International Journal of Physical Sciences Vol. 7(9), pp. 1472 - 1481, 23 February, 2012
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.1320
ISSN 1992 - 1950 ©2012 Academic Journals

Full Length Research Paper

Stability of state-proportional integral derivative (PID) feedback control system with time delay

Witchupong Wiboonjaroen and Sarawut Sujitjorn*

School of Electrical Engineering, Suranaree University of Technology, Muang District, Nakhon Ratchasima, 30000, Thailand.

Accepted 09 February 2012

This article proposes a method to compute the maximum allowable delay time ($\bar{\tau}$) for linear-time-invariant-time-delayed-systems (LTI-TDS) with state-PID feedback control. It presents the main theorem with corollary, and computing steps to obtain $\bar{\tau}$. Stability of a LTI-TDS with state- proportional integral derivative (PID) feedback is ensured if the delay time is less than $\bar{\tau}$. The proposed approach is compared with the matrix pencil method. Stabilization for the system is also proposed by using a low-pass filter. It was shown by simulation that the system is made more tolerable to delay by judicious selection of the DC gain and the time-constant of the filter. Seven case studies serve to demonstrate the effectiveness of the proposed approaches.

Key words: Routh's criterion, time-delayed, state-PID feedback, stability and stabilization, neutral systems.

INTRODUCTION

Control of systems with delays has been a very active issue for academic and industry for several decades because of many concerned practical systems including heat and chemical processes, material transport systems, etc. (Normey-Rico and Chamacho, 2007). The most common control technique for this system category is proportional integral derivative (PID) control. As microcontrollers become cheaper, embedded systems have been increasingly used worldwide. Unlike their analog counterparts, microcontroller-based control needs time for instruction-set execution, data conversion process, and data communication in the control loop. These introduce inevitable delay to a computer-controlled system, although, the plant itself is not a delayed type. Characteristic equation of such a system becomes transcendental. In effect, the number of eigenvalues becomes infinite (Michiels et al., 2002; Richard, 2003). Control design via eigenvalue assignment for this class of systems is not simple as researchers have developed the finite spectrum assignment methods to achieve this (Manitius and Olbrot, 1979; Wang et al., 1995; Brethe and Loiseau, 1998).

Delay also has a detrimental effect on stability (Niculescu, 2001). Since a system can withstand a certain delay time before becoming unstable, stability analysis is a prime interest for prediction of a tolerable delay. An early method for stability test was proposed by Rekasius (1980). The method uses exact bilinear transformation to represent the transcendental term as ratio of s-polynomials. The work has been extended to time-delayed linear-time-invariant (LTI) systems (Olgac and Sipahi, 2002). The same authors present their comparative studies among five stability analysis methods, and conclude that the Rekasius' method is the most attractive one due to simplicity, accuracy and exactness (Sipahi and Olgac, 2006). Moreover, this approach lends itself nicely to stability analysis of retarded and neutral systems (Sipahi and Olgac, 2003; Olgac and Sipahi, 2004, 2005). Interestingly, the method has been applied for computing generalized eigenvalues of certain constant matrices. It uses a matrix pencil approach that can be executed in finite steps, and applied for predicting a tolerable delay (Chen et al., 1995; Niculescu, 2001; Fu et al., 2006). Recently, a new approach based on Lambert W-function to compute eigenvalue spectrum and predict stability of a delayed system has been proposed (Asl and Ulsoy, 2003; Yi et al., 2010). Computation based on this approach is rather

*Corresponding author. E-mail: Sarawut@sut.ac.th.

complex. Furthermore, there is no guarantee for existence and convergence of a solution. From our experience of using the method via the command "fsolve" of MATLAB™, computing did not converge to a solution for a system of higher order than two.

In engineering practice, state-derivative (or state-D) feedback is a useful approach particularly to mechanical vibration control. Recently, state-PID feedback has been proposed for regulation problem of an LTI system (Sujitjorn and Wiboonjaroen, 2011). The concept is extended to an LTI system with a single delay as described by this paper. Since the stability of an LTI system is sensitive to delay and derivative component, this article presents stability analysis based on Routh's criterion in comparison with the matrix pencil approach. Various numerical examples and a case study of pendulum control are demonstrated. Moreover, stabilization of a LTI system with delay using a low-pass filter is presented. This article presents the reviews of our computing approaches, results, numerical examples with discussions and conclusion, respectively.

COMPUTING METHODS

Here, we give reviews of two computing methods used in this article to obtain a tolerable delay. Consider a linear-time-invariant with single time-delay system (LTI-TDS) represented by:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-\tau) \tag{1}$$

where $\mathbf{x}(n\times 1)$, $\mathbf{A}(n\times n), \mathbf{A_d}(n\times n) \in R$ and $\tau \in R^{+}$. $\mathbf{A}$ and $\mathbf{A_d}$ are constant matrices. The system characteristic equation is expressed by:

$$f(s,\tau) = \det\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-\tau s}\right) = 0, \qquad \tau > 0 \tag{2}$$

or in a general form:

$$CE(s,\tau) = a_n(s)e^{-n\tau s} + a_{n-1}(s)e^{-(n-1)\tau s} + ... + a_0(s) \tag{3}$$

$$= \sum_{k=0}^{n} a_k(s)e^{-k\tau s} = 0$$

where $a_k(s)$ is (n-k)th order of s-polynomial having real coefficients. The aforementioned system is stable if and only if all characteristic roots lie on the left half of the s-plane. Due to the transcendental term, the number of characteristic roots becomes infinite. These roots have been referred to in literatures as characteristic spectrum, eigenvalue spectrum or eigen-spectrum. Only the dominant branch of such spectrum was used to justify stability since the rest of the branches lie on the left side of the dominant one. The Rekasius' substitution method has been very attractive, because it helps single out the dominant branch of the eigen-spectrum, which is useful for stability analysis. The following calculation procedures are based on this approach. One which employs the Routh's criterion is referred to as direct method proposed by this article. Another one employs an algebraic calculation, and is referred to as matrix pencil method.

Direct method

Computing procedures to obtain $\bar{\tau}$:

Step 1: Define the system characteristic equation in the form of Equation 3.

Step 2: Substitute the transcendental term in Equation 3 by the Rekasius substitution, and arrange the characteristic equation in the form of,

$$\begin{aligned}\overline{CE}(s,T) &= \sum_{k=0}^{n} a_k(s)\left(\frac{1-Ts}{1+Ts}\right)^{k} = 0 \\ &= \sum_{k=0}^{n} a_k(s)(1+Ts)^{n-k}(1-Ts)^{k} = 0 \\ &= \sum_{k=0}^{2n} b_k(T)s^{k} = 0\end{aligned} \tag{4}$$

Step 3: Construct the Routh's array, $RA(T)$, according to Equation 5:

$$RA(T) = \begin{bmatrix} b_m & b_{m-2} & b_{m-4} & \cdots \\ b_{m-1} & b_{m-3} & b_{m-5} & \cdots \\ R_{m-2,1} & R_{m-2,2} & R_{m-2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ R_{1,1} & R_{1,2} & 0 & \cdots \\ R_{0,1} & 0 & 0 & \cdots \end{bmatrix} \tag{5}$$

where *m* is the maximum order of s in Equation 4.

Step 4: Iteratively compute Equation 5 for T_{ck} that results in $\pm j\omega_{ck}$ eigenvalues according to the Routh's stability criterion.

Step 5: Substitute T and ω in Equation 6 by T_{ck} and ω_{ck}, respectively.

$$\tau = \frac{2}{\omega}\left[\tan^{-1}(\omega T) + l\pi\right], \qquad l = 0,1,...\infty \tag{6}$$

Obtain $\bar{\tau} = \min(\tau)$.

Matrix pencil method

Computing procedures to obtain $\bar{\tau}$:

Step 1: Define the system characteristic equation in the form of Equation 3.

Step 2: Substitute the transcendental term in Equation 3 by the Rekasius substitution, and arrange the characteristic equation in the form of,

$$\overline{CE}(s,T) = \sum_{i=0}^{n+n_d} b_i(T)s^{n+n_d-i} = 0 \tag{7}$$

in which *n* is the system order and n_d is the commensurate degree.

Rearrange the transformed characteristic polynomial (Equation 7) in the form of Equation 8:

$$\overline{CE}(s,T)=\sum_{k=0}^{n_d} q_k(s)T^k, \quad (8)$$

where $q_k(s)=\sum_{l=0}^{n+n_d} q_{kl}s^{n+n_d-l}=q_{k0}s^{n+n_d}+q_{k1}s^{n+n_d-1}+\ldots+q_{k(n+n_d)}$, q_{kl} are constants.

Step 3: Construct the Hurwitz matrix:

$$\mathbf{H}(q_k)=\Delta_{n+n_d} \quad (9)$$

Step 4: Compute the real eigenvalues of the matrix pencil $\mathbf{\Gamma}$ for $T=\lambda_k, k=1,\ldots,m, m\le nn_d$, as:

$$\mathbf{\Gamma}(\lambda)=\mathbf{U}\lambda+\mathbf{V} \quad (10)$$

where

$$\mathbf{U}=\begin{bmatrix}\mathbf{I} & & & \\ & \ddots & & \\ & & \mathbf{I} & \\ & & & \mathbf{H}(q_{n_d})\end{bmatrix}\in\mathbf{R}^{n_d(n+n_d)\times n_d(n+n_d)} \quad (11)$$

and

$$\mathbf{V}=\begin{bmatrix}0 & -\mathbf{I} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & -\mathbf{I}\\ \mathbf{H}(q_0) & \mathbf{H}(q_1) & \cdots & \mathbf{H}(q_{n_d-1})\end{bmatrix}\in\mathbf{R}^{n_d(n+n_d)\times n_d(n+n_d)} \quad (12)$$

where **U** and **V** consist of square block matrices of order $n+n_d$.

Step 5: Compute T_{ck} for Equation 10 that results in $\pm j\omega_{ck}$ eigenvalues for Equation 8.

Step 6: Substitute T and ω in Equation 6 by T_{ck} and ω_{ck}, respectively. Obtain $\bar{\tau}=\min(\tau)$.

Note that, for both methods, Steps 1 to 4 require symbolic programming; otherwise, they need to be done by hand. Steps 5 to 6 use conventional numerical computing.

MAIN RESULTS

Tolerable delay

Consider a LTI-TDS of the form:

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t-\tau) \quad (13)$$

where $\mathbf{x}(n\times 1)$ is the state vector, $u\in R$ is the control input, $\mathbf{A}(n\times n)$ and $\mathbf{B}(n\times 1)$ are the system matrix and the control vector, respectively.

Theorem 1

Suppose that the system (Equation13) is completely controllable and t-stabilizable (Olgac and Sipahi, 2004), for the state-PID feedback that control the system characteristic equation is:

$$\overline{CE}(s,T)=\det\left((s\mathbf{I}-\mathbf{A}-\mathbf{B}(\mathbf{K_p}+\mathbf{K_i}\frac{1}{s}+\mathbf{K_d}s)\frac{1-Ts}{1+Ts}\right)=0,$$

$\tau\in R^+, T\in R$.

Proof

For the delayed control input:

$$u(t-\tau)=\mathbf{K_p}\mathbf{x}(t-\tau)+\mathbf{K_i}\int\mathbf{x}(t-\tau)dt+\mathbf{K_d}\dot{\mathbf{x}}(t-\tau),$$

the closed-loop system can be expressed by:

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\left[\mathbf{K_p}\mathbf{x}(t-\tau)+\mathbf{K_i}\int\mathbf{x}(t-\tau)dt+\mathbf{K_d}\dot{\mathbf{x}}(t-\tau)\right], \quad (14)$$

The system (Equation 13) possesses the following characteristic equation:

$$CE(s,\tau)=\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}(\mathbf{K_p}+\mathbf{K_i}\frac{1}{s}+\mathbf{K_d}s)e^{-\tau s}\right)=0, \tau\in R^+ \quad (15)$$

Substituting $e^{-\tau s}=\frac{1-Ts}{1+Ts}$, $T\in R$, we obtain:

$$CE(s,\tau)=\overline{CE}(s,T)=\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}(\mathbf{K_p}+\mathbf{K_i}\frac{1}{s}+\mathbf{K_d}s)\frac{1-Ts}{1+Ts}\right)=0 \quad (16)$$

This completes the proof.

The following is an immediate consequence of theorem 1.

Corollary 1

For a completely controllable and t-stabilizable system (Equation 13):

1. With the state-P feedback control, the characteristic polynomial is expressed by:

$$\overline{CE}(s,T)=\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}\mathbf{K_p}\frac{1-Ts}{1+Ts}\right), T\in R \quad (17)$$

2. With the state-D feedback control, the characteristic polynomial is expressed by:

$$\overline{CE}(s,T)=\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}\mathbf{K_d}s\frac{1-Ts}{1+Ts})\right),\ T\in R \qquad (18)$$

3. With the state-PD feedback control, the characteristic polynomial is expressed by:

$$\overline{CE}(s,T)=\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}(\mathbf{K_p}+\mathbf{K_d}s)\frac{1-Ts}{1+Ts}\right),\ T\in R \qquad (19)$$

4. With the state-PI feedback control, the characteristic polynomial is expressed by:

$$\overline{CE}(s,T)=\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}(\mathbf{K_p}+\mathbf{K_i}\frac{1}{s})\frac{1-Ts}{1+Ts}\right),\ T\in R \qquad (20)$$

According to the Rekasius substitution:

$$e^{-\tau s}=\frac{1-Ts}{1+Ts},\ \ \tau\in R^{+},T\in R\ ,$$

where $s=j\omega,\omega\in R$, the relationship between T and τ is:

$$\tau=\frac{2}{\omega}\left[\tan^{-1}(\omega T)+l\pi\right],\qquad l=0,1,...\infty \qquad (21)$$

Some Ts cause the eigenvalues $s=j\omega$ with infinite numbers of t, that is:

$$T_{ck}\leftrightarrow s=j\omega_{ck}\leftrightarrow\tau_{kl},k=1,2..m,l=0,1...\infty \qquad (22)$$

The maximum delay time ($\bar{\tau}$) can be figured out from:

$$\bar{\tau}=\min\left\{\frac{2}{\omega}\left[\tan^{-1}(\omega T)+l\pi\right]\right\}=\min(\tau) \qquad (23)$$

$\bar{\tau}$ results in critical or marginal stability. This means that a t-stabilizable system remains stable if $0\le\tau<\bar{\tau}$.

Stabilization

Consider a LTI system having single input of the form of Equation 13. If the system is unstable, a stabilizing state-PID feedback is designed by applying a first-order low pass filter to Equation 13, when K_f and T_f is the DC gain and the time constant of the filter. In this way, a time-delayed system can be stabilized, that is, the system is more robust to the delay time. Corollary 2 is also an immediate consequence of Theorem 1.

Corollary 2

Consider the system described by Theorem 1 with the maximum allowable delay $\tau=\bar{\tau}$. A low pass filter is added to the system. Therefore, the system characteristic equation can be express by:

$$\overline{CE}(s,T_{ck})=\det\left((s\mathbf{I}-\mathbf{A}-\mathbf{B}(\mathbf{K_p}+\mathbf{K_i}\frac{1}{s}+\mathbf{K_d}s)\frac{1-T_{ck}s}{1+T_{ck}s}.\frac{K_f}{1+T_fs}\right)=0, \qquad (24)$$

Note that T_{ck} causes a pair of imaginary characteristic roots $\left(\pm j\omega_{ck}\right)$. Furthermore,

1. With the state-P feedback control, the characteristic polynomial is expressed by:

$$\overline{CE}(s,T_{ck})=\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}\mathbf{K_p}\frac{1-T_{ck}s}{1+T_{ck}s}\frac{K_f}{1+T_fs}\right), \qquad (25)$$

2. With the state-D feedback control, the characteristic polynomial is expressed by:

$$\overline{CE}(s,T_{ck})=\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}\mathbf{K_d}s\frac{1-T_{ck}s}{1+T_{ck}s}\frac{K_f}{1+T_fs}\right), \qquad (26)$$

3. With the state-PD feedback control, the characteristic polynomial is expressed by:

$$\overline{CE}(s,T_{ck})=\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}(\mathbf{K_p}+\mathbf{K_d}s)\frac{1-T_{ck}s}{1+T_{ck}s}\frac{K_f}{1+T_fs}\right), \qquad (27)$$

4. With the state-PI feedback control, the characteristic polynomial is expressed by:

$$\overline{CE}(s,T_{ck})=\det\left(s\mathbf{I}-\mathbf{A}-\mathbf{B}(\mathbf{K_p}+\mathbf{K_i}\frac{1}{s})\frac{1-T_{ck}s}{1+T_{ck}s}\frac{K_f}{1+T_fs}\right), \qquad (28)$$

Applying the aforementioned characteristic polynomials, one can compute for the parameters of the low pass filter that stabilize the system. In other words, the system will be able to tolerate a longer delay time.

The following examples serve to demonstrate the proposed method via simulations. The first case study is explained in details. Since the other cases have similar work procedures, they are presented in brief with results.

EXAMPLES

Let us consider a t-stabilizable LTI system having:

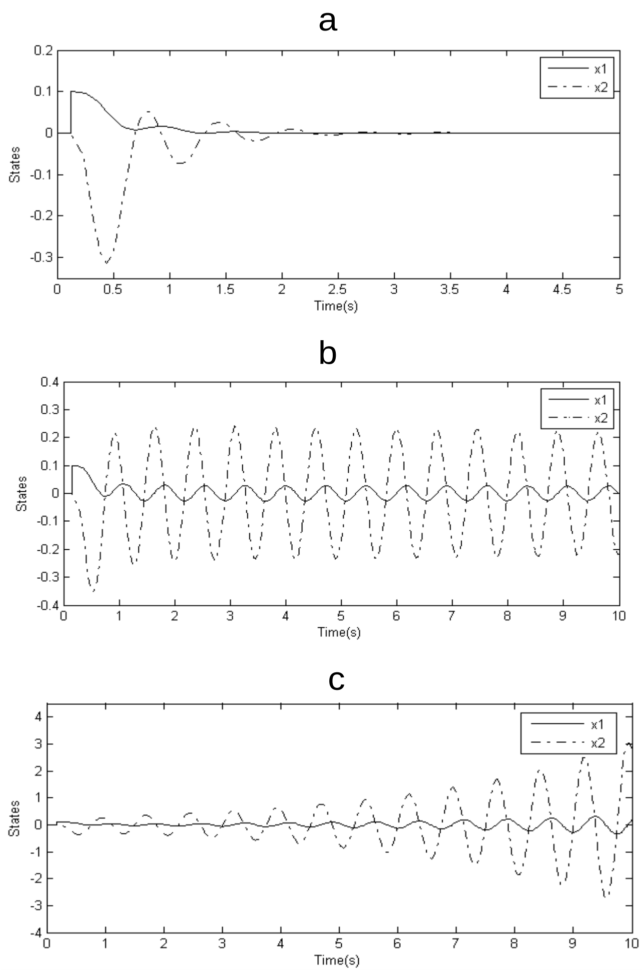

Figure 1. State responses of Example 1 delay time: (a) $\tau = 120ms < \bar{\tau}$, (b) $\tau = 155ms = \bar{\tau}$ and (c) $\tau = 170ms > \bar{\tau}$.

$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4.6985 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix}$.

The system is unstable, and has its eigenvalues at $\pm j2.1676$. Examples 1 to 5 demonstrate how we can calculate the maximum allowable delay $(\bar{\tau})$ by using the direct method proposed in comparison with matrix pencil method. Stabilization of a delayed system is illustrated by Example 6. Example 7 provides a design example of an inverted pendulum on a cart system.

Example 1: State-P feedback

Using the Ackermann's formula, a required pair of closed-loop poles at $-4 \pm j2$ can be placed via the gain matrix $\mathbf{K_p} = [61.2 \quad 32]$. The closed-loop system without delay is stable. Next, we compute the value $\bar{\tau}$.

The characteristic polynomial $CE(s,\tau)$ can be formulated as:

$$CE(s,\tau) = a_2(s)e^{-2\tau s} + a_1(s)e^{-\tau s} + a_0(s)$$

in which $a_2(s) = 0$, $a_1(s) = 8s + 15.3$ and $a_0(s) = s^2 + 4.6985$. Next, $CE(s,T)$ is obtained as:

$$CE(s,T) = \sum_{k=0}^{2n} b_k(T)s^k = b_3 s^3 + b_2 s^2 + b_1 s + b_0$$

in which $b_3(T) = 2\mathrm{T}$, $b_2(T) = 2 - 16\mathrm{T}$, $b_1(T) = 16 - 21.203\mathrm{T}$ and $b_0(T) = 39.997$. The constructed Routh's array is as follow:

$$RA(T) = \begin{bmatrix} 2T & 16 - 21.203T \\ 2 - 16T & 39.997 \\ R_{1,1} & 0 \\ R_{0,1} & 0 \end{bmatrix}.$$

Using an iterative computing, the set of *T*s can be obtained as $T = [0.0922, 0.1251]$. For $T = 0.0922$,

$$RA(T) = \begin{bmatrix} 0.1844 & 14.0451 \\ 0.5248 & 39.9970 \\ -0.0087 & 0 \\ 39.9970 & 0 \end{bmatrix},$$

and the obtained eigenvalues are $0.0008 \pm j8.7276 \cong \pm j8.7276$ and -2.8476. For $T = 0.1251$, the obtained Routh's array and eigenvalues are as follows:

$$RA(T) = \begin{bmatrix} 0.2502 & 13.3475 \\ -0.0016 & 39.9970 \\ 6267.8784 & 0 \\ 39.9970 & 0 \end{bmatrix},$$

$1.3271 \pm j7.6560$ and -2.6478, respectively. Therefore, $T_{ck} = 0.0922$ s is the critical time interval, and the imaginary-axis crossover frequencies $\omega_{ck} = \pm 8.7276$ rad/s. Finally, we obtain $\bar{\tau} = 155ms$.

Figure 1 illustrates responses of the system states. As shown in Figure 1a, the response converges to zero, as the delay time is less than the maximum allowable delay. In contrast, oscillatory and unstable responses can be observed in Figure 1b and c, as the delay times are longer than the maximum delay.

Now we present the calculation procedures based on the matrix pencil approach as follows, that is, one can write the characteristic polynomial of the system as:

$$CE(s,\tau) = \det\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-\tau s}\right)$$

$$= a_2(s)e^{-2\tau s} + a_1(s)e^{-\tau s} + a_0(s)$$

where $a_2(s) = 0$, $a_1(s) = 8s + 15.3$ and $a_0(s) = s^2 + 4.6985$. Using the Rekasius substitution, the characteristic polynomial can be rewritten as:

$$\overline{CE}(s,T) = q_0(s) + q_1(s)T + q_2(s)T^2,$$

where $q_0(s) = s^2 + 8s + 20$, $q_1(s) = s^3 - 8s^2 - 10.6s$ and $q_2(s) = 0$. The next step is to form the Hurwitz matrices, and is obtained as:

$$\mathbf{H}(q_0) = \begin{bmatrix} 0 & 8 & 0 & 0 \\ 0 & 1 & 20 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 1 & 20 \end{bmatrix}, \mathbf{H}(q_1) = \begin{bmatrix} 1 & -10.6 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 1 & -10.6 & 0 \\ 0 & 0 & -8 & 0 \end{bmatrix}$$

and $\mathbf{H}(q_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Now, we can form the matrices **U** and **V** as follows:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 8 & 0 & 0 & 1 & -10.6 & 0 & 0 \\ 0 & 1 & 20 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 1 & -10.6 & 0 \\ 0 & 0 & 1 & 20 & 0 & 0 & -8 & 0 \end{bmatrix}$$

As a result, $\mathbf{\Gamma}(\lambda) =$ 0, 1.0232 and 0.0922. Therefore, $T = 0.0922$ and $T = 1.0232$ are used. The later value of T results in eigenvalues = 7.0227, 1.6683 and -1.6683, respectively. The former one gives $0.00003 \pm j8.7281$ and -2.8477 as eigenvalues. We could say that $0.00003 \pm j8.7281 \approx \pm j8.7281 \left(\pm j\omega_{ck}\right)$, and hence, $T_{ck} = 0.0922$ and $\omega_{ck} = \pm 8.7281$ rad/s. Finally, the maximum allowable delay is obtained as $\overline{\tau} = 155ms$, which is equal to that obtained previously.

Example 2: State-D feedback

Consider the same system as the previous case. Now, the system is stabilized by using state-D feedback. The gain matrix $\mathbf{K_d} = \begin{bmatrix}-7.5 & 3.1\end{bmatrix}$ is obtained by applying the design method in (Sujitjorn and Wiboonjaroen, 2011). The closed-loop system is stable with $\tau = 0$ s. Applying the proposed direct method, we obtain $T_{ck} = 0.0399s$ as the critical time interval, and $\omega_{ck} = \pm 5.4178$ rad/s as the imaginary-axis crossover frequencies. Therefore, the maximum delay is $\overline{\tau} = 78.6$ ms. Next, we demonstrate the calculation based on the matrix pencil method. To keep our presentation short, only important data are given as follows:

$\overline{CE}(s,T) = q_0(s) + q_1(s)T + q_2(s)T^2$, $q_0(s) = s^2 + 8.34s + 20.88$, $q_1(s) = s^3 - 1.07s^2 + 2.65s$ and $q_2(s) = 0$. The Hurwitz matrices are:

$$\mathbf{H}(q_0) = \begin{bmatrix} 0 & 8.34 & 0 & 0 \\ 0 & 1 & 20.88 & 0 \\ 0 & 0 & 8.34 & 0 \\ 0 & 0 & 1 & 20.88 \end{bmatrix}, \mathbf{H}(q_1) = \begin{bmatrix} 1 & 2.65 & 0 & 0 \\ 0 & -1.07 & 0 & 0 \\ 0 & 1 & 2.65 & 0 \\ 0 & 0 & -1.07 & 0 \end{bmatrix},$$

$\mathbf{H}(q_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; and

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 8.34 & 0 & 0 & 1 & 2.65 & 0 & 0 \\ 0 & 1 & 20.88 & 0 & 0 & -1.07 & 0 & 0 \\ 0 & 0 & 8.34 & 0 & 0 & 1 & 2.65 & 0 \\ 0 & 0 & 1 & 20.88 & 0 & 0 & -1.07 & 0 \end{bmatrix}.$$

$\Gamma(\lambda) =$ -1.2636, 0 and 0.0398 are obtained. Finally, $T_{ck} = 0.0398$, $\omega_{ck} = \pm 5.4288$ rad/s, $\overline{\tau} = 78.6$ ms.

Example 3: State-PI feedback

Consider the same open-loop system. The system can be stabilized via the state-PI feedback to achieve the closed-loop poles at $-4 \pm 2j$ and -5. The corresponding gain matrices are $\mathbf{K_p} = [18.79 \quad -52]$ and $\mathbf{K_i} = [-400 \quad -240]$. $T_{ck} = 0.0538s$, $\omega_{ck} = \pm 13.45$ rad/s and $\overline{\tau} = 93.1ms$ are obtained via using the proposed direct method. The matrix pencil approach results in,

$\overline{CE}(s,T) = q_0(s) + q_1(s)T + q_2(s)T^2$, $q_0(s) = s^3 + 13s^2 + 60s + 100$, $q_1(s) = s^4 - 13s^3 - 50.60s^2 \; -100s$ and $q_2(s) = 0$.

$$\mathbf{H}(q_0) = \begin{bmatrix} 1 & 60 & 0 & 0 \\ 0 & 13 & 100 & 0 \\ 0 & 0 & 60 & 0 \\ 0 & 0 & 13 & 100 \end{bmatrix}, \mathbf{H}(q_1) = \begin{bmatrix} -13 & -100 & 0 & 0 \\ 1 & -50.60 & 0 & 0 \\ 0 & -13 & -100 & 0 \\ 0 & 0 & -50.60 & 0 \end{bmatrix},$$

$$\mathbf{H}(q_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 60 & 0 & 0 & -13 & -100 & 0 & 0 \\ 0 & 13 & 100 & 0 & 1 & -50.60 & 0 & 0 \\ 0 & 0 & 60 & 0 & 0 & -13 & -100 & 0 \\ 0 & 0 & 13 & 100 & 0 & 0 & -50.60 & 0 \end{bmatrix},$$

$T_{ck} = 0.0537$, $\omega_{ck} = \pm 13.46$ rad/s, and $\overline{\tau} = 93.1$ ms.

Example 4: State-PD feedback

The same system as of Example 1 is stabilized via the state-PD feedback method (Sujitjorn and Wiboonjaroen, 2011). The PD-gain matrices are $\mathbf{K_p} = [-61.2 \quad 0]$ and $\mathbf{K_d} = [-32 \quad 0]$. Applying the proposed direct method, we obtain $T_{ck} = 0.0922s$, $\omega_{ck} = \pm 8.7275$ rad/s and $\overline{\tau} = 155ms$. Similar situations to those of the previous cases can be observed.

As a result of applying the matrix pencil method, we obtain:

$q_0(s) = s^2 + 8s + 20$, $q_1(s) = s^3 - 8s^2 - 10.60s$, $q_2(s) = 0$,

$$\mathbf{H}(q_0) = \begin{bmatrix} 0 & 8 & 0 & 0 \\ 0 & 1 & 20 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 1 & 20 \end{bmatrix}, \mathbf{H}(q_1) = \begin{bmatrix} 1 & -10.60 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 1 & -10.60 & 0 \\ 0 & 0 & -8 & 0 \end{bmatrix},$$

$$\mathbf{H}(q_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 8 & 0 & 0 & 1 & -10.60 & 0 & 0 \\ 0 & 1 & 20 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 1 & -10.60 & 0 \\ 0 & 0 & 1 & 20 & 0 & 0 & -8 & 0 \end{bmatrix},$$

$T_{ck} = 0.0922$, $\omega_{c_k} = \pm 8.7281$ rad/s, and $\overline{\tau} = 155$ ms.

Example 5: State-PID feedback

The state-PID feedback method is applied to stabilize the same system, and the proposed direct method is applied to obtain the maximum allowable delay. As a result, the following data are obtained:

$\mathbf{K_p} = [18.794 \quad 0]$, $\mathbf{K_i} = [-400 \quad -240]$, $\mathbf{K_d} = [-52 \; , \; 0]$, $T_{ck} = 0.0538s$, $\omega_{ck} = \pm 13.4500$ rad/s and $\overline{\tau} = 93.1ms$. The

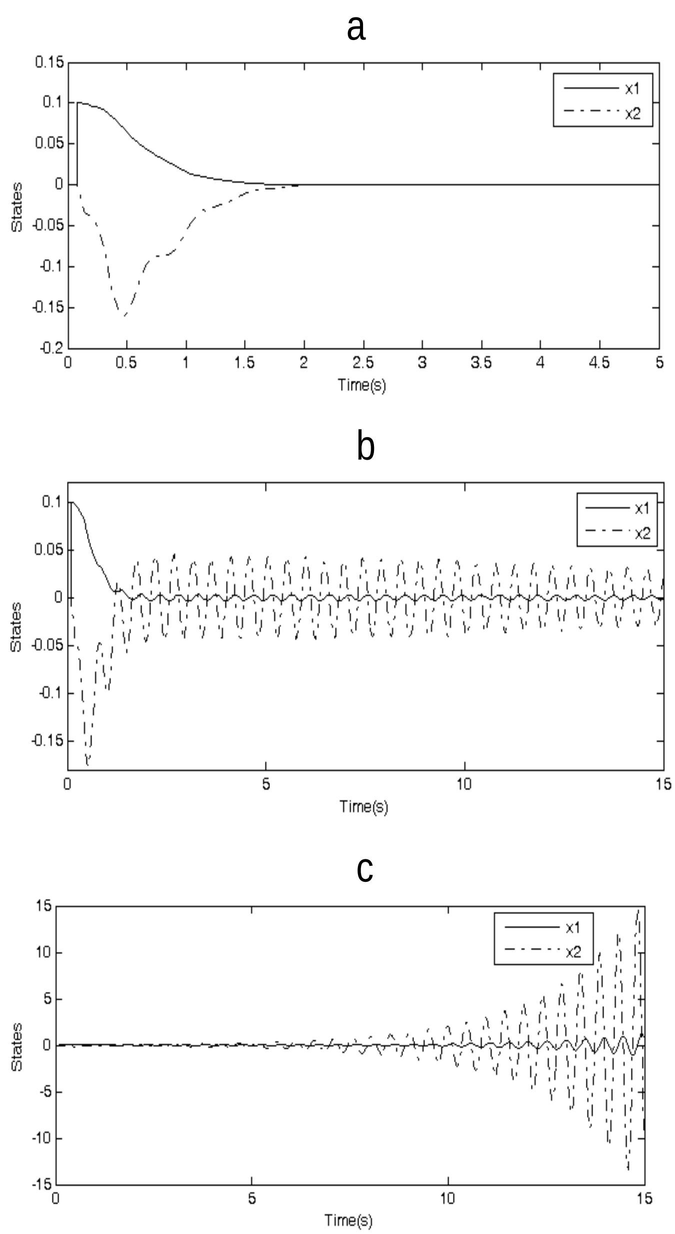

Figure 2. State responses of Example 5 delay time: (a) $\tau = 75ms < \bar{\tau}$, (b) $\tau = 93.1ms = \bar{\tau}$ and (c) $\tau = 98ms > \bar{\tau}$.

similar figures are obtained from using the matrix pencil method. Figure 2a illustrates stable responses, while Figure 2b and c illustrates unstable ones.

At this stage, we can conclude that the proposed direct method is exact, and gives the same accuracy as the existing matrix pencil method does. The calculation procedures are quite different. The matrix pencil approach needs some knowledge of matrix algebra and numerical computation. The direct method proposed needs only basic knowledge of Routh's criterion and loop iterative computing commonly taught in undergraduate level.

Example 6

This example serves to demonstrate the effectiveness of using a low pass filter to stabilize a time-delayed system. We show by numerical example that a system can be more robust to a delay by judicious selection of the DC gain and the time-constant of the filter.

Consider the situation of Example 5 in which $T_{ck} = 0.0538s$, and $\bar{\tau} = 93.1ms$. The characteristic polynomial $\overline{CE}(s, T_{ck})$ can be formulated as:

$$\overline{CE}(s, T_{ck}) = b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0$$

where $b_5 = T_f$, $b_4 = 18.59T_f + 1$, $b_3 = 4.70T_f$, $-13K_f + 18.58$, $b_2 = 87.33T_f + 186.33K_f + 4.70$, $b_1 = 9.28K_f + 87.33$ and $b_0 = 1858.74K_f$. Based on the Routh's criterion, one can derive stability conditions as follows:

$T_f > 0$, $18.58T_f + 1 > 0$,

$$\frac{215T_f^2 + (1727011\text{-}2140000\,K_f)\,T_f - 65000K_f + 92900}{92950\,T_f + 5000} > 0,$$

$$[(T_f(79749240000\,K_f^2 - 62725571526K_f + 947660) - 3391631400K_f$$
$$+ T_f^2(6023052378\,K_f + 16032547) - 3755190T_f^3 + 2422290000K_f^2$$
$$+ 4000)/(-43000T_f^2 + (428000000K_f - 345402200)T_f + 13000000K_f - 18580000) > 0,$$

$$[(34546258963800K_f + T_f\,(1108187974720000K_f^3$$
$$-1687522386738128K_f^2 + 645051990274042K_f + 8275914780)$$
$$+ T_f^2(-54999431947216K_f^2 + 52748872305290K_f$$
$$+ 140012232951) - 71783654969200K_f^2$$
$$+ 336599151200000K_f^3 + T_f^3(34079510680K_f - 32794074270)$$
$$+ 34932000)/(T_f(7974924000000K_f^2 - 6272557152600K_f$$
$$+ 94766000) - 339163140000k_f - 3755190000T_f^3$$
$$+ 242229000000K_f^2 + T_f^2(602305237800K_f + 1603254700)\text{ , } + 400000)] > 0$$

and

$185.87K_f > 0$.

The filter gain and the time-constant must be positive. Select K_f=0.5 (arbitrary), therefore the aforementioned stability conditions reduce to $0 < T_f < 0.0317$. Let $T_f = 0.001$ and the delay be $\bar{\tau} = 93.1ms$, Figure 3a

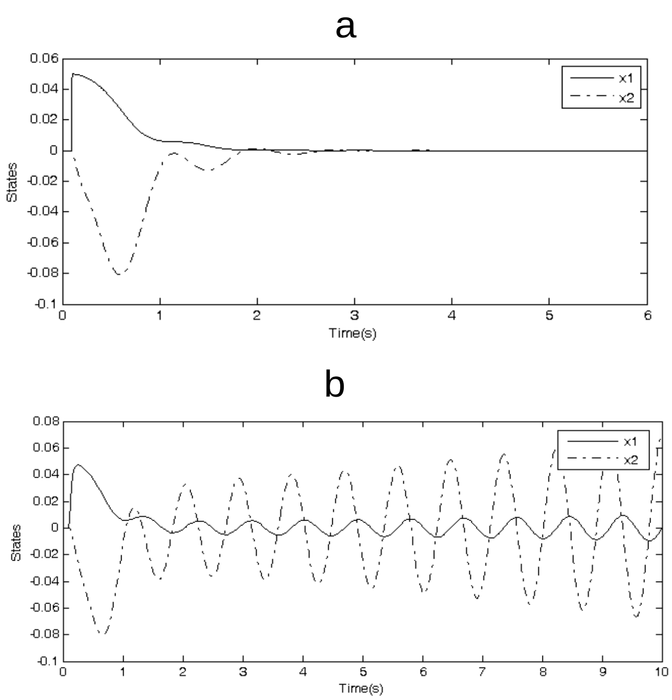

Figure 3. State responses of Example 6. (a) $T_f = 0.001$ and (b) $T_f = 0.04$

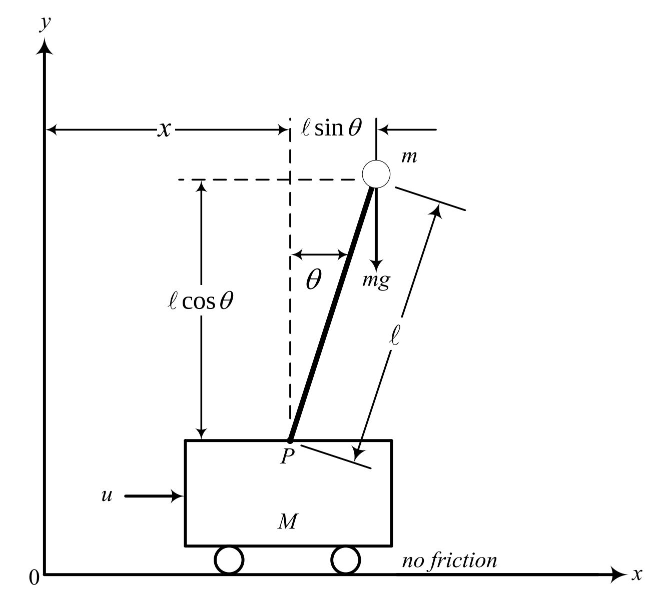

Figure 4. Inverted pendulum on cart.

illustrates the state responses of this stabilized case. If we choose $T_f = 0.04$, which defies the stability condition, the system is obviously unstable as shown by the responses in Figure 3b.

Example 7

An inverted pendulum system is adopted as an example and represented by the diagram as shown in Figure 4. Its state model is expressed by:

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix} u,$$

where $x_1 = \theta, x_2 = \theta', x_3 = x$ and $x_4 = x'$. With its poles at $0, 0$ and ± 4.5388, the system is inherently unstable. It is desirable to place the closed-loop poles at $-2 \pm 3.464j, -4, -10$ and -10. The state-PID feedback method is applied to stabilize the system, and the proposed direct method is applied to obtain the maximum allowable delay. As a result, the following data are obtained:

$\mathbf{K_p} = [-20.601\ 0\ 0\ 0]$, $\mathbf{K_I} = [7123.1490\ 1490.2386\ 1956.3781$, $1043.4560]$, $\mathbf{K_d} = [-120.6720\ \text{-}24.6841$, $\text{-}313.0368\ \text{-}49.3195]$, $T_{ck} = 0.0236$, $\omega_{ck} = \pm 27.62$ rad/s and $\bar{\tau} = 41.8ms$. The matrix pencil method fails to provide useful numerical data since the matrix $\mathbf{\Gamma}$ is singular. Figure 5a shows the unstable responses corresponding to the delay of 41.8 ms. By selecting K_f=0.5, we obtain the stability condition $0 < T_f < 0.0083$. Figure 5b depicts the stable responses when the delay is $\bar{\tau} = 41.8ms$, and $T_f = 0.0001$ (arbitrarily selected). When the filter time-constant is $T_f = 0.014$, which is out of the stable region, the system becomes unstable as demonstrated by the responses in Figure 5c.

Conclusions

This article has presented a new approach to compute the maximum allowable delay in a LTI-TDS that incorporates state-PID feedback. The proposed approach, referred to as the direct method utilizes the exact bilinear transformation known as Rekasius substitution to represent the transcendental term, and the Routh's stability criterion. Computational results are compared with those obtained from the matrix pencil method. It was found out that both methods have similar accuracies. Stabilization of a system with delay via a low pass filter is also elaborated. As shown by one control design example, the matrix pencil method cannot provide useful information, because one of the matrices involved in the

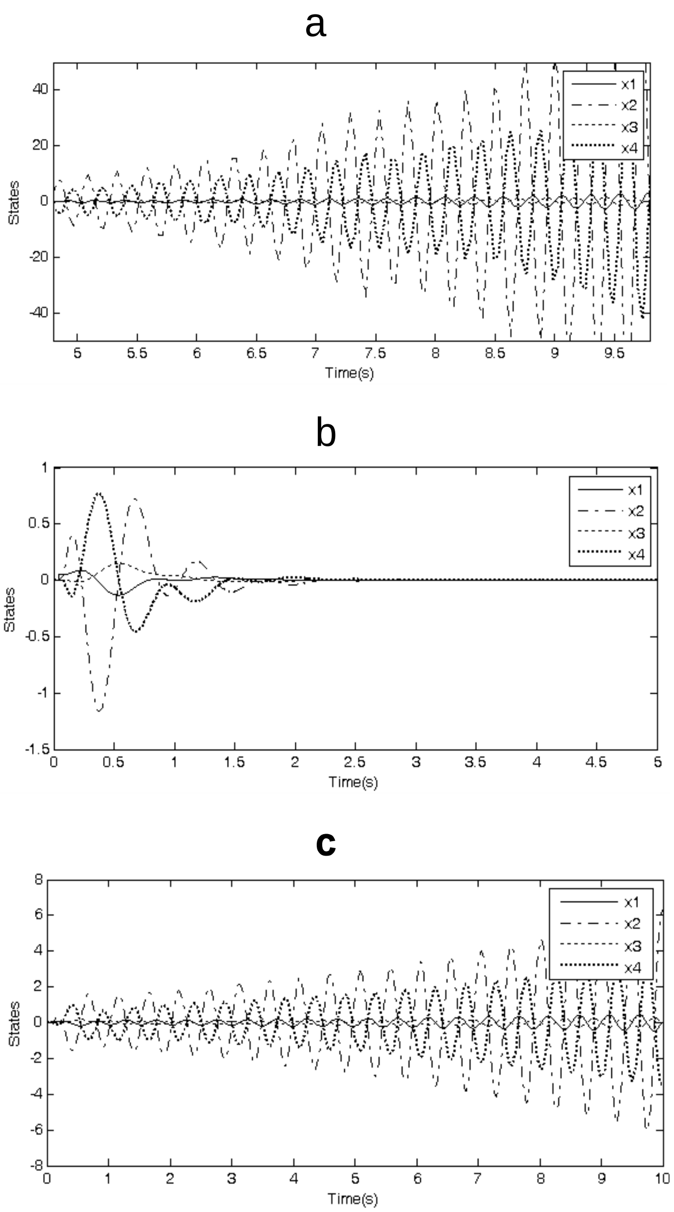

Figure 5. State responses of example 7 when $\overline{\tau} = 41.8ms$ (a) no filter, (b) filter time-constant $T_f = 0.0001$ and (c) $T_f = 0.014$.

calculation is singular. The direct method is successful with this case. Seven examples illustrated in this article confirm the effectiveness of the proposed approach.

ACKNOWLEDGEMENTS

This work was supported by Suranaree University of Technology (SUT), the Office of the Higher Education Commission under NRU project of Thailand and Ratchamangkala University of Technology Isarn, Thailand.

REFERENCES

Asl FM, Ulsoy AG (2003). Analysis of a system of linear delay differential equations. J. Dyn. Syst, Meas. Contr., 125(2): 215-223.
Brethe D, Loiseau JJ (1998). An effective algorithm for finite spectrum assignment of single-input systems with delays. Math. Comput. Simul., 45: 339-348.
Chen J, Gu G, Nett CN (1995). A new method for computing delay margins for stability of linear delay systems. Syst. Cont. Let., 26(2): 107-117.
Fu P, Niculescu SI, Chen J (2006). Stability of linear neutral time-delay systems: Exact conditions via matrix pencil solutions. IEEE Trans. Auto. Cont., 51(6): 1063-1069.
Manitius A, Olbrot AW (1979). Finite spectrum assignment problem for systems with delays. IEEE Trans. Auto. Cont., 24: 541-553.
Michiels W, Engelborghs K, Vansevenant P, Roose D (2002). Continuous pole placement for delay equations. Automatica, 38(5): 747-761.
Niculescu SI (2001). Delay Effects on Stability: A Robust Control Approach. Springer. New York, pp. 3-46.
Normey-Rico JE, Camacho EF (2007). Control of Dead-time Process. Springer. London.
Olgac N, Sipahi R (2002). An exact method for the stability analysis of time-delayed linear time-invariant (LTI) systems. IEEE Trans. Auto. Cont., 47(5): 793-797.
Olgac N, Sipahi R (2004). A practical method for analyzing the stability of neutral type LTI-time delayed systems. Automatica, 40(5): 847-853.
Olgac N, Sipahi R (2005). The cluster treatment of characteristic roots and the neutral type time-delayed systems. Trans. ASME, 127(1): 88-97.
Rekasius ZV (1980). A stability test for systems with delays (TP9-A). In Proc. Joint Automatic Control Conference. San Franscisco. USA.
Richard JP (2003). Time-delay systems: An overview of some recent advances and open problems. Automatica, 39: 1667-1694.
Sipahi R, Olgac N (2003). Degenerate cases in using direct method. J. Dyn. Syst., Meas. Cont., 125(2): 194-201.
Sipahi R, Olgac N (2006). Stability robustness of retarded LTI system with single delay and exhaustive determination of their imaginary spectra. Siam J. Cont. Opt., 45(5): 1680-1696.
Sujitjorn S, Wiboonjaroen W (2011). State-PID feedback for pole placement of LTI system. Math. Prob. Eng., 929430-DOI, 20 pages.
Wang Q, Lee TH, Tan KK (1995). Automatic tuning of finite spectrum assignment controllers for delay systems. Automatica, 31: 477-482.
Yi S, Ulsoy AG, Nelson PW (2010). Eigenvalue assignment via the Lambert W function for control of time-delay system. J. Vibr. Contr., 16(7-8): 961-982.