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*Full Length Research Paper*

# Modified variation of parameters method for solving system of second-order nonlinear boundary value problem

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**In this paper, we use the modified variation of parameters method, which is an elegant coupling of variation of parameters method and Adomian's decomposition method, for solving the solution system of nonlinear boundary value problems associated with obstacle, contact and unilateral problems. The results are calculated in terms of series with easily computable components. The suggested method is applied without any discretization, transformation and restrictive assumptions. To illustrate the implementation and efficiency of the proposed method, an example is given.**

**Key words:** Variation of parameters method, Adomian's polynomials, system of nonlinear boundary value problems.

## INTRODUCTION

In this paper, we consider the following system of second-order boundary value problem:

$$u'' = \begin{cases} f(x,u(x)) & a \le x < c \\ f(x,u(x)) + u(x)\,g(x) + r & c \le x < d \\ f(x,u(x)) & d \le x \le b \end{cases} \qquad (1)$$

with boundary conditions $u(a)=\alpha_1,\ u(b)=\alpha_2$ and continuity conditions of $u(x)$ and $u'(x)$ at internal points $c$ and $d$ of the interval $[a,b]$. Here $r$, $\alpha_1$ and $\alpha_2$ are real and finite constants and $g(x)$ is a continuous function and $f(x,u(x)) = f(u)$ is a nonlinear function. If $f(u) = f$, a linear function, then this problem had been studied extensively. To the best of our knowledge, problem (1) has not been investigated. Such type of problems arise in the study of obstacle, contact, unilateral and equilibrium problems arising in economics, transportation, nonlinear optimization, oceanography, ocean wave engineering, fluid flow through porous media, some other branches of pure and applied sciences and engineering (Al-Said et al.,1995, 2002, 2004).

In this paper, we consider the case where f in the system (1) is highly nonlinear with arbitrary choices of $r$ and $g(x)$.

Therefore, we have to use some powerful analytic or numerical technique for obtaining the approximate solution of system (1). Several techniques have been used to solve system of boundary value problems associated with obstacle problems. Noor et al. (1986) applied finite difference method for unilateral problems; Al-Said et al. (2004) used finite difference method for obstacle problems; Khalifa et al. (1990) applied quintic spline method for contact problems; Noor et al. (1994) applied quartic spline method for obstacle problems; Al-Said et al. (2002) used quartic spline method for obstacle problems, and Momani et al. (2006) used decomposition

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method for system of forth-order obstacle problems. These numerical methods provide discrete point solution. The most noticeable fact which is necessary to mention here that all these methods are proposed to solve linear system of boundary value problems associated with obstacle, unilateral and contact problems but less attention is drawn towards nonlinear system of boundary value problems. We use modified variation of parameters method to solve system of nonlinear boundary value problems associated with obstacle problems. Noor et al. (2008) have used variation of parameters method for solving a wide classes of higher orders initial and boundary value problems. Ma et al. (2004, 2008) have applied variation of parameters method for solving some non homogenous partial differential equations. Ramos (2008) had used this technique to find the frequency of some nonlinear oscillators. Ramos (2008) has also shown the equivalence of this technique with variational iteration method. It is further investigated by Noor et al. (2008) that proposed technique is totally different from variation iteration method in many aspects. The multiplier used in variation of parameters method is obtained by Wronskian technique and is totally different from Lagrange multiplier of variation of parameters method. The variation of parameters method had reduced lot of computational work involved due to this term as compared to some other existing techniques using this term which is clear advantage of proposed technique over them.

We now consider the modified variation of parameters method, which is an elegant combination of variation of parameters method and Adomian's decomposition method. It turned out that the modified variation of parameters method is very flexible. We would like to mention that the modified variation of parameters method has the efficiency of both the techniques. The use of multiplier and Adomian's polynomial together in the modified variation of parameters method increases the rate of convergence by reducing the number of iterations and successive application of integral operators. This technique makes the solution procedure simple while still maintaining the higher level of accuracy. In the present study, we implement this technique for solving a system of second-order nonlinear boundary value problems associated with obstacle, unilateral and contact problems. It is well-known that the obstacle problems can be studied via the variational inequalities (Noor, 1994, 2004, 2009, 2009a; Noor et al., 1993). It is an interesting and open problem to extend this technique for solving the variational inequalities. This may lead to further research in this field and related optimization problems. The interested readers are advised to discover novel and innovative applications for this technique.

## MODIFIED VARIATION OF PARAMETERS METHOD

To convey the basic concept of the variation of parameter method for differential equations, we consider the general differential equation in operator form.

$$Lu(x)+Ru(x)+Nu(x)=g(x), \tag{2}$$

where $L$ is a higher order linear partial operator with respect to time , $R$ is a linear partial operator of order less than $L$, $N$ is a nonlinear partial operator and $g$ is a source term.

By using the variation of parameters method as developed in Mohyud-Din et al. (2009, 2011), we have the following general solution of Equation (2) as

$$u(x)=\sum_{i=0}^{n-1}\frac{B_i x^i}{i!}+\int_0^x \lambda(x,s)\left(-Nu(s)-Ru(s)+g(s)\right)ds, \tag{3}$$

where $n$ is a order of given differential equation and $B_i$'$s$ are unknowns which are further determined by initial/boundary conditions. $\lambda(x,s)$ is multiplier, it is obtained with the help of Wronskian technique used in this method. This multiplier removes the successive application of integrals in iterative scheme and it depends upon the order of equation formula. Mohyud-Din et al. (2009, 2011), have obtained the following for finding the multiplier $\lambda(x,s)$ as

$$\lambda(x,s)=\sum_{i=1}^{n}\frac{s^{i-1}x^{n-i}(-1)^{i-1}}{(i-1)!(n-i)!}. \tag{4}$$

For different choices of $n$, one can obtain the following values of $\lambda$

$$n=1, \qquad \lambda(x,s)=1,$$
$$n=2, \qquad \lambda(x,s)=x-s,$$
$$n=3, \qquad \lambda(x,s)=\frac{x^2}{2!}-sx+\frac{s^2}{2!},$$
$$n=4, \qquad \lambda(x,s)=\frac{x^3}{3!}-\frac{sx^2}{2!}+\frac{s^2x}{2!}-\frac{s^3}{3!},$$

From (3), we have the following iterative scheme for solving the Equation (2).

$$u_{k+1}(x)=u_k(x)+\int_0^x \lambda(x,s)\left(-Nu_k(s)-Ru_k(s)+g(s)\right)ds \qquad k=0,1,2,\ldots \tag{5}$$

It is observed that the fix value of initial guess in each iteration provides the better approximation, that is $u_k(x)=u_0(x), \text{ for } k=1,2,\cdots$. However, we can modify the initial guess by dividing $u_0(x)$ in two parts and using one of them as initial guess. It is a more convenient way in case of more than two terms in $u_0(x)$. In modified variation of parameters method, we define the solution $u(x)$ by the following series $u(x)=\sum_{k=0}^{\infty}u_k(x),$ and the nonlinear terms are decomposed by infinite number of polynomials as follows

$$N(u)=\sum_{k=0}^{\infty}A_k(u_0,u_1,u_2,...,u_i),$$

where $u$ is a function of $x$ and $A_k$ are the so-called Adomian's polynomials. These polynomials can be generated for various classes of nonlinearities by specific algorithm developed in Wazwaz (2000) as follows

$$A_k=\left(\frac{1}{k!}\right)\left(\frac{d^k}{d\lambda^k}\right)N\left(\sum_{i=0}^{n}\left(\lambda^i u_i\right)\right)_{\lambda=0}, \qquad k=0,1,2,\cdots.$$

Hence, we have the following iterative scheme for finding the approximate solution of (2) as

$$u_{k+1}(x)=u_k(x)+\int_0^x\lambda(x,s)\left(-A_k-Ru_k(s)+g(s)\right)ds. \tag{6}$$

We would like to emphasize that the modified variation of parameters method (MVPM) for solving system of second-order nonlinear boundary value problems may be viewed as an important and significant refinement of the previous known methods.

## APPLICATIONS AND NUMERICAL RESULTS

Example 1

Consider following system of second-order nonlinear boundary value problem:

$$u''=\begin{cases}\frac{u^3}{3!}+\frac{u^2}{2!}+u+1, & \text{for } -1\le x<-\frac{1}{2} \text{ and } \frac{1}{2}\le x\le 1\\ \frac{u^3}{3!}+\frac{u^2}{2!}+2u, & \text{for } -\frac{1}{2}\le x<\frac{1}{2},\end{cases} \tag{7}$$

with boundary conditions $u(-1)=u(1)=0.$

We use the modified variation of parameters method for solving system of second-order nonlinear boundary value problems (1). By using the modified variation of parameters method, we have following iterative scheme to solve nonlinear system (7):

$$u_{k+1}(x)=\begin{cases}u_k(x)+\int_0^x\lambda(x,s)\left(A_k+u_k+1\right)ds, & \text{for } -1\le x<-\frac{1}{2} \text{ and } \frac{1}{2}\le x\le 1\\ u_k(x)+\int_0^x\lambda(x,s)\left(A_k+2u_k\right)ds, & \text{for } -\frac{1}{2}\le x<\frac{1}{2},\end{cases}$$

Using $\lambda(x,s)=x-s,$ since the governing equation is of second-order,

$$u_{k+1}(x)=\begin{cases}u_k(x)+\int_0^x(x-s)\left(A_k+u_k+1\right)ds, & \text{for } -1\le x<-\frac{1}{2} \text{ and } \frac{1}{2}\le x\le 1\\ u_k(x)+\int_0^x(x-s)\left(A_k+2u_k\right)ds, & \text{for } -\frac{1}{2}\le x<\frac{1}{2}.\end{cases}$$

Case 1: $-1\le x<-\frac{1}{2}.$

In this case, we implement the modified variation of parameters method as follows by taking the initial value: $u_0=c_1x+c_2.$ and obtain further iterations as follows:

$$u_1(x)=c_2+\int_0^x(x-s)\left(A_0+u_0+1\right)ds,$$

$$u_1(x)=c_2+\frac{1}{2}x^2+\frac{1}{6}x^3c_1+\frac{1}{24}x^4c_1^{\ 2}+\frac{1}{120}x^5c_1^{\ 3}.$$

$$u_{k+2}(x)=\int_0^x(x-s)\left(A_{k+1}+u_{k+1}+1\right)ds,$$

for $k=0,1,2,\ldots.$

$$u_2(x)=\frac{1}{2}c_2x^2+\frac{1}{6}c_2c_1x^3+\left(\frac{1}{24}c_1^{\ 2}c_2+\frac{1}{24}\right)x^4+\frac{1}{30}c_1x^5+\frac{11}{720}c_1^{\ 2}x^6+\frac{1}{315}c_1^{\ 3}x^7$$
$$+\frac{1}{1920}c_1^{\ 4}x^8+\frac{1}{17280}c_1^{\ 5}x^9,$$

$\vdots$

Case 2: $-\frac{1}{2}\le x<\frac{1}{2}.$

In this case, we have following approximations

$$u_0=c_3x+c_4,$$

$$u_1(x)=c_4+\int_0^x(x-s)\left(A_0+2u_0\right)ds,$$

$$u_1(x) = c_4 + \frac{1}{3}x^3 c_3 + \frac{1}{24}x^4 c_3{}^2 + \frac{1}{120}x^5 c_3{}^3.$$

$$u_{k+2}(x) = \int_0^x (x-s)(A_{k+1} + 2u_{k+1})ds, \text{ for } k = 0,1,2,\ldots.$$

$$u_2(x) = c_4 x^2 + \frac{1}{6}c_3 c_4 x^3 + \frac{1}{24}c_3{}^2 c_4 x^4 + \frac{1}{30}c_3 x^5 + \frac{1}{72}c_3{}^2 x^6 + \frac{3}{560}c_3{}^3 x^7 + \frac{1}{1920}c_3{}^4 x^8 + \frac{1}{17280}c_3{}^5 x^9,$$

$$\vdots$$

Case 3: $\frac{1}{2} \le x \le 1$.

In this case, we proceed as follows

$$u_0 = c_5 x + c_6,$$

$$u_1(x) = c_6 + \int_0^x (x-s)(A_0 + u_0 + 1)ds,$$

$$u_1(x) = c_2 + \frac{1}{2}x^2 + \frac{1}{6}x^3 c_1 + \frac{1}{24}x^4 c_1{}^2 + \frac{1}{120}x^5 c_1{}^3.$$

$$u_{k+2}(x) = \int_0^x (x-s)(A_{k+1} + u_{k+1} + 1)ds,$$

for $k = 0,1,2,\ldots.$

$$u_2(x) = \frac{1}{2}c_6 x^2 + \frac{1}{6}c_6 c_5 x^3 + \left(\frac{1}{24}c_5{}^2 c_6 + \frac{1}{24}\right)x^4 + \frac{1}{30}c_5 x^5 + \frac{11}{720}c_5{}^2 x^6 + \frac{1}{315}c_5{}^3 x^7 + \frac{1}{1920}c_5{}^4 x^8 + \frac{1}{17280}c_5{}^5 x^9,$$

$$\vdots$$

By using the modified variation of parameters method, we have following formula for getting series solution in the whole domain from the above cases

$$u(x) = \begin{cases} \sum_{k=0}^{\infty} u_k(x) & \text{for} -1 \le x \le -\frac{1}{2}, \\ \sum_{k=0}^{\infty} u_k(x) & \text{for } -\frac{1}{2} \le x \le \frac{1}{2}, \\ \sum_{k=0}^{\infty} u_k(x) & \text{for } \frac{1}{2} \le x \le 1. \end{cases}$$

Hence, we have the following series solution after two iterations

$$u(x) = \begin{cases} c_2 + c_1 x + \left(\frac{1}{2}c_2 + \frac{1}{2}\right)x^2 + \left(\frac{1}{6}c_1 + \frac{1}{6}c_2 c_1\right)x^3 + \left(\frac{1}{24}c_1{}^2 c_2 + \frac{1}{24}c_1{}^2 + \frac{1}{24}\right)x^4 + \left(\frac{1}{30}c_1 + \frac{1}{120}c_1{}^3\right)x^5 \\ + \frac{11}{720}c_1{}^2 x^6 + \frac{1}{315}c_1{}^3 x^7 + \frac{1}{1920}c_1{}^4 x^8 + \frac{1}{17280}c_1{}^5 x^9, \quad \text{for } -1 \le x < -\frac{1}{2}, \\ c_4 + c_3 x + c_4 x^2 + \left(\frac{1}{3}c_3 + \frac{1}{6}c_3 c_4\right)x^3 + \left(\frac{1}{24}c_3{}^2 c_4 + \frac{1}{24}c_3{}^2\right)x^4 + \left(\frac{1}{30}c_3 + \frac{1}{120}c_3{}^3\right)x^5 \\ + \frac{1}{72}c_3{}^2 x^6 + \frac{3}{560}c_3{}^3 x^7 + \frac{1}{1920}c_3{}^4 x^8 + \frac{1}{17280}c_3{}^5 x^9, \quad \text{for } -\frac{1}{2} \le x < \frac{1}{2}, \\ c_6 + c_5 x + \left(\frac{1}{2}c_6 + \frac{1}{2}\right)x^2 + \left(\frac{1}{6}c_5 + \frac{1}{6}c_6 c_5\right)x^3 + \left(\frac{1}{24}c_5{}^2 c_6 + \frac{1}{24}c_5{}^2 + \frac{1}{24}\right)x^4 + \left(\frac{1}{30}c_5 + \frac{1}{120}c_5{}^3\right)x^5 \\ + \frac{11}{720}c_5{}^2 x^6 + \frac{1}{315}c_5{}^3 x^7 + \frac{1}{1920}c_5{}^4 x^8 + \frac{1}{17280}c_5{}^5 x^9, \quad \text{for } \frac{1}{2} \le x \le 1. \end{cases} \tag{8}$$

Now we will use boundary conditions and continuity conditions at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$. Hence we have the following system of nonlinear equations

$$-\frac{6}{5}c_1 + \frac{3}{2}c_2 + \frac{41}{720}c_1{}^2 - \frac{29}{2520}c_1{}^3 + \frac{1}{1920}c_1{}^4 - \frac{1}{17280}c_1{}^5 - \frac{1}{6}c_2 c_1 + \frac{1}{24}c_2 c_1{}^2 + \frac{13}{24} = 0,$$

$$.04166666667c_5{}^2 c_6 + .00005787037031c_5{}^5 + 1.500000000c_6 + 1.2c_5 + 1.5c_6 + .05694444445c_5{}^2 + .0005208333334c_5{}^4 + .01150793651c_5{}^3 + .1666666667c_5 c_6 + .5416666667 = 0,$$

$$.5427083333c_3 - 1.250000000c_4 - .002821180554c_3{}^2 + .0003022693461c_3{}^3 + .02083333333c_3 c_4 - .002604166670c_3{}^2 c_4 + 4.069010416\times10^{6} c_3{}^4 - 9.042245370\times10^{7} c_3{}^5 + 2.034505209\times10^{6} c_1{}^4 - 1.130280672\times10^{7} c_1{}^5 - .0002852182544c_1{}^3 - .02083333333c_1 c_2 + .002842881948c_1{}^2 + .002604166670c_1{}^2 c_2 - .5218749999c_1 + 2.034505209\times10^{6} c_1{}^4 - 9.042245370\times10^{7} c_3{}^5 + .1276041667 + 1.125000000c_2 = 0,$$

$$-.1276041667 - .5218749999c_5 - 1.125000000c_6 - .02083333333c_5 c_6 + .002604166670c_3{}^2 c_4 - .002604166670c_5{}^2 c_6 - .002842881948c_5{}^2 - .0002852182542c_5{}^3 - 2.034505209\times10^{6} c_5{}^4 - 1.130280672\times10^{7} c_5{}^5 + 1.130280672\times10^{7} c_3{}^5 + 2.034505209\times10^{6} c_3{}^4 + .0003022693461c_3{}^3 + .5427083333c_3 + .002821180554c_3{}^2 + 1.250000000c_4 + .02083333333c_3 c_4 = 0,$$

$$-.1250000000c_3 c_4 - .02083333333c_1{}^2 c_2 - .003190104169c_3{}^3 + .02343749999c_3{}^2 + 1.135416667c_1 + .1250000000c_1 c_2 + .02083333333c_3{}^2 c_4 - .5000000000c_2 - .02369791666c_1{}^2 - .5208333330 + .002951388889c_1{}^3 - 3.255208329\times10^{5} c_1{}^5 + 2.034505208\times10^{6} c_1{}^5 - 1.260416667c_3 + c_4 + 3.255208329\times10^{5} c_3{}^4 - 2.034505208\times10^{6} c_3{}^5 = 0,$$

$$.1250000000c_3 c_4 + .003190104169c_3{}^3 + .02343749999c_3{}^2 + .02083333333c_3{}^2 c_4 + 1.260416667c_3 + c_4 + 3.255208329\times10^{-5} c_3{}^4 + 2.034505208\times10^{-6} c_3{}^5 - 1.135416667c_5 - .5000000000c_6 - .02369791666c_5{}^2 - 2.951388892\times10^{-3} c_5{}^3 - 3.255208329\times10^{-5} c_5{}^4 - 2.034505208\times10^{-6} c_5{}^5 - .1250000000c_5 c_6 - .02083333333c_5{}^2 c_6 - .5208333333 = 0. \tag{9}$$

By using Newton's method for system of nonlinear equations (9), we have following values for unknown constants:

$$c_1 = .5472307373, \quad c_2 = .0702184452, \quad c_3 = 0, \quad c_4 = -.0631417028, \quad c_5 = -.5472307373, \quad c_6 = .0702184453 \tag{10}$$

By using values of unknowns from (10) into (8), we have

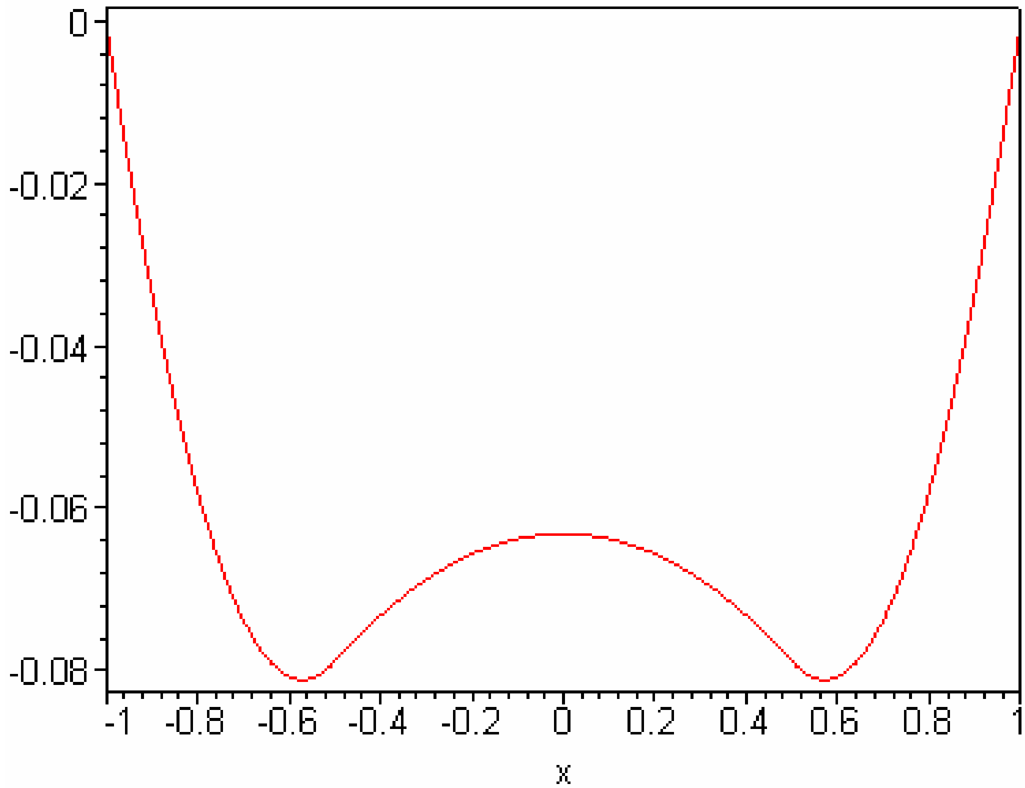


**Figure 1.** Graphical representation of analytical solution of system of second-order nonlinear boundary value problem (7) by using modified variation of parameters method.

following analytic solution of system of second-order nonlinear boundary value problem associated with obstacle problem (7). See Figure 1 for the graphical representation of the solution.

$$u(x)=\begin{cases} .0702184452+.5472307373\,x+.5351092226\,x^2+.09760940481\,x^3 \\ +.05502038331\,x^4+.01960664563\,x^5+.004575105941\,x^6+.0005202365917\,x^7 \\ +4670686349\times10^{-5}\,x^8+2.839936817\times10^{-6}\,x^9, & \text{for}\quad -1\le x<-\frac{1}{2} \\ -.0631417028-.0631417028\,x^2, & \text{for}\quad -\frac{1}{2}\le x<\frac{1}{2} \\ .0702184453-.5472307373\,x+.5351092226\,x^2+.09760940482\,x^3 \\ +.05502038331\,x^4+.01960664563\,x^5+.004575105941\,x^6+.0005202365917\,x^7 \\ +4670686349\times10^{-5}\,x^8+2.839936817\times10^{-6}\,x^9, & \text{for}\quad \frac{1}{2}\le x\le 1. \end{cases}$$

## Conclusion

In this paper, we have used the modified variation of parameters method (MVPM), which is a combination of variation of parameters method and Adomian's decomposition method, for solving system of second-order nonlinear boundary value problem. It is worth mentioning that we have solved a nonlinear system of boundary value problem by our proposed technique while most of the methods in the literature were proposed to solve linear system of boundary value problems associated with obstacle problems. We give an example of the system which is highly nonlinear in its nature. After applying our proposed technique we obtained series solution as well as its graphical representation over the whole domain.

We remark that our proposed method is well suited for such physical problems as it provides solution in less number of iterations. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods. The use of multiplier gives this technique a clear edge over the decomposition method by removing successive application of integrals. Therefore, it may be concluded that modified variation of parameters method is very powerful and efficient in finding the analytical solutions for a wide class of system of nonlinear boundary value problems.

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