2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right) \quad (7)$$
$$\alpha_{12}=\alpha_4+\sqrt{\alpha_8},\qquad \alpha_{13}=\alpha_5-\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)$$

In some problems, $\alpha_3=0$. For this type of problems when

$$\lim_{\alpha_3\to 0}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-\alpha_3)s=L_n^{\alpha_{10}-1}(\alpha_{11}s) \quad (8)$$

And

$$\lim_{\alpha_3\to 0}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}=e^{\alpha_{13}s} \quad (9)$$

the solution given in Equation 6 becomes as (Tezcan and Sever, 2009):

$$\psi(s)=s^{\alpha_{12}}e^{\alpha_{13}s}L_n^{\alpha_{10}-1}(\alpha_{11}s) \quad (10)$$

Dirac equation with scalar and vector ring-shaped non-central harmonic oscillator potential

The Dirac equation with scalar potential $S(\vec{r})$ and vector potential $V(\vec{r})$ is $[\hbar=c=1]$ (Gereiner, 2000)

$$\left[\vec{\alpha}.\vec{p}+\beta\left(M+S(\vec{r})\right)\right]\Psi(\vec{r})=\left[E-V(\vec{r})\right]\Psi(\vec{r}) \quad (11)$$

Where, E is the relativistic energy of the system and $\vec{p}=-i\vec{\nabla}$ is the three-dimensional momentum operator. $\vec{\alpha}$ and β are the 4×4 usual Dirac matrices given as;

$$\vec{\alpha}=\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix},\quad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\qquad i=1,2,3 \quad (12)$$

Where I is the 2×2 unitary matrix and the 2×2 three Pauli matrices σ_i are given by

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad (13)$$

In Pauli-Dirac representation, let

$$\Psi(\vec{r})=\begin{pmatrix}\varphi(\vec{r})\\ \chi(\vec{r})\end{pmatrix} \quad (14)$$

Substituting Equation 12-14 into Equation 11, we obtain

$$\vec{\sigma}.\vec{p}\chi(\vec{r})=\left[E-V(\vec{r})-M-S(\vec{r})\right]\varphi(\vec{r}) \quad (15a)$$

$$\vec{\sigma}.\vec{p}\varphi(\vec{r})=\left[E-V(\vec{r})+M+S(\vec{r})\right]\chi(\vec{r}) \quad (15b)$$

When scalar potential $S(\vec{r})$ is equal to the vector potential $V(\vec{r})$, Equation 15 become:

$$\vec{\sigma}.\vec{p}\chi(\vec{r})=\left[E-M-2V(\vec{r})\right]\varphi(\vec{r}) \quad (16a)$$

$$\chi(\vec{r})=\frac{\vec{\sigma}.\vec{p}}{E+M}\varphi(\vec{r}) \quad (16b)$$

Substituting Equation 16b into Equation 16a, one can obtain

$$\left[p^2+2(E+M)V(\vec{r})\right]\varphi(\vec{r})=\left[M^2-E^2\right]\varphi(\vec{r}) \quad (17)$$

and substituting potential (1) into Equation (17), we have

$$\left[-\nabla^2+2(M+E)\left(\frac{A}{2}r^2+\frac{B}{r^2}+\frac{\eta+\beta\cos^2\theta+\gamma\cos^4\theta}{r^2\sin^2\theta\cos^2\theta}\right)\right]\varphi(r,\theta,\phi) =\left[E^2-M^2\right]\varphi(r,\theta,\phi) \quad (18)$$

Let,

$$\varphi(r,\theta,\phi)=\frac{u(r)}{r}H(\theta)\Phi(\phi) \quad (19)$$

Separating the variables in Equation 18, we obtain

$$\frac{d^2u(r)}{dr^2}-\left\{2(M+E)\left(\frac{A}{2}r^2+\frac{B}{r^2}\right)+\frac{\lambda}{r^2}-\left(E^2-M^2\right)\right\}u(r)=0 \quad (20a)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH(\theta)}{d\theta}\right)+\left[\lambda-\frac{m^2}{\sin^2\theta}-\frac{2(E+M)(\eta+\beta\cos^2\theta+\gamma\cos^4\theta)}{\sin^2\theta\cos^2\theta}\right]H(\theta)=0 \quad (20b)$$

$$\frac{d^2\Phi(\phi)}{d\phi^2}+m^2\Phi(\phi)=0 \quad (20c)$$

Where λ and m^2 are separation constants. It is well known that the solution of Equation 20c is

$$\Phi_m(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi} \qquad m=0,\pm1,\pm2,... \quad (21)$$

ANALYTICAL SOLUTIONS OF RADIAL AND POLAR ANGEL PARTS OF DIRAC EQUATION

Solution of polar angle part

To derive eigenvalues and eigenfunctions of polar angle part of Dirac equation (Eq. (20b)), we use an appropriate transformation as $s=\cos^2\theta$ and it reduced as

$$\frac{d^2H(s)}{ds^2}+\frac{1-3s}{2s(1-s)}\frac{dH(s)}{ds}+\frac{1}{4s^2(1-s)^2}\left[-(\lambda+\gamma')s^2+(\lambda-m^2-\beta')x-\eta'\right]H(s)=0 \quad (22)$$

Where, $\gamma'=2(E+M)\gamma$, $\beta'=2(E+M)\beta$ and $\eta'=2(E+M)\eta$.

Comparing the above equation with Equation 3, the following expressions are obtained:

$$\alpha_1=\frac{1}{2}, \qquad \xi_1=\frac{1}{4}(\lambda+\gamma')$$
$$\alpha_2=\frac{3}{2}, \qquad \xi_2=\frac{1}{4}(\lambda-m^2-\beta') \quad (23)$$
$$\alpha_3=1, \qquad \xi_3=\frac{1}{4}\eta'$$

And

$$\alpha_4=\frac{1}{4}, \qquad \alpha_5=-\frac{1}{4}$$
$$\alpha_6=\frac{1}{16}+\frac{1}{4}(\lambda-\gamma'), \qquad \alpha_7=-\frac{1}{8}-(\lambda-m^2-\beta') \quad (24)$$

20b

$$\alpha_8=\frac{1}{16}+\frac{1}{4}\eta', \qquad \alpha_9=\frac{1}{4}(\eta'+\beta'+\gamma'+m^2)$$

from Equations 23 and 24, and 5, we obtain:

20c

$$\lambda=\left(1+2\tilde{n}+\sqrt{m^2+\eta'+\beta'+\gamma'}\right)\left(1+2\tilde{n}+\sqrt{m^2+\eta'+\beta'+\gamma'}+\sqrt{\frac{1}{4}+\eta'}\right)+(\eta'-\beta') \quad (25)$$

Where, $\tilde{n}$ is non-negative integer. For the wave functions of the polar part, from Equations 6 and 7, one obtains:

$$\begin{aligned}
\alpha_{10} &= \frac{1}{16} + \sqrt{\frac{1}{4} + \eta'} \\
\alpha_{11} &= 2 + 2\left(\sqrt{\frac{1}{4}(\eta' + \beta' + \gamma' + m^2)} + \sqrt{\frac{1}{16} + \frac{1}{4}\eta'}\right) \\
\alpha_{12} &= \frac{1}{4} + \sqrt{\frac{1}{16} + \frac{1}{4}\eta'} \\
\alpha_{13} &= -\frac{1}{4} - \left(\sqrt{\frac{1}{4}(\eta' + \beta' + \gamma' + m^2)} + \sqrt{\frac{1}{16} + \frac{1}{4}\eta'}\right)
\end{aligned} \tag{26}$$

and from Equation 4, we obtain:

$$\begin{aligned}
H(s) &= s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} P_{\tilde{n}}^{\left(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s) \\
&= s^{\frac{1}{4}(1+\sqrt{1+4\eta'})}(1-s)^{\frac{1}{2}\sqrt{m^2+\eta'+\beta'+\gamma'}} P_{\tilde{n}}^{\left(\sqrt{1/4+\eta'}, \sqrt{m^2+\eta'+\beta'+\gamma'}\right)}(1-2s)
\end{aligned} \tag{27}$$

Or equivalently

$$H(\theta) = C_{\tilde{n}}(\cos\theta)^{\frac{1}{2}(1+\sqrt{1+4\eta'})}(\sin\theta)^{\sqrt{m^2+\eta'+\beta'+\gamma'}} P_{\tilde{n}}^{\left(\sqrt{1/4+\eta'}, \sqrt{m^2+\eta'+\beta'+\gamma'}\right)}(1-2\cos^2\theta) \tag{28}$$

Where, $C_{\tilde{n}}$ is normalization constant.

Solution of radial part

To derive eigenvalues and eigenfunctions of radial part of Dirac equation (Equation 20a), we follow similar method as given in subsection 4.1, and transformation $s = r^2$. Then Equation 20a reduces to;

$$\frac{d^2u(s)}{ds^2} + \frac{1}{2s}\frac{du(s)}{ds} - \frac{1}{4s^2}\left[A's^2 - \varepsilon^2 s + \lambda + 2B'\right]u(s) = 0 \tag{29}$$

Where, $A' = 2(E+M)\gamma$, $B' = 2(E+M)B$ and $\varepsilon^2 = E^2 - M^2$. By using the methodology of previous subsection, one can obtain the energy eigenvalue of radial part as follows:

$$\sqrt{E+M}\,(E-M) = 2\sqrt{A}\left(n + 1 + \sqrt{2(E+M)B + \lambda + \frac{1}{4}}\right) \tag{30}$$

For effect of angle-dependent part on radial solutions, we substitute Eq. (26) into Eq. (30) and obtain

$$\begin{aligned}
&\sqrt{E+M}(E-M) = 2\sqrt{A}\Bigg(n+1+ \\
&\times\sqrt{2(E+M)B + \left(1+2\tilde{n}+\sqrt{m^2+\eta'+\beta'+\gamma'}\right)\left(1+2\tilde{n}+\sqrt{m^2+\eta'+\beta'+\gamma'}+\sqrt{\frac{1}{4}+\eta'}\right)+(\eta'-\beta')+\frac{1}{4}}\Bigg)
\end{aligned} \tag{31}$$

When, $\eta = \beta = \gamma = 0$, the potential (1) reduced to the pseudoharmonic potential (38) reduced to (Aydoğdu and R. Sever, 2009).

$$\begin{aligned}
&\sqrt{E+M}\,(E-M) \\
&\qquad = 2\sqrt{A}\left(n+1+\sqrt{2(M+E)B + \left(\tilde{n}+m+\frac{1}{2}\right)^2}\right)
\end{aligned} \tag{32}$$

Also, when $B = \eta = \beta = \gamma = 0$, the potential (1) becomes harmonic oscillator potential and its energy eigenvalue function reduced as follows (Gereiner, 2000):

$$\sqrt{E+M}\,(E-M) = 2\sqrt{A}\left(n + l + \frac{3}{2}\right) \tag{33}$$

Where, $l = \tilde{n} + m$. The radial eigenfunctions cane be obtain as

$$u(r) = D_n r^{\frac{1}{2}+\sqrt{2(M+E)B+\lambda+\frac{1}{4}}} e^{-\frac{1}{2}\sqrt{A'}r^2} L_n^{\sqrt{2(M+E)B+\lambda+\frac{1}{4}}}\left(\sqrt{A'}r^2\right) \tag{34}$$

where D_n is normalization constant. Finally, we can write $\varphi(\vec{r})$ as

$$\begin{aligned}
\varphi(\vec{r}) &= \frac{u(r)}{r}H(\theta)\Phi(\phi) \\
&= \frac{N}{\sqrt{2\pi}} r^{-\frac{1}{2}+\sqrt{2(M+E)B+\lambda+\frac{1}{4}}} e^{-\frac{1}{2}\sqrt{A'}r^2} L_n^{\sqrt{2(M+E)B+\lambda+\frac{1}{4}}}\left(\sqrt{A'}r^2\right) \\
&\quad \times(\cos\theta)^{\frac{1}{2}(1+\sqrt{1+4\eta'})}(\sin\theta)^{\sqrt{m^2+\eta'+\beta'+\gamma'}} P_{\tilde{n}}^{\left(\sqrt{1/4+\eta'}, \sqrt{m^2+\eta'+\beta'+\gamma'}\right)}(1-2\cos^2\theta)e^{im\phi}
\end{aligned} \tag{35}$$

Where, N is the normalization constant. From Equations 16b and 14, the spinor wave function becomes

$$\begin{aligned}
\Psi(\vec{r}) &= \begin{pmatrix}\varphi(\vec{r}) \\ \chi(\vec{r})\end{pmatrix} = \frac{N}{\sqrt{2\pi}}\begin{pmatrix}1 \\ \frac{\vec{\sigma}.\vec{p}}{E+M}\end{pmatrix} r^{\frac{1}{2}+\sqrt{2(M+E)B+\lambda+\frac{1}{4}}} e^{-\frac{1}{2}\sqrt{A'}r^2} L_n^{\sqrt{2(M+E)B+\lambda+\frac{1}{4}}}\left(\sqrt{A'}r^2\right) \\
&\quad \times(\cos\theta)^{\frac{1}{2}(1+\sqrt{1+4\eta'})}(\sin\theta)^{\sqrt{m^2+\eta'+\beta'+\gamma'}} P_{\tilde{n}}^{\left(\sqrt{1/4+\eta'}, \sqrt{m^2+\eta'+\beta'+\gamma'}\right)}(1-2\cos^2\theta)e^{im\phi}
\end{aligned} \tag{36}$$

At the end, when $S(\vec{r}) = -V(\vec{r})$, one can find the lower spinor component of the Dirac equation. To avoid repetition, we use below transformations in Equation 16a-16b as (Aydoğdu and R. Sever, 2009):

$$\begin{aligned}
\phi(r) &\to \chi(r) \\
\chi(r) &\to -\phi(r) \\
V(r) &\to -V(r) \\
E &\to -E
\end{aligned} \tag{37}$$

CONCLUSION

In this paper, by using the generalized parametric Nikiforov-Uvarov method, the Dirac equation has been studied for a spherically pseudoharmonic oscillatory ring-shaped potential. The analytical expressions for energy eigenvalues and eigenfunctions have been found. We point that these results may have interesting applications in the study of different quantum mechanical systems and atomic physics (Hartmann and Schuch, 1980; Carpido-Bernido and Bernido, 1989).

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