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Full Length Research Paper

Euler savary formula for the one parameter motions in the complex plane C

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Müller (1978), in the Euclidean plane E^2 introduced the one parameter planar motions and obtained the relation between absolute, relative, sliding velocities (and accelerations). During one parameter planar motion in the Euclidean plane E^2, the Euler-Savary formula was expressed by Müller (Blaschke and Müller, 1956). Also Blaschke and Müller (1956) and Tutar et al. (2001) provided the relation between the velocities (and accelerations) in the sense of the complex under the one parameter motions in the complex plane $\mathbb{C} = \{x + iy \mid x, y \in IR, i^2 = -1\}$. In this paper we have defined canonical relative system of one parameter motions in the complex plane, C. With the aid of this relative system we have obtained the Euler-Savary formula giving the relation between the curvatures of the trajectory curves of one parameter motions in the complex plane c.

Key words: Euler savary formula, one parameter motion, complex plane.

INTRODUCTION

Complex numbers were first discovered by Cardan, who called them “fictitious”, during his attempts to find solutions to cubic equations, (Conway, 1986). The solutions of a general cubic equation may require intermediate calculations containing the square roots of negative numbers, even when the final solutions are real numbers, a situation known as casus irreduciblis. This ultimately led to the fundamental theorem of algebra, which shows that complex numbers, it is always possible to find solutions to polynomial equations of degree one or higher, (Ahlfors, 1979; Pedoe, 1988).

Complex numbers are used in many different fields including applications in engineering, electromagnetism, quantum physics, applied mathematics, and chaos theory, (Anastopoulos, 2006; Benioff, 2007; Duma, 1988; Philippsen and Guenthner, 2000).

The Euler-Savary formula is well-known theorem that is used in serious fields of study of engineering and mathematics, (Alexander and Maddocks, 1988; Buckley and Whitfield, 1949; Dooner and Griffis, 2007; Ito and Takahaski, 1999; Pennock and Raje, 2004). For each mechanism type a simple graphical procedure is outlined to determine the circles of inflections and cusps, which are useful to compute the curvature of any point of the mobile plane through the Euler-Savary formula.

ONE PARAMETER MOTIONS IN COMPLEX PLANE, C

Let A and E be moving and E' be fixed complex planes and $\{B; a_1, a_2\}$, $\{O; e_1, e_2\}$ and $\{O'; e_1', e_2'\}$ be the coordinate systems for these planes, respectively. Suppose that φ and ψ are the rotation angles of one parameter planar motions A/E and A/E', respectively. Let us consider a point X with the coordinates of (x_1, x_2) in moving plane A. If we denote the vectors $\overline{BX}$, $\overline{OB}$ and $\overline{O'B}$ with the complex numbers

$$\tilde{X} = x_1 + ix_2, \qquad b = b_1 + ib_2, \qquad b' = b_1' + ib_2'$$

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On the moving coordinate system of A, respectively, then we have

$$X = \left(b + \tilde{X}\right) e^{i\varphi} \tag{1}$$

And

$$X' = \left(b' + \tilde{X}\right) e^{i\psi} \tag{2}$$

Where the complex numbers X and X' denote the point X with respect to the coordinate systems of E and E', respectively. Let's find the velocities of one parameter motion with the help of the differentiation the equations (1) and (2)

$$dX = \left(\sigma + i\tau\tilde{X} + d\tilde{X}\right) e^{i\varphi} \tag{3}$$

So that here (Tutar et al., 2001)

$$\sigma = \sigma_1 + i\sigma_2 = db + ib\, d\varphi, \qquad \tau = d\varphi \tag{4}$$

Therefore the relative velocity vector of X with respect to E is;

$$V_r = \frac{dX}{dt}.$$

Differentiating equation (2) yield us

$$d'X = \left(\sigma' + i\tau'\tilde{X} + d\tilde{X}\right) e^{i\psi} \tag{5}$$

Where (Tutar et al., 2001)

$$\sigma' = \sigma_1' + i\sigma_2' = d'b + ib'\, d\psi, \quad \tau' = d\psi . \tag{6}$$

Thus, the absolute velocity vector, that is, the velocity vector of X with respect to E', is;

$$V_a = \frac{d'X}{dt}.$$

Here $\sigma_1, \sigma_2, \sigma_1', \sigma_2', \tau$ and τ' are the Pfaffian forms of one parameter motion with respect to t.

If $V_r = 0$ and $V_a = 0$ then the point X is fixed in the planes E and E', respectively. Thus, the conditions that the point is fixed in planes E and E' become

$$d\tilde{X} = -\sigma - i\tau\tilde{X} \tag{7}$$

And

$$d\tilde{X} = -\sigma' - i\tau'\tilde{X} \tag{8}$$

respectively. Substituting equation (7) into equation (5), considering that the sliding velocity vector of the point X is $V_f = d_f X / dt$, we see that

$$d_f X = \left[\left(\sigma' - \sigma\right) + i\left(\tau' - \tau\right)\tilde{X}\right] e^{i\psi} . \tag{9}$$

Therefore we can easily see that (Tutar et al., 2001).

$$d'X = d_f X + dX . \tag{10}$$

To avoid the cases of pure translation, we suppose that $\dot{\varphi} \neq 0$ and $\dot{\psi} \neq 0$. In the one parameter planar motion the rotation pole is characterized by vanishing sliding velocity. Thus, if we take $d_f X = 0$ and considering equation (9). We find that the pole point P of one parameter planar motion E/E' is

$$P = i\frac{\sigma' - \sigma}{\tau' - \tau} \tag{11}$$

Such that $\overline{BP} = P = p_1 + ip_2$ (Tutar et al., 2001). If we pass to the Euclidean coordinates, then the last equation to be

$$p_1 = -\frac{\sigma_2' - \sigma_2}{\tau' - \tau}, \qquad p_2 = \frac{\sigma_1' - \sigma_1}{\tau' - \tau} . \tag{12}$$

In one parameter complex planar motion E/E', moving and fixed pole curves determine the geometric locus of the point P in planes E and E', respectively. In other words; (P) and (P') are the representation of the moving and fixed pole curves, respectively.

EULER-SAVARY FORMULA FOR ONE PARAMETER MOTIONS IN C

The Euler-Savary formula is the relation between the curvatures of the trajectory curves under one parameter motion E/E'. In the one parameter planar motion, the Euler-Savary formula was expressed in Euclidean plane E^2 by Blaschke and Müller (1956). The formula is very important for Engineering and Mathematics. There are a lot of applications in engineering of the formula, (Buckley and Whitfield, 1949; Pennock and Raje, 2004).

Now, we will study the formula in the one parameter motion in complex plane C. In this section we choose the relative system $\{B; a_1, a_2\}$ satisfying the following conditions:

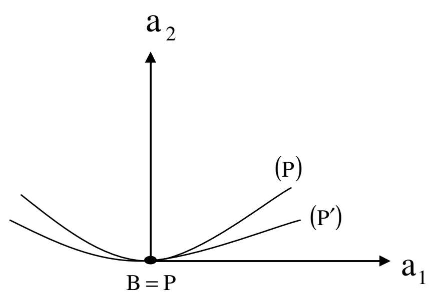

Figure 1. Pole curves of (P) and (P').

i) The initial point B of the system is the instantaneous rotation pole P (that is $P = B$).
ii) The axis $\{B; a_1\}$ coincides with the common tangent of the pole curves (P) and (P') (Figure 1).

If we consider the condition i), then, from equations (11) and (12) we see that $p_1 = p_2 = 0$.

Thus

$$\sigma_1' = \sigma_1, \qquad \sigma_2' = \sigma_2. \tag{13}$$

From equations (4) and (6), we get

$$\begin{aligned} db &= (db + ib\,d\varphi)e^{i\varphi} = \sigma_1 a_1 + \sigma_2 a_2 = \sigma e^{i\varphi} \\ d'b &= (db' + ib'\,d\psi)e^{i\psi} = \sigma_1' a_1 + \sigma_2' a_2 = \sigma' e^{i\psi}. \end{aligned} \tag{14}$$

Taking equation (13) together with equation (14) into consideration we reach that

$$dp = dp' = db = d'b. \tag{15}$$

This means that the moving pole curve (P) and fixed pole curve (P') roll on each other without sliding. Considering the condition ii) yields us that $\sigma_2 = \sigma_2' = 0$. If we choose $\sigma_1 = \sigma_1' = \sigma$ then the differential equations for the canonical relative system $\{P; a_1, a_2\}$ become

$$da_1 = \tau a_2 = i\tau e^{i\varphi}, da_2 = -\tau a_1 = -\tau e^{i\varphi}, dp = \sigma a_1 = \sigma e^{i\varphi} \tag{16}$$

$$d'a_1 = \tau' a_2 = i\tau' e^{i\psi}, d'a_2 = -\tau' a_1 = -i\tau' e^{i\psi}, d'p = \sigma a_1 = \sigma e^{i\psi} \tag{17}$$

Where; $\sigma = ds$ is the scalar arc element of the pole curves (P) and (P'), τ is the central cotangent angle, that is, two neighboring tangents angle of (P). Hence the curvature of the moving pole curve (P) at the point P is τ/σ. Similarly τ' is a central cotangent angle and the curvature of the fixed pole curve (P') at the point P is τ'/σ. Therefore the curvature radii of the pole curves (P) and (P') are

$$r = \frac{\sigma}{\tau} \tag{18}$$

And

$$r' = \frac{\sigma}{\tau'} \tag{19}$$

respectively. Moving plane E rotates the infinitesimal instantaneous angle of the $d\phi = \tau' - \tau$ around the rotation pole P within the time scale dt with respect to fixed plane E'. Therefore the angular velocity of rotational motion of E with respect to E' becomes

$$\frac{\tau' - \tau}{ds} = \frac{d\phi}{ds} = \dot{\phi}. \tag{20}$$

From equations (18), (19) and the last equation, we obtain

$$\frac{\tau' - \tau}{ds} = \frac{d\phi}{ds} = \frac{1}{r'} - \frac{1}{r}. \tag{21}$$

Suppose that for the direction of unit tangent vector a_1, $\frac{ds}{dt} > 0$. In these cases since the curvature centre of the moving pole curve (P) stays in the left-hand side of the directed pole curve $\{P; \overrightarrow{a_1}\}$, we can easily see that $r > 0$. Similarly $r' > 0$.

Let us investigate the differentiation of point X which has the coordinates of x_1, x_2 with respect to the canonical relative system. Considering the condition ii) and equations (3) and (5) we obtain

$$dX = \left(\sigma_1 + i\tau\tilde{X} + d\tilde{X}\right)e^{i\varphi} \qquad , \sigma = \sigma_1. \tag{22}$$

And

$$d'X = \left(\sigma_1 + i\tau'\tilde{X} + d\tilde{X}\right)e^{i\psi} \tag{23}$$

respectively. Thus the condition that the point X to be the fixed in the moving plane E and the fixed plane E' are

$$d\tilde{X} = -\sigma_1 - i\,\tau\tilde{X} \tag{24}$$

And

$$d'\tilde{X} = -\sigma_1 - i\,\tau'\tilde{X} \tag{25}$$

respectively. Therefore the sliding velocity of the motion is written to be

$$d_f X = i\left(\tau' - \tau\right)\tilde{X}\,e^{i\psi}\,. \tag{26}$$

Now, we search for the curvature centres of trajectory curves which are drawn in the fixed plane by the points of moving plane in the movement E/E'.

According to the canonical relative system, let the coordinates of the points X (at E) and X' (at E') be $\left(x_1, x_2\right)$ and (x_1', x_2'), respectively. The points X, X' and the instantaneous rotation pole P stay on a line, that is on an instantaneous trajectory normal related to X at every time t. In general a curvature centre with respect to a point of a plane curve stays on the normal of the curve with respect to that point. However, this curvature centre thought to be the limit of the meeting point of the two neighbouring point that are on the curve. Thus the vectors

$$\overline{PX} = x_1 + ix_2 = X$$

$$\overline{PX'} = x_1' + ix_2' = X' \tag{27}$$

Have the same direction which passes the rotation pole P. Therefore, for the points

X And X' we write

$$\frac{X}{X'} = \lambda \in IR\,. \tag{28}$$

Differentiating the last equations gives us

$$dX\;X' - X\;dX' = 0\,. \tag{29}$$

Substituting equations (24) and (25) into equation (29) we obtain

$$\sigma\left(X - X'\right) + i\,X\,X'\left(\tau' - \tau\right) = 0\,. \tag{30}$$

If we pass to polar coordinates that is

$$X = a\,e^{i\alpha} \tag{31}$$

$$X' = a'\,e^{i\alpha} \tag{32}$$

Then we obtain from equation (30)

$$\sigma\left(a - a'\right) + i\,a\,a' e^{i\alpha}\left(\tau' - \tau\right) = 0\,. \tag{33}$$

Where; a and a', represent the distance of the points X and X' on the complex plane from rotation pole P. If we consider last equation together with equation (21) we reach

$$\left(\frac{1}{a'} - \frac{1}{a}\right) i\,e^{-i\alpha} = \frac{1}{r'} - \frac{1}{r} = \frac{d\phi}{ds}\,. \tag{34}$$

Here, r and r' are the radii of curvature of pole curves P and P', respectively. ds represents the scalar arc element and $d\phi$ represents the infinitesimal angle of the motion of the pole curves. The equation (34) is called Euler-Savary formula for one parameter motion in complex plane, C. Therefore the following theorem can be given.

Theorem: Let E and E' be moving and fixed complex planes. A point X in E in a one parameter planar movement $\left(E/E'\right)$ draws a trajectory in plane E' for which the curvature centre is at the point X'. In the reverse movement E'/E point X' in E' draws a trajectory in plane E for which the curvature centre is at the point X. The relation between the point X and X' is given by Euler-Savary formula equation (34).

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