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Full Length Research Paper

Hybrid evolutionary computational approach: Application to van der pol oscillator

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We presented a method for van der Pol oscillators using artificial neural network optimized by evolutionary computational approach. A trail solution of the oscillator is written as a feed-forward neural network containing adjustable adaptive parameters. The optimization of the networks is performed by genetic algorithms in an unsupervised way. The proposed scheme is tested successfully by applying on both the stiff and non-stiff problems. A Monte Carlo simulation is performed for the reliability and effectiveness of the scheme. It is shown that the obtained results are in good agreement with Runge Kutta numerical method.

Key words: Van der pol oscillator, genetic algorithms, neural network modeling.

INTRODUCTION

The dynamics of van der pol oscillators occurs in wide range applications of electrical and electronics engineering (Itoh, 2001; Atay, 1998). The system that we investigated has been used as an example in the study of limit cycles and self-sustained oscillatory phenomena in the nonlinear systems. This nonlinear system is represented by so called van der pol equation. It is catalogued as a good example of both non-linear and stiff system at the same time which is tedious to solve (Li, 2010). Earlier work in the area of oscillators has been done by various investigators. Minorsky (1962) has studied two van der pol oscillators with small coupling. Hayashi and Kuramitsu (1974) used an averaging method to study van der pol damped linear oscillators. Linkens (1974, 1976) has used the method of harmonic balance to study large group of van der pol oscillators. A perturbation method is used to study the steady state behavior of two van der pol oscillators (Storti and Rand, 1982), which were fiddly and complex in solving by analytical methods, so the need for approximating the solution arises. A number of algorithms such as Runge-Kutta, finite difference, etc are available for the approximation of the solutions at discrete grid of time (Kunz and Luebbers, 1993; Dormand and Prince, 1980). The limitations like, discrete pre-defined locations of the solution space, increase of computational complexity with the number of sampling points and problem of rounding of error (Saloma, 1993) gave birth to solutions with neural networks (NN) optimized by various methods. The strengths and applicability of stochastic methods along with NN in non-linear systems and fractional systems were importance in the recent years (Jang et al., 2000; Raja et al., 2010; Li and Wang, 2006; Zhao and Chen, 2002; Khan et al., 2011). After a comprehensive survey, it has been noticed that, the approximation of van der pol oscillator by evolutionary computation (EC) may be an area to be explored.

In this paper, we have implemented the NN optimized by EC technique for van der pol oscillators given as

$$\ddot{y} + \mu\,(y^2 - 1)\,\dot{y} + y = 0 \qquad 0 \le t \le 2\,, \tag{1}$$

With the following initial conditions

$$y(0) = 2, \quad \dot{y}(0) = 0 \tag{2}$$

Where μ is the factor defining stiffness of the oscillator. The oscillator is considered to be non-stiff for the smaller

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values of μ and is stiff for very large values. The van der pol oscillator can be represented by second order non-linear differential equation. Previously, this has been solved by reducing the non-linear second order differential equation to two coupled differential equations of the first order. But we have directly solved various test cases of stiff and non-stiff oscillator by NN optimized using EC techniques, explicitly genetic algorithm (GA). The approximated results by GA, interior point algorithm (IPA) and GA hybridized with IPA are compared with the results of Runge Kutta (RK45) method. We have provided the solution for entire continuous finite time domain instead of discrete grid of time unlike the classical numerical techniques. A comprehensive statistical analysis is constructed to see the effectiveness of the given scheme. The interest in this problem is due to its vast application in engineering of non-linear systems and dynamics to tackle with systems which were not controllable in late 1980's but can be managed by the advent of recent state of art computational algorithms. Also the present application of NN and stochastic algorithms in electronics engineering is used from the point of view of a new approach for solving abnormal systems.

MATERIALS AND METHODS

The method of optimization, like GA is explained along with the mathematical NN modeling. The process used for the training of weights is also given in this study.

Evolutionary computation

Evolutionary systems were independently studied by various scientists in 1960's based on the idea that evolution could be used as an optimization tool for engineering problems (Lehre and Witt, 2010). The core of all these systems was evolving a candidate solution using operators inspired by natural genetic variation and natural selection. Rechenberg introduces evolution strategies, a method used to optimize real valued parameters for devices like airfoils (Mitchell, 1998). Several other researchers during 1950s developed evolution inspired algorithms for optimization and machine learning. Box (1957), Friedman (1959), Bledsoe (1961), Bremermann (1962) and Reed, Toombs and Baricelli all worked in the area of EC like evolution strategies and GA (Mitchell and Forect, 1994). From 2000's onward, there has been widespread application of various EC methods and the boundaries are studied between GA, evolution strategies and other evolutionary strategies. The GA is a heuristic search that mimics the process of natural evolution (Goldberg and David, 1989). This heuristic approach is used in routine to generate useful solutions to optimization and search problems which are computationally complex to solve (Srinivas and Patnaik, 1994). GAs belong to the larger class of evolutionary algorithms (EA), which generate solutions to optimization problems using techniques, such as inheritance, mutation, selection and crossover (Zhang et al., 2007). Generally, the algorithm terminates, when, either a maximum number of generations have been produced, or a satisfactory fitness level has been achieved. The major advantage of using GA is that, it is robust, simple, efficient and does not get trapped in the poor region of search space or local minimum like classical numerical methods (Raja et al., 2010). The generic flow diagram of the evolutionary algorithm used for optimization is provided in Figure 1.

Neural network mathematical modeling

An approximate mathematical model has been developed using feed-forward Artificial Neural Networks (ANN), which is well known as universal function approximator (Funahashi, 1989; Hornik et al., 1990; Junaid et al., 2009; Cybenko, 1989) and is used widely in diverse fields. Any network suitably trained to approximate a mapping satisfying some ordinary differential equation (ODE) having function that may also approximate the differential equation (DE) (Meada and Fernandez, 1994). For the following continuous mapping is employed.

$$u(x) = \sum_{i=1}^{n} \alpha_i \phi\ (w_i x + b_i), \tag{3}$$

$$\dot{u}(x) = \sum_{i=1}^{n} \alpha_i \dot{\phi}\ (w_i x + b_i), \tag{4}$$

$$\ddot{u}(x) = \sum_{i=1}^{n} \alpha_i \ddot{\phi}\ (w_i x + b_i), \tag{5}$$

For u, $\dot{u}$ and $\ddot{u}$, respectively, where φ being the activation function normally taken as log sigmoid for hidden layers and linear function for output layer.

$$\phi(t) = \frac{1}{1+e^{-t}}, \tag{6}$$

$$\phi(t) = t, \tag{7}$$

Where α_i, w_i, and b_i are real-valued bounded adaptive parameters and n is the number of neurons in the ANN architecture.

Fitness function

The unsupervised error function e is formulated by the linear combination of networks (3) to (5) for any problem in the form given in (1) as

$$e = \frac{1}{1+e_j} \quad j = 1,2,\ldots\ldots, \tag{8}$$

Where j is the cycle index and the function e_j is defined as the mean of sum of square error.

$$e_j = e_1 + e_2\big|_j, \tag{9}$$

Where e_1 error is associated with the equation and is given as

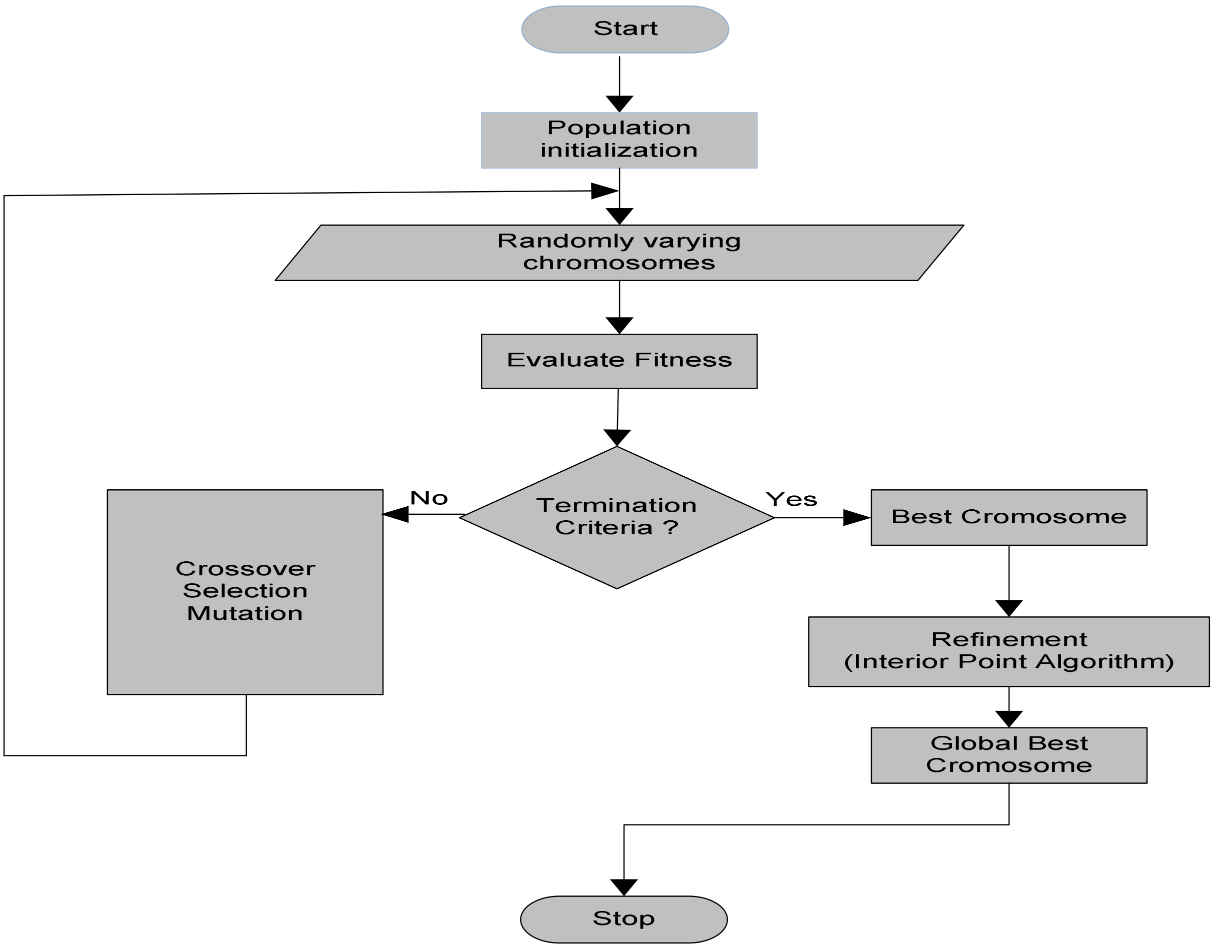


Figure 1. Generic flow diagram of genetic algorithm.

$$e_1 = \frac{1}{m+1}\sum_{i=0}^{m}[\ddot{u}(t_i) + \mu\left(u(t_i)^2 - 1\right)\dot{u}(t_i) + u(t_i)]^2 \,, \quad (10)$$

Where *T = mh,* m is the total number of steps and h defines the size of the step, *t* is taken between *(0,T)*. Greater the value of *m* more will be the accuracy but at the cost of greater computational complexity of algorithm. Setting the value of *m* is bit fiddly, because it is a parameter that decides a compromise between accuracy and computational cost. Similarly e_2 is linked with initial conditions and is written as,

$$e_2 = \frac{1}{2}\left\{(u(0)-2)^2 + \dot{u}(0)^2\right\}, \quad (11)$$

It is quite evident that subject to the availability of unknown weights for which the function e_j approaches zero, then the value of unsupervised error *e* approaches 1. Hence, the solution *u(t)* of the equation approximates the model given in (1). The ANN architecture for the expression (1) is shown in Figure 2 and is called differential equation neural network (DE-NN).

Learning procedure

Now our intent is to provide the necessary details about the learning procedure for adaptive parameters during the simultaneous training of networks represented by (3 to 5). The learning methodology is based on the GA hybridized with IPA. The algorithm runs iteratively for the optimization of adaptive parameters. The structure of the given algorithm is described briefly in the following steps:

Step 1: Initialization population

An initial population of M chromosomes is generated in a bounded range with the help of a random number generator.

Step 2: Generation of subpopulations

The populations divide the q subpopulations, each with $\frac{M}{q}$ chromosomes.

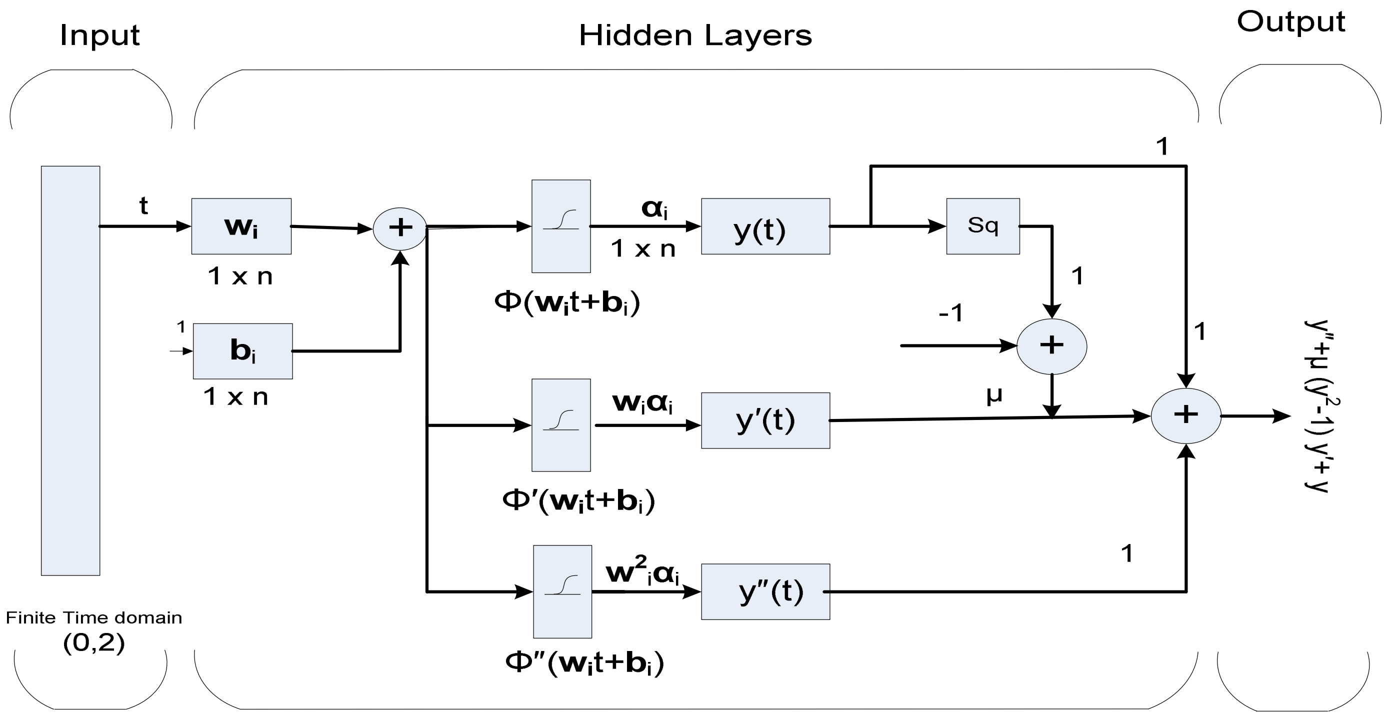


Figure 2. Neural network architecture of van der pol oscillator.

Step 3: Fitness evaluation and ranking

Calculate the fitness of each chromosome of the subpopulation using Equation (8). Individual were rank according to the subpopulation on the basis of the maximum fitness.

Step 4: (Termination criteria)

Terminate basis on the following criteria

Predefined fitness $e^j \leq 10^{-10}$ is achieved.
Maximum number generations completed.

Step 5: (Reproduction using GA operators)

Following methodology is implied for GA operators
Crossover is performed using scattered function.
Adaptive feasible function is used for mutation.
Selection is carried out on the base of heuristic function.
.

Step 6: (Improvement)

The fitness of chromosomes in sub population is improved by repeating Steps 3 to 5 with newly reproduced individual from step 5.

Step 7: (Repetition)

Update the chromosomes of each sub population by repeating Steps 3 to step 6. Store the best chromosome of the population.

Step 8: (Local search)

The chromosome from Step 7 is given as a start point to interior point algorithm for rapid local convergence.

Step 9: (Storage)

Store the global best individual along with its fitness value from Step 8 for this independent execution of the algorithm.

Step 10: (Statistical analysis)

Repeat Step 1 to 9 for sufficient large numbers, in order to have a statistical analysis of the results.

RESULTS AND DISCUSSION

In this section, the results of detailed simulation are presented by the proposed scheme. We have considered two test cases for non-stiff oscillator and two for stiff. The comparative analysis is provided with RK45.

Van der pol non-stiff oscillator

Consider an oscillator model of the form (1) for non-stiff case by taking $\mu = 1$ and $\mu = 3$. This problem is solved by DE-NN algorithm. The number of neurons in each hidden

Table 1. Parameters setting of the algorithms.

GA		IPA	
Parameter	**Setting**	**Parameter**	**Setting**
Population size	240	Sub problem algorithm	Idl factorization
Chromosome size	30	Chromosome size	30
No. of runs	2000	Minimum perturbation	1e-8
Selection	Stochastic uniform	Maximum perturbation	0.1
Scaling function	Rank	X-tolerance	1e-15
Reproduction	Elite count of 2 crossover fraction 0.7	Scaling	Objective and constraints
Mutation	Adaptive feasible	Derivative type	Central difference
Crossover	Scattered	Hessian	BFGS
Migration direction	Forward	Initial barrier parameter	0.1
Migration interval	20	Maximum function evaluation	50000
Hybridization	IPA	Maximum Iteration	1000

layer is taken as *n = 10* which results in *30* unknown adaptive weights (α_i, w_i and b_i). The optimization of these weights is carried out using built-in function of GA using GA and direct search toolbox of MATLAB. The parameter setting used for the execution of the algorithm is given in Table 1. The input of the training set is taken from time $t \in (0, 2)$ with a step size of *h=0.1.* It means that the total time steps *m = 20* so the fitness function for non stiff cases at $\mu = 1$ and $\mu = 3$ are formulated below:

$$e_j = \frac{1}{21}\sum_{i=1}^{21}[\ddot{u}(t_i) + (u(t_i)^2 - 1)\dot{u}(t_i) + u(t_i)]^2 + \frac{1}{2}[\{u(0)-2\}^2 + \{\dot{u}(0)\}^2]\Bigg|_j, \quad j = 1,2,3\ldots, \tag{13}$$

$$e_j = \frac{1}{21}\sum_{i=1}^{21}[\ddot{u}(t_i) + 3(u(t_i)^2 - 1)\dot{u}(t_i) + u(t_i)]^2 + \frac{1}{2}[\{u(0)-2\}^2 + \{\dot{u}(0)\}^2]\Bigg|_j, \quad j = 1,2,3\ldots, \tag{14}$$

Where *j* is the iteration index and $\ddot{u}, \dot{u}$ and u are the networks given in (3 to 5), respectively. One of the unknown weights learned by GA are provided in Table 2 which has the fitness values *4.3217E-04* and *6.7077E-04* for *μ=1* and *μ=3,* respectively are provided in Table 2. The one of the best chromosomes attained by GA-IPA are also given in Table 2 and also the fitness values *4.3217E-04* and *6.7077E-04* for *μ=1* and *μ=3,* respectively. This is worth to note that the results obtained by GA-IPA are more precise than those achieved by IPA or GA. The adaptive weights are taken up to an accuracy of 10 decimal places, because by using the weights of the mentioned accuracy level can provide the desired solutions. These weights and the parameter setting given in Table 1 can be used in Equation (3) to obtain the solution of the equation readily for any input time *t* between *0* and *2* on the continuous grid. The solution of y_{GA} y_{IPA} and $y_{GA\text{-}IPA}$ are determined using the weight given in Table 2 for *μ=1* on the interval (0, 2) with step 0.1. The results are summarized in Table 3. In order to make the comparison, the results are also obtained for RK45 on the same ranges inputs domain. The results for the solution y_{rk45} are also provided in Table 3.

It is clear from the results that the accuracy of the GA-IPA is better as compared to GA and IPA and is also in good agreement with the state of art numerical solver. The absolute error of the RK45 with GA, IPA and GA-IPA is listed and is evident that the absolute error of GA-IPA is in the range *E-08* to *E-05.*The behavior of the derivative of the solution using the same weights as given in Table 2 is seen. The results are also provided for the derivative of the solution from *0* to *1* with a step of *0.1* in Table 4. It can be seen from table that the absolute error is in the range *E-03* to *E-05, E-05* to *E-06* and *E-05* to *E-07* for GA, IPA and GA-IPA, respectively from RK45. The effectiveness of the proposed scheme is tested for non-stiff van der Pol oscillator at *μ=3* as well for inputs *t* between *0* to *2* with a step size of *0.1*. The adaptive parameters trained by GA and GA-IPA are also provided in Table 2. By using these weights the results of the problem are given with absolute error in Table 5 and its derivative in Table 6, respectively. For the second non-stiff condition the value of the absolute error is in line with the range of *E-03, E-05* to *E-06* and *E-06* to *E-08* for GA, IPA and GA-IPA, respectively from RK45. So it this can be concluded that the results attained by GA-IPA are the

Table 2. Adaptive parameters obtained by DE-NN networks.

	μ=1			μ=3		
	w_i	α_i	β_i	w_i	α_i	β_i
GA	0.8453261042	-0.4011661839	1.3620270226	0.3312201351	-1.0623077914	0.5750409795
	-0.5971684692	0.7263677524	0.7557357130	-0.8149353217	1.0024050738	2.7219971074
	3.6166798222	0.6177622934	1.0367813700	1.0066182284	0.4659396762	0.5278975371
	1.3946054222	-0.5147163833	-0.3336453208	-1.5995136970	0.2758782816	4.5951103401
	-0.9700850034	1.7824544146	2.9351614986	-0.2903398783	1.0782618973	0.3663869480
	9.0841060478	0.2445612189	2.5754169671	0.0724065760	1.0824790284	0.3386196337
	0.8352946883	-1.7644306840	2.5427879135	2.8918366775	-0.3556667244	0.8103846571
	1.5405272613	-4.5194604493	-4.7611223703	4.6761358608	0.0986752919	-0.3050141921
	-3.0832029211	1.2494958374	8.2785213477	9.9999850963	0.3823374438	2.5511569007
	-0.2634523420	0.2012972617	-0.2608087501	0.6611531635	-0.4559170160	0.1215298773
GA-IPA	-2.8901507009	1.2991330862	-1.3864986210	-0.1680311688	1.3029426975	1.2291823021
	-1.8423980679	1.1566279413	3.4827264402	-0.6596716052	-1.8300534918	-2.7679617140
	-0.6171883408	1.4117053435	-0.2522594102	-2.1367442682	0.9001358585	7.2326536215
	-2.8061185229	3.4786140479	7.5790075797	-0.9356820610	0.6832914852	-0.4534421025
	-1.8315678821	-0.8100368766	-1.7968729165	-0.6602796040	-0.0818863263	-0.1189985862
	0.8431074104	-3.6317819333	2.7122634135	-0.9240396221	0.9338323979	-1.4099264330
	-2.5565147982	-0.3917875420	0.7682924141	-0.9179052836	-0.6859923169	-1.5292444162
	-3.4909517134	-0.4057274792	-0.2341527758	1.0866654896	-0.7297961220	-2.4014057911
	-1.2408002259	2.9304478749	-1.0080853566	-9.5164924589	-3.8208063513	-5.1634884011
	-3.8081236873	-3.7915947696	-2.4845407680	-1.9566612579	-0.1528119580	-3.3611685123

Table 3. Comparison of the results for μ=1.

t	y_{rk45}	y_{GA}	y_{IPA}	$y_{GA\text{-}IPA}$	$\lvert y_{rk45} - y_{GA}\rvert$	$\lvert y_{rk45} - y_{IPA}\rvert$	$\lvert y_{rk45} - y_{GA\text{-}IPA}\rvert$
0.0	2.00000000	1.99856580	2.00000072	1.99999993	1.43420E-03	7.17921E-07	7.30998E-08
0.1	1.99093553	1.98940627	1.99093359	1.99093171	1.52927E-03	1.94811E-06	3.82916E-06
0.2	1.96696167	1.96551353	1.96695173	1.96694877	1.44814E-03	9.94267E-06	1.29049E-05
0.3	1.93182891	1.93056081	1.93183124	1.93182788	1.26810E-03	2.33301E-06	1.02753E-06
0.4	1.88817543	1.88693815	1.88817762	1.88817220	1.23728E-03	2.18679E-06	3.23425E-06
0.5	1.83771055	1.83629196	1.83771993	1.83771202	1.41859E-03	9.37805E-06	1.46847E-06
0.6	1.78154354	1.77977779	1.78155431	1.78154551	1.76575E-03	1.07683E-05	1.97494E-06
0.7	1.72030907	1.71816438	1.72032295	1.72031441	2.14470E-03	1.38730E-05	5.33362E-06
0.8	1.65432239	1.65188882	1.65433729	1.65432839	2.43356E-03	1.49057E-05	6.00259E-06
0.9	1.58364203	1.58109764	1.58365847	1.58364795	2.54439E-03	1.64444E-05	5.91987E-06
1.0	1.50812775	1.50568067	1.50814595	1.50813342	2.44709E-03	1.81985E-05	5.67248E-06
1.1	1.42746201	1.42529749	1.42748267	1.42746893	2.16452E-03	2.06526E-05	6.91085E-06
1.2	1.34116026	1.33939552	1.34118322	1.34116945	1.76474E-03	2.29633E-05	9.19626E-06
1.3	1.24856580	1.24721965	1.24858957	1.24857654	1.34615E-03	2.37758E-05	1.07466E-05
1.4	1.14883389	1.14781450	1.14885690	1.14884460	1.01939E-03	2.30133E-05	1.07071E-05
1.5	1.04090799	1.04002070	1.04093014	1.04091790	8.87291E-04	2.21462E-05	9.90988E-06
1.6	0.92348870	0.92246842	0.92351246	0.92349960	1.02028E-03	2.37601E-05	1.09004E-05
1.7	0.79500506	0.79357298	0.79503194	0.79501869	1.43208E-03	2.68760E-05	1.36262E-05
1.8	0.65359380	0.65154164	0.65362244	0.65361035	2.05216E-03	2.86396E-05	1.65572E-05
1.9	0.49712104	0.49440554	0.49714747	0.49713753	2.71550E-03	2.64291E-05	1.64903E-05
2.0	0.32327549	0.32009833	0.32330481	0.32329465	3.17716E-03	2.93224E-05	1.91554E-05

Table 4. Comparison of the results for μ=1 for derivative of the solution.

t	y'_{rk45}	y'_{GA}	y'_{IPA}	$y'_{GA\text{-}IPA}$	$\lvert y'_{rk45} - y'_{GA} \rvert$	$\lvert y'_{rk45} - y'_{IPA} \rvert$	$\lvert y'_{rk45} - y'_{GA\text{-}IPA} \rvert$
0.0	0.00000000	-0.00114873	-0.00000240	-0.00000020	1.14873E-03	2.40137E-06	1.98882E-07
0.1	-0.17265716	-0.17281555	-0.17266561	-0.17268455	1.58384E-04	8.44737E-06	2.73896E-05
0.2	-0.30074920	-0.29904554	-0.30072722	-0.30072955	1.70366E-03	2.19857E-05	1.96525E-05
0.3	-0.39741857	-0.39611663	-0.39739065	-0.39740167	1.30194E-03	2.79205E-05	1.68968E-05
0.4	-0.47287582	-0.47363006	-0.47284072	-0.47286800	7.54242E-04	3.51020E-05	7.82024E-06
0.5	-0.53454215	-0.53738298	-0.53451576	-0.53453420	2.84083E-03	2.63951E-05	7.95439E-06
0.6	-0.58776806	-0.59162772	-0.58774655	-0.58774689	3.85966E-03	2.15086E-05	2.11672E-05
0.7	-0.63638906	-0.63991873	-0.63637473	-0.63637272	3.52967E-03	1.43270E-05	1.63344E-05
0.8	-0.68325567	-0.68535387	-0.68324441	-0.68325480	2.09820E-03	1.12602E-05	8.66959E-07
0.9	-0.73058417	-0.73065788	-0.73057024	-0.73059055	7.37045E-05	1.39338E-05	6.37924E-06
1.0	-0.78023113	-0.77825195	-0.78020993	-0.78022751	1.97918E-03	2.12021E-05	3.62187E-06
1.1	-0.83388953	-0.83033409	-0.83386395	-0.83387001	3.55544E-03	2.55821E-05	1.95187E-05
1.2	-0.89323655	-0.88896419	-0.89321937	-0.89321462	4.27236E-03	1.71843E-05	2.19336E-05
1.3	-0.96005656	-0.95614478	-0.96005766	-0.96004903	3.91178E-03	1.09979E-06	7.53042E-06
1.4	-1.03633652	-1.03388937	-1.03634880	-1.03634411	2.44715E-03	1.22832E-05	7.58999E-06
1.5	-1.12434236	-1.12426871	-1.12434194	-1.12434539	7.36521E-05	4.23529E-07	3.02337E-06
1.6	-1.22665683	-1.22941887	-1.22662765	-1.22663506	2.76204E-03	2.91819E-05	2.17662E-05
1.7	-1.34612586	-1.35148256	-1.34609605	-1.34609380	5.35669E-03	2.98105E-05	3.20623E-05
1.8	-1.48566559	-1.49243817	-1.48567132	-1.48565076	6.77258E-03	5.73543E-06	1.48327E-05
1.9	-1.64768879	-1.65375241	-1.64770127	-1.64768621	6.06362E-03	1.24775E-05	2.58710E-06
2.0	-1.83301956	-1.83578618	-1.83295719	-1.83297212	2.76662E-03	6.23651E-05	4.74402E-05

Table 5. Comparison of the results for μ=3.

t	Y_{rk45}	y_{GA}	y_{IPA}	$y_{GA\text{-}IPA}$	$\lvert y_{rk45} - y_{GA} \rvert$	$\lvert y_{rk45} - y_{IPA} \rvert$	$\lvert y_{rk45} - y_{GA\text{-}IPA} \rvert$
0.0	2.00000000	2.00434418	1.99998880	1.99999897	4.34418E-03	1.12034E-05	1.02883E-06
0.1	1.99243120	1.99636319	1.99241680	1.99243075	3.93199E-03	1.44000E-05	4.52743E-07
0.2	1.97611798	1.97972162	1.97610013	1.97611804	3.60364E-03	1.78553E-05	5.75850E-08
0.3	1.95608236	1.95958659	1.95606618	1.95608255	3.50423E-03	1.61779E-05	1.84967E-07
0.4	1.93428980	1.93781173	1.93427735	1.93429248	3.52193E-03	1.24455E-05	2.68847E-06
0.5	1.91150856	1.91505031	1.91149929	1.91151411	3.54174E-03	9.27176E-06	5.55032E-06
0.6	1.88803750	1.89159194	1.88802615	1.88804027	3.55444E-03	1.13491E-05	2.77388E-06
0.7	1.86396902	1.86758765	1.86396110	1.86397434	3.61863E-03	7.91655E-06	5.32270E-06
0.8	1.83933722	1.84309758	1.83932736	1.83934081	3.76036E-03	9.86336E-06	3.59540E-06
0.9	1.81412506	1.81811209	1.81411302	1.81412869	3.98703E-03	1.20400E-05	3.62215E-06
1.0	1.78830451	1.79257404	1.78828923	1.78830873	4.26952E-03	1.52802E-05	4.21758E-06
1.1	1.76183618	1.76639976	1.76181702	1.76184051	4.56358E-03	1.91650E-05	4.32976E-06
1.2	1.73466993	1.73949493	1.73464937	1.73467514	4.82500E-03	2.05619E-05	5.20977E-06
1.3	1.70675175	1.71176484	1.70673119	1.70675610	5.01310E-03	2.05606E-05	4.35260E-06
1.4	1.67801469	1.68312028	1.67799820	1.67801888	5.10559E-03	1.64872E-05	4.18640E-06
1.5	1.64838677	1.65348065	1.64837542	1.64838970	5.09388E-03	1.13470E-05	2.92721E-06
1.6	1.61777986	1.62277568	1.61777525	1.61778342	4.99583E-03	4.60659E-06	3.55941E-06
1.7	1.58609666	1.59094668	1.58609547	1.58610069	4.85002E-03	1.19591E-06	4.02462E-06
1.8	1.55321849	1.55794779	1.55321704	1.55322447	4.72930E-03	1.44760E-06	5.97906E-06
1.9	1.51901036	1.52374780	1.51900192	1.51901623	4.73744E-03	8.44710E-06	5.86988E-06
2.0	1.48330742	1.48833231	1.48329080	1.48331234	5.02489E-03	1.66235E-05	4.91511E-06

Table 6. Comparison of the results for μ=3 for derivative of the solution.

t	y'_{rk45}	y'_{GA}	y'_{IPA}	$y'_{GA\text{-}IPA}$	$\lvert y'_{rk45} - y'_{GA} \rvert$	$\lvert y'_{rk45} - y'_{IPA} \rvert$	$\lvert y'_{rk45} - y'_{GA\text{-}IPA} \rvert$
0.0	0.00000000	-0.00483443	0.00002224	-0.00000039	4.83443E-03	2.22353E-05	3.93393E-07
0.1	-0.13195437	-0.13604061	-0.13198880	-0.13191467	4.08624E-03	3.44359E-05	3.96989E-05
0.2	-0.18667471	-0.18876531	-0.18665755	-0.18665661	2.09060E-03	1.71615E-05	1.81046E-05
0.3	-0.21106495	-0.21119335	-0.21102731	-0.21104691	1.28407E-04	3.76386E-05	1.80412E-05
0.4	-0.22359729	-0.22333476	-0.22357928	-0.22358445	2.62530E-04	1.80033E-05	1.28367E-05
0.5	-0.23153170	-0.23143873	-0.23153283	-0.23153664	9.29736E-05	1.12692E-06	4.94363E-06
0.6	-0.23779269	-0.23748202	-0.23777440	-0.23778406	3.10675E-04	1.82983E-05	8.62938E-06
0.7	-0.24348153	-0.24250673	-0.24349453	-0.24350011	9.74797E-04	1.30022E-05	1.85740E-05
0.8	-0.24918395	-0.24731452	-0.24920293	-0.24919136	1.86943E-03	1.89846E-05	7.41515E-06
0.9	-0.25509971	-0.25249411	-0.25513285	-0.25510097	2.60560E-03	3.31480E-05	1.26506E-06
1.0	-0.26136922	-0.25840905	-0.26140817	-0.26136592	2.96018E-03	3.89474E-05	3.30219E-06
1.1	-0.26808694	-0.26523543	-0.26811448	-0.26808008	2.85151E-03	2.75369E-05	6.86313E-06
1.2	-0.27532025	-0.27301991	-0.27533013	-0.27532148	2.30034E-03	9.88430E-06	1.22788E-06
1.3	-0.28316258	-0.28173103	-0.28314035	-0.28316664	1.43155E-03	2.22308E-05	4.05614E-06
1.4	-0.29169068	-0.29129483	-0.29164399	-0.29170043	3.95853E-04	4.66895E-05	9.74188E-06
1.5	-0.30102218	-0.30161510	-0.30095730	-0.30102463	5.92922E-04	6.48804E-05	2.45504E-06
1.6	-0.31127038	-0.31258223	-0.31121613	-0.31126595	1.31185E-03	5.42472E-05	4.42716E-06
1.7	-0.32259839	-0.32407458	-0.32257755	-0.32258305	1.47619E-03	2.08411E-05	1.53384E-05
1.8	-0.33518215	-0.33595553	-0.33522068	-0.33517158	7.73378E-04	3.85246E-05	1.05754E-05
1.9	-0.34926231	-0.34806870	-0.34934695	-0.34926537	1.19360E-03	8.46445E-05	3.06623E-06
2.0	-0.36512148	-0.36023313	-0.36517939	-0.36513147	4.88835E-03	5.79136E-05	9.99094E-06

best which represents the supremacy of GA-IPA algorithm on GA and IPA. In Table 6, the results of the derivative of the solution at *μ=3* are listed, which represents the deep strength of the proposed scheme, because the same chromosomes are used to find the results against the derivative as taken for the solution of the problem.

It is quite evident from the table that the value of the absolute error is in line with the range of *E-03* to *E-04, E-05* to *E-06* and *E-05* to *E-07* for GA, IPA and GA-IPA, respectively, from RK45. This describes the strength of GA-IPA on other mentioned solvers. By decreasing the period of oscillation and seeing the behavior of oscillation for a long span, the results can be achieved at a low accuracy. We can increases the accuracy by tuning the weights for a longer span but this will cost a large computational complexity in term of time and space. The reliability of the stochastic algorithm is being validated by a comprehensive statistical analysis. In which, 50 independent runs of the proposed methodology are carried with the parameter setting of Table 1. The references of the analysis are the mean, standard deviation (STD), the best and the worst values of the absolute error of proposed method from RK45. The best and the worst are considered as ones with minimum and maximum absolute error from $Y_{rk45.}$ The statistical mode of mean and STD determine the spreadness in the results. The results are provided in Table 7 for inputs between 0 to *2* with a step size of *0.5* for both of the non-stiff cases*.* The value of the fitness functions are computed for 50 independent runs to have a close look on the optimization behavior of various input times. The value of the functions e_j for GA, IPA and GA-IPA are plotted in the descending order for *μ=1* in the Figure 3 (a) and for *μ=3* in the Figure 3(b), respectively. The results are plotted on the semi-log scale as the difference between the results for various inputs times are almost negligible.

Van der pol stiff oscillator

We have discussed two cases in van der pol stiff oscillator. For a stiff problem, solutions can change on a time scale that is very short compared to the interval of integration, but the solution of interest changes on a much longer time scale. Methods not designed for stiff problems are ineffective on intervals where the solution changes slowly, because they use time steps small enough to resolve the fastest possible change. When the stiffness parameter μ is increased to 1000, the solution to the van der Pol equation changes dramatically and exhibits oscillation on a much longer time scale.

Approximating the solution of the initial value problem becomes a more difficult task. We have solved the problem by taking $\mu = 500$ and $\mu = 1000$ on the same pattern as done in the non-stiff case. The input of the training set is taken from time *t ϵ (0, 2)* with a step size of

Table 7. Statistical analysis of the solution and its derivative by proposed scheme.

		μ=1				μ=3			
T	Algorithm	Best	Worst	Mean	STD	Best	Worst	Mean	STD
	GA	1.43E-03	2.05E-01	6.25E-02	4.80E-02	3.05E-03	6.84E-02	2.26E-02	1.71E-02
0	IPA	7.31E-08	2.82E-03	1.06E-04	4.01E-04	9.25E-09	5.19E-05	7.24E-06	1.00E-05
	GA-IPA	2.11E-07	1.34E-04	2.49E-05	2.72E-05	1.32E-07	4.64E-04	5.21E-05	7.28E-05
	GA	1.42E-03	2.23E-01	6.86E-02	5.28E-02	1.33E-03	5.30E-02	1.81E-02	1.30E-02
0.5	IPA	4.11E-07	2.48E-03	1.13E-04	3.56E-04	1.56E-07	6.85E-05	1.43E-05	1.64E-05
	GA-IPA	3.63E-07	1.01E-04	2.69E-05	2.77E-05	2.74E-06	5.75E-04	6.81E-05	9.12E-05
	GA	2.45E-03	2.67E-01	8.49E-02	6.37E-02	1.97E-03	5.89E-02	2.08E-02	1.46E-02
1	IPA	1.86E-06	2.66E-03	1.29E-04	3.84E-04	1.55E-07	8.00E-05	1.48E-05	1.97E-05
	GA-IPA	1.07E-06	1.53E-04	4.39E-05	3.88E-05	1.39E-06	6.31E-04	8.31E-05	1.01E-04
	GA	1.88E-04	3.74E-01	1.24E-01	8.92E-02	2.68E-03	6.41E-02	2.30E-02	1.58E-02
1.5	IPA	3.18E-06	3.08E-03	1.77E-04	4.53E-04	4.16E-07	8.75E-05	2.14E-05	2.15E-05
	GA-IPA	1.35E-06	2.11E-04	5.73E-05	5.13E-05	2.16E-06	7.12E-04	9.07E-05	1.15E-04
	GA	1.53E-03	5.85E-01	1.98E-01	1.42E-01	2.09E-03	7.80E-02	2.71E-02	1.93E-02
2	IPA	2.09E-06	4.39E-03	2.68E-04	6.46E-04	2.01E-07	1.03E-04	2.48E-05	2.65E-05
	GA-IPA	2.38E-06	3.43E-04	9.94E-05	8.65E-05	6.49E-07	9.10E-04	1.16E-04	1.46E-04

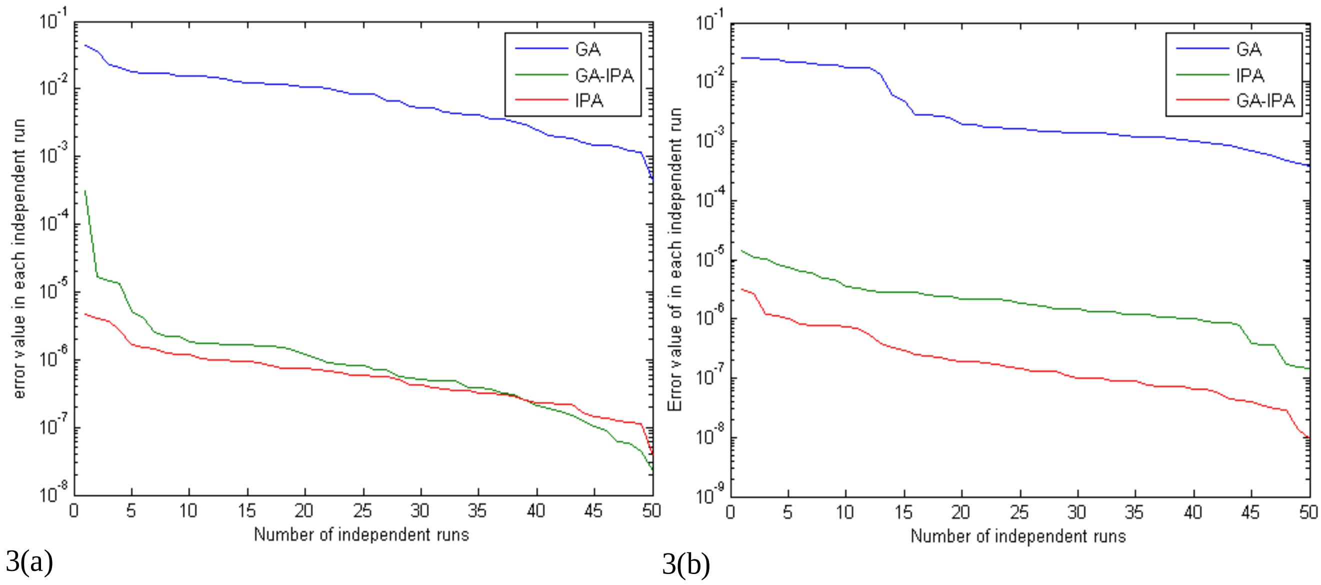


Figure 3. The behavior of the optimization error for 50 independent runs.

h=0.1. It means that the total time steps are *m = 20.* So the fitness function for stiff cases at $\mu = 500$ and $\mu = 1000$ are formulated below:

$$e_j = \frac{1}{21}\sum_{i=0}^{20}[\ddot{u}(t_i) + 500(u(t_i)^2 - 1)\dot{u}(t_i) + u(t_i)]^2 + \frac{1}{2}[\{u(0) - 2\}^2 + \{\dot{u}(0)\}^2]\Big|_j, \quad j = 1,2,3\ldots, \tag{15}$$

Table 8. Statistical analysis of the solution and its derivative by proposed scheme.

		μ=1				μ=3			
T	**Algorithm**	**Best**	**Worst**	**Mean**	**STD**	**Best**	**Worst**	**Mean**	**STD**
0	GA	3.26E-04	5.11E-02	1.93E-02	1.36E-02	5.88E-04	2.02E-01	5.13E-02	7.64E-02
	IPA	6.34E-08	6.60E-04	6.99E-05	1.58E-04	2.25E-08	4.65E-05	6.00E-06	9.05E-06
	GA-IPA	3.78E-07	1.34E-04	2.03E-05	2.57E-05	1.77E-07	2.89E-04	3.26E-05	5.23E-05
0.5	GA	1.04E-03	5.30E-02	1.80E-02	1.38E-02	9.30E-05	1.33E-02	4.45E-03	3.18E-03
	IPA	1.09E-07	1.61E-03	1.03E-04	2.40E-04	2.92E-07	1.64E-04	4.20E-05	3.60E-05
	GA-IPA	3.84E-07	2.60E-04	4.95E-05	5.35E-05	1.13E-06	3.45E-04	1.03E-04	6.95E-05
1	GA	1.80E-04	1.44E-01	5.24E-02	3.37E-02	1.25E-04	1.38E-02	6.15E-03	2.61E-03
	IPA	2.83E-06	7.29E-04	9.20E-05	1.28E-04	5.28E-07	1.38E-04	2.74E-05	3.37E-05
	GA-IPA	1.82E-06	4.14E-04	7.39E-05	9.12E-05	4.02E-06	4.91E-04	1.10E-04	8.77E-05
1.5	GA	7.37E-05	2.84E-01	1.04E-01	7.14E-02	7.57E-05	1.43E-02	4.03E-03	3.77E-03
	IPA	3.38E-07	9.25E-04	2.11E-04	2.37E-04	1.48E-06	2.09E-04	4.55E-05	4.42E-05
	GA-IPA	2.17E-07	4.39E-04	1.39E-04	1.00E-04	8.05E-06	3.04E-04	1.42E-04	6.92E-05
2	GA	1.69E-03	6.48E-01	2.26E-01	1.60E-01	2.89E-03	4.97E-02	1.93E-02	1.14E-02
	IPA	9.79E-06	1.74E-03	2.90E-04	3.53E-04	9.99E-06	1.71E-04	5.75E-05	3.43E-05
	GA-IPA	9.84E-06	6.31E-04	1.83E-04	1.26E-04	4.82E-05	6.22E-04	1.49E-04	9.20E-05

$$e_j = \frac{1}{21}\sum_{i=0}^{20}[\ddot{u}(t_i) + 1000\ (u(t_i)^2 - 1)\dot{u}(t_i) + u(t_i)]^2 + \frac{1}{2}[\{u(0)-2\}^2 + \{\dot{u}(0)\}^2]\Big|_j, \quad j = 1,2,3\dots, \tag{16}$$

Where j be the iteration index and $\ddot{u}$, $\dot{u}$ and u are the networks given in (3 to 5), respectively. One of the unknown weights learned by GA which provide the fitness values *1.82072E-04* and *8.6912E-05* for *μ=50*.

0 and *μ=1000,* respectively are provided in Table 9. One of the best chromosomes attained by GA-IPA is also given in Table 2. For this case the fitness values are *9.0870E-07* and *4.5396E-07* for *μ=500* and *μ=1000,* respectively. The adaptive weights are taken up to an accuracy of 10 decimal places to get a good accuracy and ease in the reproduction of the results. The solution of y_{GA} y_{IPA} and $y_{GA\text{-}IPA}$ are determined using the weight given in Table 9 for *μ=500* on the interval (0, 2) with step 0.1. The results are summarized in Table 10. In order to make the comparison, the results are also obtained for RK45 on the same ranges of the inputs. The results for the solution y_{rk45} are also provided in Table 10. It is clear from the results that the accuracy of the GA-IPA is better as compared to GA or IPA alone and is also in a good agreement with the state of art numerical solver. The absolute error of GA-IPA in comparison to RK45 is in the range of *E-05.*

The results are also provided for the derivative of the solution as well for the same input points by using the same weights as given in Table 9. The absolute of error of GA-IPA is less as compared to GA and IPA in comparison with RK45, which show the strength of the GA-IPA algorithm. The results of the derivative of the solution for the stiffness factor *500* are given in Table 11. There is always a tradeoff between the level of accuracy and the step size used in the approximation procedure. The effectiveness of the proposed scheme is also for *μ=1000* as well for inputs between *0* to *2* with a step size *0.1*. The adaptive parameters trained by GA and GA-IPA are also provided in Table 9. By using these weights the results of the problem along with absolute errors are given in Table 12 and its derivative in Table 13, respectively. The reliability of the stochastic algorithm is being tested by a comprehensive statistical analysis. In which, 50 independent runs of the proposed methodology is carried with the parameter setting of Table 1. The references of the analysis are the mean, STD, the best and the worst values of the absolute error of proposed method from RK45. The results are provided in Tables 14 and 15 for some of the inputs between 0 to *2* for both stiff cases*.* The value of the fitness functions are computed for 50 independent runs to have a close look on the optimization behavior of various input times. The value of the functions e_j for GA, IPA and GA-IPA are plotted in the descending order for *μ=500* in Figure 4 (a) and for *μ=1000* in the Figure 4(b), respectively. The results are plotted on the semi-log scale as the difference between the results for various inputs times are almost negligible. It is quite evident from Figure 4 that the optimization

Table 9. Adaptive parameters obtained by DE-NN networks.

	μ=500			*μ=1000*		
	w_i	α_i	β_i	w_i	α_i	β_i
GA	0.7527096645	0.1862053267	0.3654400314	0.1141257491	0.0377515785	2.9209935697
	1.0465559797	0.6891535682	0.0627992234	0.0807704982	0.1893975271	0.9725291549
	0.8485928003	0.2612432984	1.2466074493	0.0205415788	1.4307581392	1.8118541283
	-0.9765511877	0.2300192727	0.5044516489	0.1056166693	0.5601390582	1.8976221713
	1.1947150990	0.7024971819	-0.5194648590	0.1918623476	-0.5621821807	0.0981422911
	0.9323993611	1.6773505695	0.0039455975	0.0115658659	0.4274465969	1.0033073239
	0.8621008861	0.8824853562	0.3105447091	0.6222549241	0.0120562978	0.9176148698
	0.7033175039	-0.2449521067	-0.0391499721	0.2749566037	0.2120383320	0.0710802217
	0.2588161607	0.9063877837	1.4224897743	0.5639631544	0.0212104669	1.1938222781
	0.1211192555	0.3976889017	0.6369191533	0.5449252481	-0.0598190828	0.9822889301
GA-IPA	0.711285649	-3.65E-06	-0.20149591	0.3580028924	0.0317720676	1.7260250307
	0.645974777	0.004989814	0.889547460	0.4121950120	-0.2106818066	0.8249257600
	0.201191050	0.208202329	3.063171364	-0.0355134271	0.6303786326	1.3087086803
	0.292641281	0.062669691	1.674428041	-0.0245508798	0.4332772525	1.7438866640
	-0.018983625	1.168009758	1.853548662	0.2813287368	0.1761584824	2.3547165653
	0.118449060	0.257229237	1.312785952	0.0572577506	0.7137221600	2.0970560342
	0.376056629	0.094329233	0.240445049	0.4434120503	0.0965946045	0.5331224754
	0.483440301	0.042088139	2.173924004	0.6919841191	0.0080808876	0.5650787931
	0.287407257	-0.349835012	0.918172326	0.0169215383	0.4653219244	0.8384971612
	-3.8081236873	-3.7915947696	-2.4845407680	-1.9566612579	-0.1528119580	-3.3611685123

Table 9. Adaptive parameters obtained by DE-NN networks.

	μ=500			*μ=1000*		
	w_i	α_i	β_i	w_i	α_i	β_i
GA	0.7527096645	0.1862053267	0.3654400314	0.1141257491	0.0377515785	2.9209935697
	1.0465559797	0.6891535682	0.0627992234	0.0807704982	0.1893975271	0.9725291549
	0.8485928003	0.2612432984	1.2466074493	0.0205415788	1.4307581392	1.8118541283
	-0.9765511877	0.2300192727	0.5044516489	0.1056166693	0.5601390582	1.8976221713
	1.1947150990	0.7024971819	-0.5194648590	0.1918623476	-0.5621821807	0.0981422911
	0.9323993611	1.6773505695	0.0039455975	0.0115658659	0.4274465969	1.0033073239
	0.8621008861	0.8824853562	0.3105447091	0.6222549241	0.0120562978	0.9176148698
	0.7033175039	-0.2449521067	-0.0391499721	0.2749566037	0.2120383320	0.0710802217
	0.2588161607	0.9063877837	1.4224897743	0.5639631544	0.0212104669	1.1938222781
	0.1211192555	0.3976889017	0.6369191533	0.5449252481	-0.0598190828	0.9822889301
GA-IPA	0.711285649	-3.65E-06	-0.20149591	0.3580028924	0.0317720676	1.7260250307
	0.645974777	0.004989814	0.889547460	0.4121950120	-0.2106818066	0.8249257600
	0.201191050	0.208202329	3.063171364	-0.0355134271	0.6303786326	1.3087086803
	0.292641281	0.062669691	1.674428041	-0.0245508798	0.4332772525	1.7438866640
	-0.018983625	1.168009758	1.853548662	0.2813287368	0.1761584824	2.3547165653
	0.118449060	0.257229237	1.312785952	0.0572577506	0.7137221600	2.0970560342
	0.376056629	0.094329233	0.240445049	0.4434120503	0.0965946045	0.5331224754
	0.483440301	0.042088139	2.173924004	0.6919841191	0.0080808876	0.5650787931
	0.287407257	-0.349835012	0.918172326	0.0169215383	0.4653219244	0.8384971612
	-3.8081236873	-3.7915947696	-2.4845407680	-1.9566612579	-0.1528119580	-3.3611685123

Table 10. Comparison of the results for μ=500.

t	y_{rk45}	y_{GA}	y_{IPA}	$y_{GA\text{-}IPA}$	$\lvert y_{rk45} - y_{GA} \rvert$	$\lvert y_{rk45} - y_{IPA} \rvert$	$\lvert y_{rk45} - y_{GA\text{-}IPA} \rvert$
0.0	2.00000000	1.99984218	1.99995851	2.00002439	1.57822E-04	4.14895E-05	2.43853E-05
0.1	1.99986755	1.99970753	1.99982507	1.99989105	1.60022E-04	1.08488E-04	2.35058E-05
0.2	1.99973419	1.99957470	1.99969176	1.99975769	1.59494E-04	1.75125E-04	2.35010E-05
0.3	1.99960082	1.99944266	1.99955851	1.99962431	1.58160E-04	2.41697E-04	2.34906E-05
0.4	1.99946744	1.99931073	1.99942526	1.99949092	1.56708E-04	3.08262E-04	2.34801E-05
0.5	1.99933404	1.99917849	1.99929198	1.99935751	1.55545E-04	3.74863E-04	2.34732E-05
0.6	1.99920062	1.99904576	1.99915863	1.99922409	1.54864E-04	4.41526E-04	2.34715E-05
0.7	1.99906719	1.99891249	1.99902520	1.99909067	1.54701E-04	5.08262E-04	2.34749E-05
0.8	1.99893375	1.99877876	1.99889170	1.99895723	1.54992E-04	5.75073E-04	2.34826E-05
0.9	1.99880029	1.99864468	1.99875813	1.99882378	1.55614E-04	6.41944E-04	2.34929E-05
1.0	1.99866682	1.99851039	1.99862452	1.99869032	1.56424E-04	7.08855E-04	2.35040E-05
1.1	1.99853333	1.99837605	1.99849088	1.99855684	1.57280E-04	7.75782E-04	2.35137E-05
1.2	1.99839982	1.99824176	1.99835726	1.99842334	1.58064E-04	8.42698E-04	2.35208E-05
1.3	1.99826630	1.99810762	1.99822367	1.99828983	1.58687E-04	9.09577E-04	2.35239E-05
1.4	1.99813277	1.99797367	1.99809012	1.99815629	1.59096E-04	9.76402E-04	2.35224E-05
1.5	1.99799922	1.99783995	1.99795664	1.99802274	1.59272E-04	1.04316E-03	2.35167E-05
1.6	1.99786566	1.99770643	1.99782321	1.99788917	1.59225E-04	1.10987E-03	2.35077E-05
1.7	1.99773208	1.99757309	1.99768981	1.99775558	1.58992E-04	1.17655E-03	2.34976E-05
1.8	1.99759849	1.99743986	1.99755638	1.99762198	1.58632E-04	1.24325E-03	2.34895E-05
1.9	1.99746488	1.99730666	1.99742284	1.99748837	1.58220E-04	1.31005E-03	2.34879E-05
2.0	1.99733126	1.99717341	1.99728908	1.99735475	1.57842E-04	1.37706E-03	2.34988E-05

Table 11. Comparison of the results for μ=500 for derivative of the solution.

t	y'_{rk45}	y'_{GA}	y'_{IPA}	$y'_{GA\text{-}IPA}$	$\lvert y'_{rk45} - y'_{GA} \rvert$	$\lvert y'_{rk45} - y'_{IPA} \rvert$	$\lvert y'_{rk45} - y'_{GA\text{-}IPA} \rvert$
0.0	0.00000000	-0.00136007	-0.00133541	-0.0013331	1.36007E-03	1.33541E-03	1.33313E-03
0.1	-0.00133348	-0.00133536	-0.00133363	-0.0013335	1.87953E-06	6.66989E-04	8.33914E-09
0.2	-0.00133346	-0.00132291	-0.00133267	-0.0013337	1.05577E-05	6.65740E-04	2.53336E-07
0.3	-0.00133367	-0.00131895	-0.00133238	-0.0013339	1.47214E-05	6.65716E-04	2.17537E-07
0.4	-0.00133404	-0.00132037	-0.00133259	-0.001334	1.36723E-05	6.65688E-04	2.43795E-08
0.5	-0.00133431	-0.00132468	-0.00133314	-0.0013341	9.62744E-06	6.65636E-04	1.91463E-07
0.6	-0.00133446	-0.00133004	-0.00133388	-0.0013342	4.41220E-06	6.67052E-04	2.44744E-07
0.7	-0.00133445	-0.00133518	-0.00133467	-0.0013343	7.38352E-07	6.68337E-04	1.32772E-07
0.8	-0.00133440	-0.00133932	-0.00133540	-0.0013344	4.91504E-06	6.67580E-04	2.34490E-08
0.9	-0.00133442	-0.00134207	-0.00133595	-0.0013346	7.65320E-06	6.68354E-04	1.42371E-07
1.0	-0.00133456	-0.00134336	-0.00133628	-0.0013347	8.79654E-06	6.70292E-04	1.42726E-07
1.1	-0.00133499	-0.00134334	-0.00133633	-0.0013349	8.34725E-06	6.69460E-04	1.13899E-07
1.2	-0.00133546	-0.00134227	-0.00133612	-0.0013351	6.80594E-06	6.68772E-04	4.01235E-07
1.3	-0.00133590	-0.00134048	-0.00133569	-0.0013353	4.58471E-06	6.69043E-04	6.47798E-07
1.4	-0.00133626	-0.00133834	-0.00133512	-0.0013354	2.08392E-06	6.68094E-04	8.08925E-07
1.5	-0.00133645	-0.00133616	-0.00133453	-0.0013356	2.89374E-07	6.67283E-04	8.13191E-07
1.6	-0.00133662	-0.00133423	-0.00133411	-0.0013358	2.39036E-06	6.67025E-04	8.10821E-07
1.7	-0.00133663	-0.00133278	-0.00133407	-0.001336	3.84880E-06	6.66804E-04	6.77578E-07
1.8	-0.00133670	-0.00133201	-0.00133468	-0.0013361	4.68793E-06	6.67234E-04	6.34469E-07
1.9	-0.00133666	-0.00133205	-0.00133625	-0.0013361	4.60543E-06	6.68870E-04	5.35788E-07
2.0	-0.00133624	-0.00133302	-0.00133912	-0.0013361	3.21744E-06	6.71812E-04	1.23669E-07

Table 12. Comparison of the results for μ=1000.

t	Y_{rk45}	y_{GA}	y_{IPA}	$y_{GA\text{-}IPA}$	$\lvert y_{rk45} - y_{GA} \rvert$	$\lvert y_{rk45} - y_{IPA} \rvert$	$\lvert y_{rk45} - y_{GA\text{-}IPA} \rvert$
0.0	2.00000000	1.99926999	2.00000978	2.00000518	7.30006E-04	9.78240E-06	5.17519E-06
0.1	1.99993355	1.99920379	1.99994313	1.9999384	7.29759E-04	9.57434E-06	4.85084E-06
0.2	1.99986688	1.99913720	1.99987644	1.9998717	7.29680E-04	9.55856E-06	4.81811E-06
0.3	1.99980021	1.99907034	1.99980974	1.99980509	7.29868E-04	9.53405E-06	4.88522E-06
0.4	1.99973353	1.99900330	1.99974304	1.99973852	7.30223E-04	9.51277E-06	4.99255E-06
0.5	1.99966684	1.99893619	1.99967634	1.99967195	7.30657E-04	9.50065E-06	5.10669E-06
0.6	1.99960016	1.99886905	1.99960966	1.99960537	7.31101E-04	9.49959E-06	5.21360E-06
0.7	1.99953346	1.99880196	1.99954297	1.99953877	7.31503E-04	9.50753E-06	5.30689E-06
0.8	1.99946677	1.99873495	1.99947629	1.99947215	7.31825E-04	9.52066E-06	5.38364E-06
0.9	1.99940007	1.99866803	1.99940961	1.99940552	7.32040E-04	9.53639E-06	5.44418E-06
1.0	1.99933337	1.99860123	1.99934292	1.99933886	7.32137E-04	9.55100E-06	5.48806E-06
1.1	1.99926666	1.99853455	1.99927623	1.99927218	7.32118E-04	9.56026E-06	5.51385E-06
1.2	1.99919996	1.99846796	1.99920952	1.99920548	7.31993E-04	9.56331E-06	5.52245E-06
1.3	1.99913324	1.99840146	1.99914280	1.99913876	7.31782E-04	9.55983E-06	5.51412E-06
1.4	1.99906653	1.99833501	1.99907608	1.99907201	7.31516E-04	9.54975E-06	5.48807E-06
1.5	1.99899981	1.99826857	1.99900934	1.99900525	7.31231E-04	9.53538E-06	5.44472E-06
1.6	1.99893308	1.99820211	1.99894260	1.99893847	7.30970E-04	9.51971E-06	5.38413E-06
1.7	1.99886635	1.99813557	1.99887586	1.99887166	7.30781E-04	9.50616E-06	5.30606E-06
1.8	1.99879962	1.99806891	1.99880912	1.99880483	7.30717E-04	9.49900E-06	5.21061E-06
1.9	1.99873289	1.99800205	1.99874239	1.99873798	7.30832E-04	9.50261E-06	5.09786E-06
2.0	1.99866615	1.99793496	1.48329080	1.48331234	5.02489E-03	1.66235E-05	4.91511E-06

Table 13. Comparison of the results for μ=1000 for derivative of the solution.

t	y'_{rk45}	y'_{GA}	y'_{IPA}	$y'_{GA\text{-}IPA}$	$\lvert y'_{rk45} - y'_{GA} \rvert$	$\lvert y'_{rk45} - y'_{IPA} \rvert$	$\lvert y'_{rk45} - y'_{GA\text{-}IPA} \rvert$
0.0	0.00000000	-0.00065959	-0.0006664	-0.00066628	6.59590E-04	6.66394E-04	6.66280E-04
0.1	-0.00066664	-0.00066418	-0.0006677	-0.00066675	2.46378E-06	1.05766E-06	1.11533E-07
0.2	-0.00066693	-0.00066747	-0.0006665	-0.00066697	5.38633E-07	4.76676E-07	3.58056E-08
0.3	-0.00066666	-0.00066965	-0.0006658	-0.00066702	2.98585E-06	8.35089E-07	3.58831E-07
0.4	-0.00066690	-0.00067088	-0.0006657	-0.00066699	3.98450E-06	1.22880E-06	8.67310E-08
0.5	-0.00066750	-0.00067135	-0.0006657	-0.00066692	3.84431E-06	1.77045E-06	5.85267E-07
0.6	-0.00066683	-0.00067120	-0.0006659	-0.00066685	4.37068E-06	9.44242E-07	2.37900E-08
0.7	-0.00066634	-0.00067059	-0.0006661	-0.00066681	4.25805E-06	2.63326E-07	4.77632E-07
0.8	-0.00066782	-0.00066967	-0.0006663	-0.00066681	1.85809E-06	1.54198E-06	1.00542E-06
0.9	-0.00066760	-0.00066857	-0.0006665	-0.00066685	9.76392E-07	1.11761E-06	7.52485E-07
1.0	-0.00066598	-0.00066742	-0.0006667	-0.00066692	1.43605E-06	7.05305E-07	9.30389E-07
1.1	-0.00066687	-0.00066633	-0.0006669	-0.00066701	5.47822E-07	2.69638E-08	1.35358E-07
1.2	-0.00066735	-0.00066539	-0.0006671	-0.00066711	1.96129E-06	2.41838E-07	2.38199E-07
1.3	-0.00066665	-0.00066471	-0.0006673	-0.00066722	1.93889E-06	6.72731E-07	5.70845E-07
1.4	-0.00066702	-0.00066436	-0.0006675	-0.00066731	2.66365E-06	5.07492E-07	2.86816E-07
1.5	-0.00066725	-0.00066441	-0.0006677	-0.00066738	2.83865E-06	4.90637E-07	1.26740E-07
1.6	-0.00066709	-0.00066492	-0.000668	-0.00066741	2.16293E-06	8.65748E-07	3.25190E-07
1.7	-0.00066727	-0.00066594	-0.0006682	-0.00066741	1.32717E-06	8.96002E-07	1.39082E-07
1.8	-0.00066745	-0.00066750	-0.0006684	-0.00066736	5.17514E-08	9.29288E-07	8.68283E-08
1.9	-0.00066738	-0.00066962	-0.0006686	-0.00066727	2.24332E-06	1.21053E-06	1.07414E-07
2.0	-0.00066731	-0.00067232	-0.0006688	-0.00066714	5.00761E-06	1.48550E-06	1.73305E-07

Table 14. Statistical analysis of the solution by proposed scheme.

		μ=500				*μ=1000*			
t	**Algorithm**	**Best**	**Worst**	**Mean**	**STD**	**Best**	**Worst**	**Mean**	**STD**
0	GA	1.58E-04	1.07E+00	1.33E-01	1.84E-01	7.30E-04	1.05E+00	4.24E-01	4.06E-01
	IPA	4.15E-05	3.06E+00	1.21E-01	9.79E-01	9.78E-06	1.05E+00	2.30E-01	4.38E-01
	GA-IPA	1.10E-05	3.06E+00	1.44E-02	9.79E-01	5.18E-06	4.80E+01	2.90E-02	7.72E+00
0.5	GA	1.56E-04	1.06E+00	1.33E-01	1.83E-01	3.98E-04	1.04E+00	4.23E-01	4.04E-01
	IPA	3.75E-04	3.05E+00	1.01E-01	9.77E-01	9.50E-06	1.04E+00	2.29E-01	4.35E-01
	GA-IPA	4.21E-05	3.05E+00	1.47E-02	9.77E-01	5.11E-06	4.80E+01	2.90E-02	7.72E+00
1	GA	1.56E-04	1.05E+00	1.32E-01	1.82E-01	6.56E-05	1.04E+00	4.22E-01	4.03E-01
	IPA	7.09E-04	3.04E+00	1.02E-01	9.75E-01	9.55E-06	1.03E+00	2.28E-01	4.32E-01
	GA-IPA	4.23E-05	3.04E+00	1.43E-02	9.75E-01	5.49E-06	4.80E+01	2.89E-02	7.72E+00
1.5	GA	1.59E-04	1.04E+00	1.92E-01	1.81E-01	2.69E-04	1.03E+00	4.20E-01	4.00E-01
	IPA	1.04E-03	3.03E+00	1.42E-01	9.72E-01	9.54E-06	1.02E+00	2.26E-01	4.29E-01
	GA-IPA	4.26E-05	3.03E+00	1.21E-02	9.72E-01	5.44E-06	4.80E+01	2.89E-02	7.72E+00
2	GA	1.58E-04	1.02E+00	1.32E-01	1.79E-01	3.63E-05	1.02E+00	4.18E-01	3.98E-01
	IPA	1.38E-03	3.00E+00	1.41E-01	9.66E-01	9.52E-06	1.01E+00	2.22E-01	4.22E-01
	GA-IPA	4.22E-05	3.00E+00	1.12E-02	9.66E-01	4.97E-06	4.80E+01	2.88E-02	7.72E+00

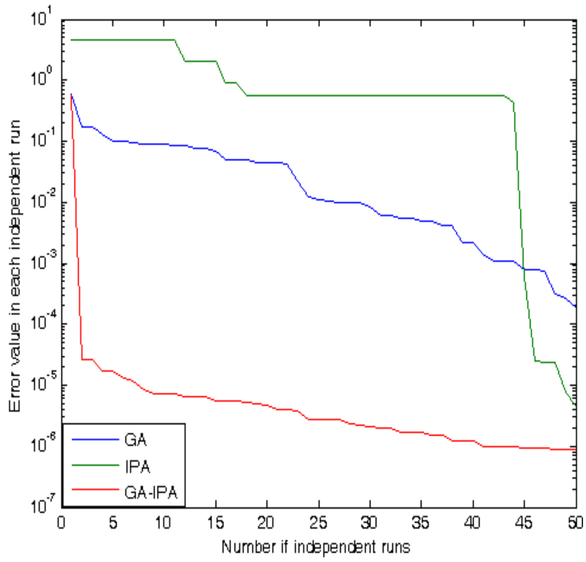
Table 15. Statistical analysis of the derivative of the solution by proposed scheme.

		μ=500				*μ=1000*			
t	**Algorithm**	**Best**	**Worst**	**Mean**	**STD**	**Best**	**Worst**	**Mean**	**STD**
0	GA	1.13E-03	1.69E-02	1.69E-03	2.21E-03	5.43E-04	1.43E-02	3.79E-03	4.83E-03
	IPA	2.89E-06	6.69E-01	3.84E-02	1.29E-01	6.63E-04	1.07E-02	2.85E-03	4.16E-03
	GA-IPA	2.43E-06	6.69E-01	3.84E-02	1.29E-01	5.08E-08	7.43E-01	2.21E-02	1.04E-01
0.5	GA	2.26E-04	1.96E-02	1.44E-02	7.01E-03	2.40E-04	1.46E-02	3.65E-03	5.10E-03
	IPA	9.12E-06	1.60E-02	5.52E-04	2.26E-03	4.15E-08	1.28E-02	2.81E-03	5.34E-03
	GA-IPA	1.17E-06	2.03E-02	1.48E-02	7.33E-03	4.30E-07	1.40E-02	8.90E-03	5.77E-03
1	GA	2.02E-04	2.42E-02	1.70E-02	8.45E-03	1.57E-06	1.54E-02	4.34E-03	6.24E-03
	IPA	1.11E-06	2.12E-02	5.65E-04	2.99E-03	5.94E-08	1.52E-02	3.33E-03	6.33E-03
	GA-IPA	3.95E-07	2.48E-02	1.75E-02	8.79E-03	7.26E-08	1.69E-02	1.07E-02	6.94E-03
1.5	GA	3.24E-05	3.50E-02	2.31E-02	1.18E-02	7.40E-05	2.32E-02	5.66E-03	8.44E-03
	IPA	2.89E-07	3.02E-02	7.78E-04	4.25E-03	5.55E-09	2.04E-02	4.41E-03	8.39E-03
	GA-IPA	6.60E-07	3.57E-02	2.35E-02	1.21E-02	1.06E-07	2.40E-02	1.39E-02	9.31E-03
2	GA	6.43E-04	1.42E-01	6.28E-02	3.83E-02	1.22E-04	3.24E-02	7.10E-03	1.08E-02
	IPA	3.22E-06	3.88E-02	1.03E-03	5.46E-03	1.16E-08	5.91E-02	1.11E-02	2.12E-02
	GA-IPA	2.88E-06	1.42E-01	6.32E-02	3.86E-02	5.31E-07	9.52E-02	3.27E-02	2.61E-02

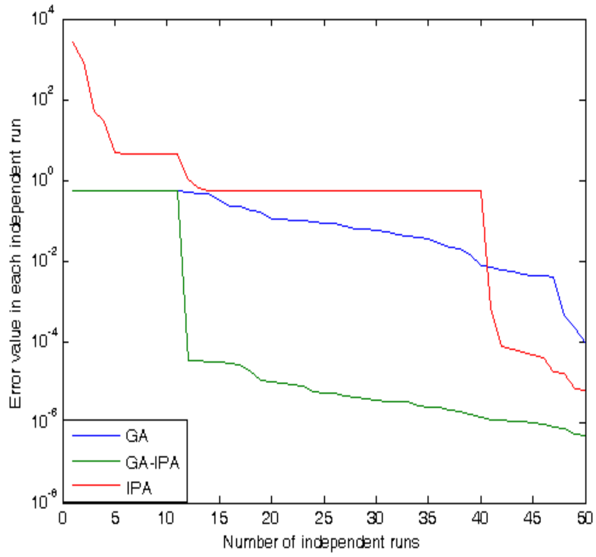
behavior for the approximation of the results for GA-IPA is more precise as compared to GA and IPA.

Conclusions

An alternate stochastic in nature method, has been used for numerical treatment of van der pol oscillator. It has been presented for both cases of stiff and non-stiff conditions. The accuracy of the proposed method is found to be in good agreement with the standard state of art numerical solver RK45 method. A large number of Monte Carlo simulations are performed to validate there liability and effectiveness of the proposed scheme. The



4(a)



4(b)

Figure 4. The behavior of the optimization error for 50 independent runs.

strength of designed scheme over classical numerical methods is that, it can provide the results on continuous time finite domain of inputs instead of predefined discrete grid of points. Moreover, the given scheme provides the simplicity of concept, ease in implementation and wider application domain in engineering.

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