Full Length Research Paper

Exponential chaos lag synchronization of nonlinear bloch equations

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This paper proposes a new exponential lag synchronization (ELS) method for chaotic behavior in nonlinear Bloch equations. An ELS controller that is based on Lyapunov theory and linear matrix inequality (LMI) approach is presented to guarantee the exponential synchronization of drive and response systems. The proposed controller can be obtained by solving a convex optimization problem represented by the LMI. A simulation study is presented to demonstrate the effectiveness of the proposed synchronization scheme.

Key words: Exponential lag synchronization (ELS), non-linear Bloch equations, linear matrix inequality (LMI), Lyapunov theory.

INTRODUCTION

During the last two decades, synchronization in chaotic dynamic systems has received a great deal of interest among scientists from various research fields since Pecora and Carroll (1990) introduced a method to synchronize two identical chaotic systems with different initial conditions. It has been widely explored in a variety of fields including physical, chemical and ecological systems (Chen and Dong, 1998). In the literature, various synchronization schemes, such as variable structure control (Wang and Su, 2004), OGY method (Ott et al., 1990), parameters adaptive control (Park, 2005; Wang et al., 2003), observer-based control (Yang and Chen, 2002), active control (Bai and Lonngen, 1997; Bai and Lonngren, 2000), time-delay feedback approach (Park, 2005), $\mathcal{H}_\infty$ approach (Ahn, 2009a, 2009c), backstepping design technique (Wu and Lu, 2003; Hu et al., 2005), fuzzy logic approach (Ahn, 2009b, 2009d), and so on, have been successfully applied to the chaos synchronization.

The interaction of the two-level atom or the spin with the electric or magnetic field usually described by the nonlinear Bloch equation is very important for the under-

standing of the underlying physical processes of nuclear magnetic resonance, magnetic resonance imaging, electron spin resonance, and two-level laser propagation (Yang et al., 2007; Ziolkowski et al., 1995; Zou et al., 2004). The basic process can be viewed as the combination of a precession about a magnetic field and of a relaxation process. Abergel (2002) demonstrated that the set of nonlinear Bloch equations would admit chaotic solutions for a certain set of numerical values assigned to the system constants and initial conditions. Ucar et al. (2003) extend the calculation of Abergel (2002) and demonstrate that an active control method can synchronize two of these nonlinear Bloch equations. Recently, some control schemes, such as adaptive control (Park, 2006) and stability criterion method (Ghosh et al., 2008), were proposed for synchronizing chaotic behavior in nonlinear Bloch equations.

In the typical synchronization regimes, lag synchronization has been proposed as the coincidence of the states of chaotic systems in which one of the systems is delayed by a finite time. Many experimental investigations and computer simulations of chaos synchronization in uni-directionally coupled external cavity semiconductor lasers (Shahverdiev et al., 2002; Taherion and Lai, 1999; Barsella and Lepers, 2002) have demonstrated the presence of lag time between the drive and response lasers intensities. Similar experiments for chaotic circuits

(Huang et al., 2001) have also demonstrated the complete synchronization, that is, the states of two chaotic systems remain identical in the course of temporal evolution and this is practically impossible for the presence of the signal transmission time and evolution time of response system itself. Thus, knowledge of the lag synchronization is of considerable practical importance. Recently, some control methods, such as observer based scheme (Li et al., 2005), impulsive control (Li et al., 2005), projective approach (Zhang and Lu, 2008; Li, 2009), and adaptive control (Tang et al., 2008; Wang et al., 2009), have been applied to the lag synchronization for chaotic systems. To the best of our knowledge, however, for the lag synchronization of chaotic behavior in nonlinear Bloch equations, there is no result in the literature so far, which still remains open and challenging.

In this paper, a new controller for the exponential lag synchronization (ELS) of chaotic behavior in nonlinear Bloch equations is proposed. This controller is a new contribution to the topic of chaos synchronization. Theoretical proof shows that the ELS controller can make the closed-loop lag synchronization error system exponentially synchronized. Based on Lyapunov method and linear matrix inequality (LMI) approach, an existence criterion for the proposed controller is represented in terms of the LMI. The LMI problem can be solved efficiently by using recently developed convex optimization algorithms (Boyd et al., 1994).

This paper is organized as follows. An introduction of nonlinear Bloch equations and problem formulations is discussed. Next, an LMI problem for the ELS of chaotic behavior in nonlinear Bloch equations is proposed and a numerical example is given and finally, conclusion is presented.

NONLINEAR BLOCH EQUATIONS AND PROBLEM FORMULATION

In dimensionless units, the dynamic model of nonlinear modified Bloch equations with feedback field (Abergel, 2002) is given by

$$
\begin{cases}
\dot{x}_d(t) = \delta y_d(t) + \lambda z_d(t)(x_d(t)\sin\psi - y_d(t)\cos\psi) - \dfrac{x_d(t)}{\tau_2}, \\
\dot{y}_d(t) = -\delta x_d(t) - z_d(t) + \lambda z_d(t)(x_d(t)\cos\psi + y_d(t)\sin\psi) - \dfrac{y_d(t)}{\tau_2}, \\
\dot{z}_d(t) = y_d(t) - \lambda\sin\psi(x_d^2(t) + y_d^2(t)) - \dfrac{z_d(t)-1}{\tau_1},
\end{cases}
\tag{1}
$$

Where; δ, λ and ψ are the system parameters and τ_1 and τ_2 are the longitudinal time and transverse relaxation times, respectively. The subscript 'd' indicates that the system will be considered as the drive (or master) system.

Abergel has extensively investigated the dynamics of the system (1) for a fixed subset of the system parame- ters (δ, λ, τ_1, τ_2) and for a space area range of the radiation damping feedback ψ (Abergel, 2002). In particular, the regions of the radiation damping feedback ψ that would admit chaotic behavior were obtained. For details of other dynamic properties of the system (1), refer to Abergel (2002), Lonngren and Bai (2003). The synchronization problem of system (1) is considered by using the drive-response configuration. The system (1) is considered as the drive system. According to the drive-response concept, the controlled response system is given by

$$
\begin{cases}
\dot{x}_r(t) = \delta y_r(t) + \lambda z_r(t)(x_r(t)\sin\psi - y_r(t)\cos\psi) - \dfrac{x_r(t)}{\tau_2} + u_1(t), \\
\dot{y}_r(t) = -\delta x_r(t) - z_r(t) + \lambda z_r(t)(x_r(t)\cos\psi + y_r(t)\sin\psi) - \dfrac{y_r(t)}{\tau_2} + u_2(t), \\
\dot{z}_r(t) = y_r(t) - \lambda\sin\psi(x_r^2(t) + y_r^2(t)) - \dfrac{z_r(t)-1}{\tau_1} + u_3(t),
\end{cases}
\tag{2}
$$

Where; $u_1(t)$, $u_2(t)$ and $u_3(t)$ are the nonlinear controllers.

Define the lag synchronization error as

$$
\begin{cases}
e_1(t) = x_r(t) - x_d(t - \tau), \\
e_2(t) = y_r(t) - y_d(t - \tau), \\
e_3(t) = x_r(t) - z_d(t - \tau),
\end{cases}
\tag{3}
$$

Where; $\tau > 0$ is the synchronization lag. Then the following lag synchronization error system are obtained:

$$
\begin{cases}
\dot{e}_1(t) = \delta e_2(t) - \dfrac{1}{\tau_2}e_1(t) + \lambda z_r(t)(x_r(t)\sin\psi - y_r(t)\cos\psi) \\
\qquad - \lambda z_d(t-\tau)(x_d(t-\tau)\sin\psi - y_d(t-\tau)\cos\psi) + u_1(t), \\
\dot{e}_2(t) = -\delta e_1(t) - e_3(t) - \dfrac{1}{\tau_2}e_2(t) + \lambda z_r(t)(x_r(t)\cos\psi + y_r(t)\sin\psi) \\
\qquad - \lambda z_d(t-\tau)(x_d(t-\tau)\cos\psi + y_d(t-\tau)\sin\psi) + u_2(t), \\
\dot{e}_3(t) = e_2(t) - \dfrac{1}{\tau_1}e_3(t) - \lambda\sin\psi(x_r^2(t) + y_r^2(t)) + \lambda\sin\psi(x_d^2(t-\tau) + y_d^2(t-\tau)) \\
\qquad + u_3(t),
\end{cases}
\tag{4}
$$

which is rewritten as

$$
\dot{e}(t) = Ae(t) + f(t, t-\tau) + u(t),
\tag{5}
$$

Where $e(t)$, $u(t)$, A and $f(t, t - \tau)$ are defined by

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \quad A = \begin{bmatrix} -\dfrac{1}{\tau_2} & \delta & 0 \\ -\delta & -\dfrac{1}{\tau_2} & -1 \\ 0 & 1 & -\dfrac{1}{\tau_1} \end{bmatrix},$$

$$f(t, t-\tau)$$

$$= \begin{bmatrix} \lambda z_r(t)(x_r(t)\sin\psi - y_r(t)\cos\psi) - \lambda z_d(t-\tau)(x_d(t-\tau)\sin\psi - y_d(t-\tau)\cos\psi) \\ \lambda z_r(t)(x_r(t)\cos\psi + y_r(t)\sin\psi) - \lambda z_d(t-\tau)(x_d(t-\tau)\cos\psi + y_d(t-\tau)\sin\psi) \\ -\lambda\sin\psi(x_r^2(t) + y_r^2(t)) + \lambda\sin\psi(x_d^2(t-\tau) + y_d^2(t-\tau)) \end{bmatrix}.$$

Definition 1

Exponential lag synchronization: The error system Equation 5, is exponentially lag-synchronized if the lag synchronization error $e(t)$ satisfies

$$\| e(t) \| < M e^{-Nt}, t \geq 0, \tag{6}$$

Where M and N are positive scalars.

MAIN RESULT

In this section, we present the LMI problem for achieving the exponential lag synchronization of Nonlinear Bloch Equations.

Theorem 1

For a given $Q = Q^T > 0$, if there exist $X = X^T > 0$ and Y such that

$$\begin{bmatrix} AX + XA^T + Y + Y^T & X \\ X & -Q^{-1} \end{bmatrix} < 0, \tag{7}$$

Then the exponential lag synchronization is achieved under the controller

$$u(t) = YX^{-1} \begin{bmatrix} x_r(t) - x_d(t-\tau) \\ y_r(t) - y_d(t-\tau) \\ z_r(t) - z_d(t-\tau) \end{bmatrix} - \begin{bmatrix} \lambda z_r(t)(x_r(t)\sin\psi - y_r(t)\cos\psi) \\ \lambda z_r(t)(x_r(t)\cos\psi + y_r(t)\sin\psi) \\ -\lambda\sin\psi(x_r^2(t) + y_r^2(t)) \end{bmatrix}$$

$$\begin{aligned} &-\lambda z_d(t-\tau)(x_d(t-\tau)\sin\psi - y_d(t-\tau)\cos\psi) \\ &-\lambda z_d(t-\tau)(x_d(t-\tau)\cos\psi + y_d(t-\tau)\sin\psi) \\ &+\lambda\sin\psi(x_d^2(t-\tau) + y_d^2(t-\tau)) \end{aligned} \Bigg]. \tag{8}$$

Proof

The closed-loop error system with the control input:

$$u(t) = Ke(t) - f(t, t-\tau)$$

Where $K \in \Re^{3\times 3}$ is the gain matrix of the control input $u(t)$ and can be written as

$$\dot{e}(t) = (A + K)e(t). \tag{9}$$

Consider a Lyapunov function

$$V(e(t)) = e^T(t)Pe(t) \tag{10}$$

Where; $P = P^T > 0$.

Note that $V(e(t))$ satisfies the following Rayleigh inequality (Strang, 1986):

$$\lambda_{\min}(P) \| e(t) \|^2 \leq V(e(t)) \leq \lambda_{\max}(P) \| e(t) \|^2 \tag{11}$$

Where; $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the maximum and minimum eigenvalues of the matrix.

The time derivative of $V(e(t))$ along the trajectory of Equation 9 is

$$\dot{V}(e(t)) = \dot{e}(t)^T Pe(t) + e^T(t)P\dot{e}(t)$$
$$= e^T(t)[A^T P + PA + PK + K^T P]e(t).$$

If the following matrix inequality is satisfied

$$A^T P + PA + PK + K^T P + Q < 0, \tag{12}$$

we have

$$\dot{V}(e(t)) < -e^T(t)Qe(t) \quad \leq -\lambda_{\min}(Q) \| e(t) \|^2$$
$$\leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V(e(t)). \tag{13}$$

From Equation 13, we obtain

$$V(e(t)) < V(e(0))e^{-\kappa t}$$

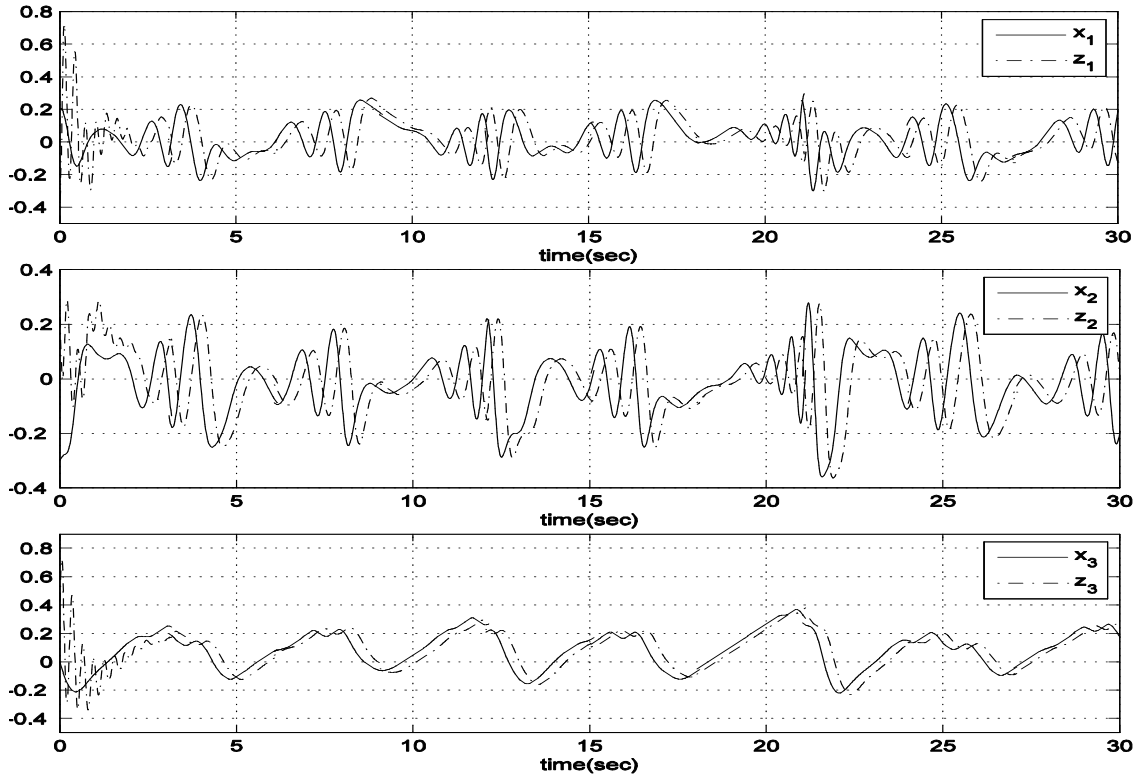Where $\kappa = \dfrac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} > 0.$

Figure 1. State trajectories

Using Equation 11, we have

$$\| e(t) \| < \sqrt{\frac{V(e(0))}{\lambda_{min}(P)}} e^{-\kappa t} = \sqrt{\frac{e^T(0)Pe(0)}{\lambda_{min}(P)}} e^{-\frac{\kappa}{2}t}, \tag{15}$$

Since

$$\sqrt{\frac{e^T(0)Pe(0)}{\lambda_{min}(P)}} > 0, \qquad \frac{\kappa}{2} > 0, \tag{16}$$

The relation (15) guarantees the exponential lag synchronization. From Schur complement, the matrix inequality (12) is equivalent to

$$\begin{bmatrix} A^TP + PA + PK + K^TP & I \\ I & -Q^{-1} \end{bmatrix} < 0. \tag{17}$$

Pre- and post-multiplying (17) by $diag(P^{-1}, I)$ and introducing change of variables such as $X = P^{-1}$ and $Y = KP^{-1}$, Equation 17 is equivalently changed into the LMI (7). Then the gain matrix of the control input $u(t)$ is given by $K = YX^{-1}$. This completes the proof.

Remark 1

The LMI problem given in Theorem 1 is   to   determine

whether the solution exists or not. It is called the feasibility problem. The LMI problem can be solved efficiently by using recently developed convex optimization algorithms (Boyd et al., 1994). In this paper, in order to solve the LMI problem, we utilize MATLAB LMI Control Toolbox (Gahinet et al., 1995), which implements state-of-the-art interior-point algorithms.

NUMERICAL EXAMPLE

In order to verify and demonstrate the effectiveness of the proposed method, the simulation result obtaining from synchronizing nonlinear Bloch equations under different initial conditions is discussed. For the numerical simulation, the system parameters ψ, τ_1, τ_2, δ, and λ are fixed as 0.173, 5, 2.5, -0.4π, and 35, respectively, so that the Bloch equations exhibit a chaotic behavior. Let $\tau = 0.3$ and $Q = I$, where $I \in R^{3 \times 3}$ is an identity matrix. Applying Theorem 1 to the Bloch equations yields

$$X = \begin{bmatrix} 0.7826 & 0.0000 & 0.0000 \\ 0.0000 & 0.7826 & 0.0000 \\ 0.0000 & 0.0000 & 0.7826 \end{bmatrix}, \quad Y = \begin{bmatrix} -0.8609 & -5.0830 & 15.6762 \\ 5.0830 & -0.8609 & -0.7231 \\ -15.6762 & 0.7231 & -1.0174 \end{bmatrix}. \tag{18}$$

Figure 1 shows state trajectories for drive and response systems when the initial conditions are given by
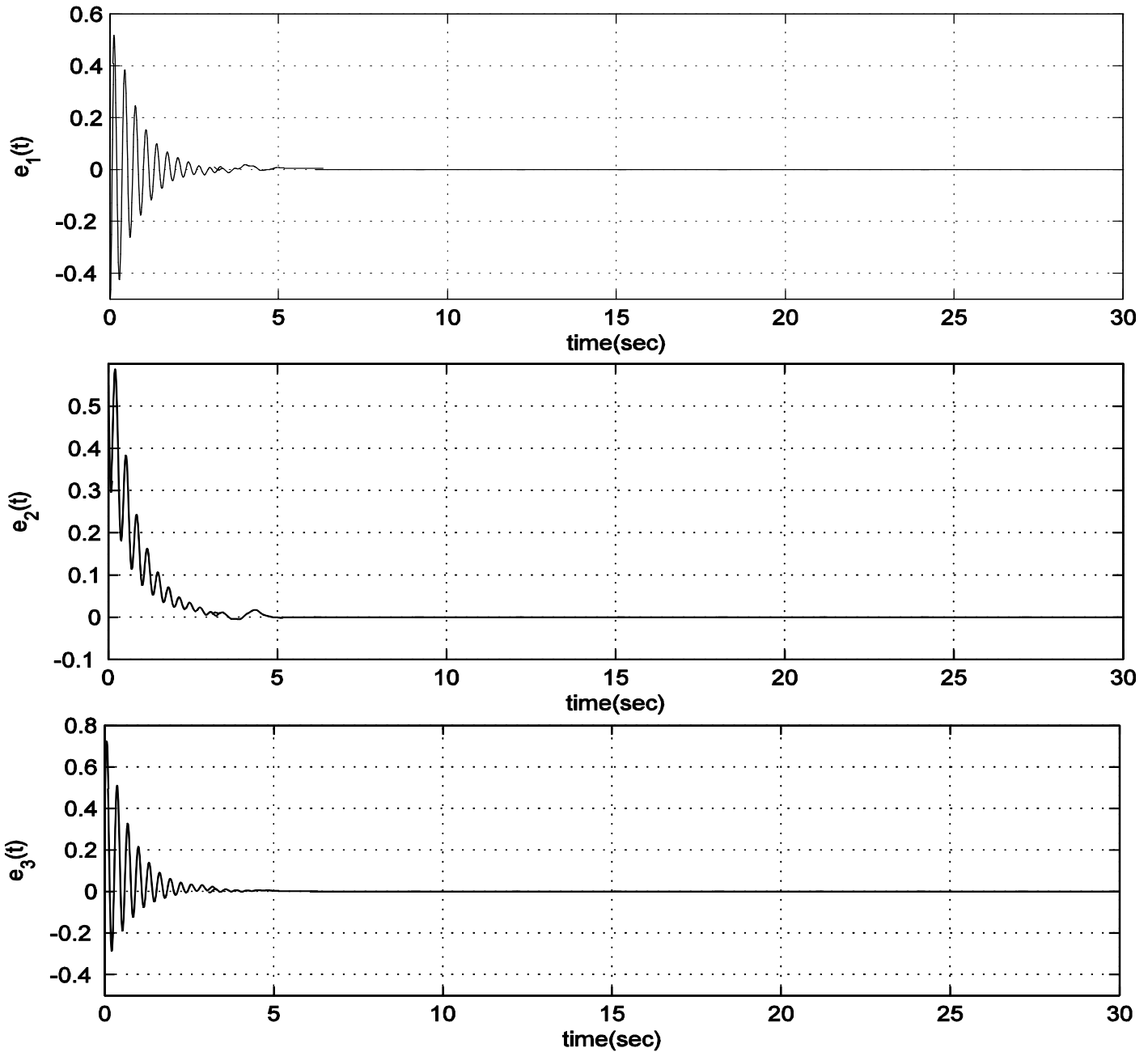
Figure 2. Synchronization errors

$$\begin{bmatrix} x_d(0) \\ y_d(0) \\ z_d(0) \end{bmatrix} = \begin{bmatrix} 0.2 \\ -0.3 \\ 0 \end{bmatrix}, \begin{bmatrix} x_r(0) \\ y_r(0) \\ z_r(0) \end{bmatrix} = \begin{bmatrix} -0.4 \\ 0.25 \\ 0.37 \end{bmatrix}. \tag{19}$$

From Figure 1, it can be seen that drive and response systems are indeed achieving chaos lag synchronization. Figure 2 shows that the lag synchronization error $e(t)$ converges to zero exponentially fast. Simulation results reveal that the response system controlled using the proposed method performs well. The effectiveness and

accuracy of the proposed method are verified.

Conclusion

In this paper, we propose a new ELS scheme for chaotic behavior in nonlinear Bloch equations. Based on Lyapunov theory and LMI formulation, the proposed method can guarantee the ELS between the drive and response systems. The ELS controller can be easily obtained by solving the LMI problem. Furthermore, a nu-

merical simulation is given to illustrate the effectiveness of the proposed scheme.

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