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Short Communication

Finitisticness via filters

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In this paper we generalized Finitisticness of topological spaces via filters, introduced the concept of F-finitistic space and studied its various basic properties.

Key words: F- finitistic space, topological space, open refinement, filter.

INTRODUCTION AND PRELIMINARIES

The concept of finitistic space in general topology was introduced in 1960 by Swan (1960). The purpose of considering these spaces was to simplify and extend Borel's proof (Borel, 1955) of the classical Smith's Fixed point theorems for spheres (Smith, 1938, 1939, 1941, 1945). Of course Swan did not name these spaces as finitistic. The term "Finitistic" was used by Bredon in his Book (Bredon, 1972) and since than it has become a firmly established term. In general topology the study of finitistic spaces became interesting with the basic paper of Deo and Tripathi (1982) in which they proved a Characterization Theorem for non-finitistic paracompact spaces. This Theorem is known as Deo-Tripathi Theorem. Several works appeared there after (Deo and Tripathi, 1982) in this field. Almost all the possible properties of finitistic spaces in general topology have been studied and concepts like countably finitistic, locally finitistic and completely finitistic spaces have been introduced and their properties were almost settled completely.

A nonempty family F of nonempty subsets of X is said to be a Filter (Willard, 1970) on X if (i) $A\in F$ and $A\subset B\Rightarrow B\in F$ (ii) A, $B\in F \Rightarrow A\cap B\in F$. The order (Pears, 1975) of a family $\{U_\lambda:\lambda\in\Delta\}$ of subsets, not all empty, of some set X is the largest integer n for which there exists a subset M of Δ with n+1 elements such that $\cap_{\lambda\in M}U_\lambda$ is nonempty, or is ∞ if there is no such largest integer. A family of empty subsets has order -1. A Topological Space (X, T) is said to be finitistic (Bredon, 1972). If each open cover of (X, T) has a finite order open refinement.

Let X be a nonempty set. An ideal on X is a nonempty collection I of subsets of X such that (i) $U\in I$ and $V\subset U \Rightarrow V\in I$ (ii) U, $V\in I\Rightarrow U\cup V\in I$. A topological space (X, T) is said to be I-compact (Rancin, 1972) if for every open cover $\mu = \{U_\lambda:\lambda\in\Delta\}$ of (X, T) there exists a finite subfamily $\nu = \{V_1, V_2, V_3, \ldots, V_k\}$ of μ such that $(\cup_{i=1}^{k}V_\alpha)' \in I$. A topological space (X, T) is said to be I-Finitistic (Ahmed, 2008). where I is an ideal on X if for each open cover $\mu = \{U_\lambda:\lambda\in\Delta\}$ of (X,T) there exists a finite order subfamily $\nu = \{V_\alpha:\alpha\in\Delta_1\}$ of T such that $(\cup_{\alpha\in\Delta_1}V_\alpha)'\in I$ and for each $V_\alpha\in\nu$ there exists some $U_\lambda\in\mu$ such that $V_\alpha\subset U_\lambda$. Let (X, T) be a topological Space and F be a filter on X. (X,T) is said to be F-compact (Ahmed, 2008) if for every open cover μ of (X,T), there exists a finite subfamily $\nu = \{U_1, U_2, U_3, \ldots, U_k\}$ of μ such that $X - (\cup_{i=1}^{k}U_i)\notin F$.

F- Finitistic space

Definition 1

Let (X, T) be a topological space and F be a filter on X. (X, T) is said to be F-Finitistic if for each open cover μ of (X, T) there exists a finite order subfamily $\nu = \{V_\alpha:\alpha\in\Delta\}$ of open subsets (X, T) such that $X - (\cup_{\alpha\in\Delta}V_\alpha)\notin F$ and for each $V_\alpha\in\nu$ there exists some $U_\lambda\in\mu$ such that $V_\alpha\subset U_\lambda$.

Theorem 1

Every finitistic space is F-Finitistic.

Abbreviation: AMS, Mathematics subject classification: 54A10, 54D10.

Proof

Let (X, T) be a finitistic topological space and F be a filter on X. We have to show that (X, T) is F-Finitistic. Let μ be any open cover of (X, T). Since (X, T) is finitistic, therefore μ has a finite order open refinement say $\nu = \{V_\alpha:\alpha\in\Delta_1\}$. Since $\nu = \{V_\alpha:\alpha\in\Delta_1\}$ is an open cover of (X, T), therefore $X - (\cup_{\alpha\in\Delta1}V_\alpha) = \varnothing\notin F$. Also clearly for each $V_\alpha\in\nu$ there exists some $U_\lambda\in\mu$ such that $V_\alpha\subset U_\lambda$ because ν is a refinement of μ. Hence (X, T) is F-Finitistic.

Remark 1

Converse of above theorem is not true. See following example:

Example 1

Let X = N (Set of all natural numbers) and $T_n = \{\varnothing\} \cup\{A\subset N: n\in A\}$. Then clearly (X, T_n) is a topological space. Let $F_n = \{A\subset N: n\in A\}$. Then clearly F_n is a Filter on X. Here (X, T_n) is not finitistic because the open cover {[n-1, n+1], [n-2, n+2], [n-3, n+3], [n-4, n+4], [n-5, n+5],…} of (X, T_n) has no finite order open refinement. We show that (X, T) is F_n-Finitistic. Let μ be any open cover of (X, T). Then there exists some $U_n\in\mu$ such that $n\in U_n$. Then $\nu = \{\varnothing, U_n\}$ is a zero order subfamily of T_n such that $X - \varnothing \cup U_n = X - U_n \notin F_n$ and clearly for each $V_\alpha\in\nu$ there exists some $U_\lambda\in\mu$ such that $V_\alpha \subset U_\lambda$. Hence (X, T) is F-Finitistic.

Note that here [n-1, n+1] = {$x\in N$ such that $n-1\leq x \leq n+1$}

Theorem 2

Every F- Compact topological space is F- Finitistic.

Remark 2

If (X, T) is a discrete or an indiscrete topological space, then (X, T) is F-Finitistic where F is any Filter on X.

Remark 3

If (X, T) is a topological space where X is finite or T is finite, then (X, T) is F-Finitistic where F is any Filter on X.

Theorem 3

Let (X, T) be a topological space and F_1, F_2 be two Filters on X such that $F_1\subset F_2$. Then (X, T) is F_2-Finitistic $\Rightarrow$(X, T) is F_1-Finitistic.

Theorem 4

F-Finitisticness is a topological Property.

Proof

Let f: $(X, T_1) \rightarrow (Y, T_2)$ be a homeomorphism and (X, T_1) is F-Finitistic Space. Here F is a Filter on X. It can be easily shown that $f(F) = \{f(A) : A\in F\}$ is a Filter on Y. We have to show that (Y, T_2) is F- Finitistic .Let μ be any open cover of (Y, T_2). Then it can be easily checked that $\nu = \{f^{-1}(U_\lambda): U_\lambda\in\mu\}$ is an open cover of (X, T_1).Since (X, T_1) is F-Finitistic, there exists a finite order subfamily say $\nu = \{V_\alpha:\alpha\in\Delta_1\}$ of T_1 such that $X - (\cup_{\alpha\in\Delta1}V_\alpha)\notin F$ and for each $V_\alpha\in\nu$ there exists some $f^{-1}(U_\lambda)\in\mu$ such that $V_\alpha\subset f^{-1}(U_\lambda)$. Let $X - (\cup_{\alpha\in\Delta1}V_\alpha) = A\notin F$. Then $f(A) \notin f(F)$. Since f: $(X,T_1) \rightarrow (Y,T_2)$ is a homeomorphism, therefore it can be easily seen that $\nu_1 = \{f(V_\alpha): V_\alpha\in\nu\}$is a finite order subfamily of T_2 such that $Y - (\cup \nu_1) \notin f(F)$ and for each $f(V_\alpha)\in\nu$ there exists some $U_\lambda\in\mu$ such that $f(V_\alpha)\subset U_\lambda$. Hence (Y, T_2) is f (F) - Finitistic.

Remark 4

The above result is not true for continuous functions. See the following example:

Example 2

Let X be an infinite set. Let T be a topology on X such that (X, T) is not F-Finitistic. Let D be the discrete topology on X. Then clearly (X, D) is F-Finitistic. Define f: (X, D) $\rightarrow$ (X, T) as f(x) = x for all $x\in X$. Then clearly f: (X, D) $\rightarrow$ (X, T) is a continuous function. Here (X, T) is continuous image of (X, D). Here (X, D) is F-finitistic but (X, T) is not F-Finitistic.

Theorem 5

Every closed subspace of F-Finitistic Space is F-Finitistic.

Proof

Let (X, T) be an F-Finitistic topological space and $(Y, T|_Y)$ be a closed subspace of (X, T). We have to show that $(Y, T|_Y)$ is F- Finitistic. Let μ be any open cover of $(Y, T|_Y)$.

Then for each $V_\lambda\in\mu$, there exists some $U_\lambda\in T$ such that $V_\lambda = U_\lambda\cap Y$. Since $(Y, T|_Y)$ is a closed subspace of (X, T), therefore $Y'\in T$. Clearly $\nu = \{U_\lambda: V_\lambda = U_\lambda\cap Y$ and $V_\lambda\in\mu\}$

$\cup\{Y'\}$ is an open cover of (X, T). Since (X, T) is F – Finitistic, there exists a finite order subfamily $\nu_1 = \{W_\alpha : \alpha \in \Delta_1\}$ of T such that $X - (\cup_{\alpha\in\Delta 1} W_\alpha) \notin F$ and for each $W_\alpha \in \nu$ there exists some $S_\lambda \in \nu$ such that $W_\alpha \subset S_\lambda$ .Then clearly $\mu_1 = \{ W_\alpha \cap Y : W_\alpha \in \nu_1 \}$is a finite order subfamily of $T|_Y$ such that $Y - (\cup_{W\alpha\cap Y} \in \mu_1 (W_\alpha \cap Y)) \notin F$ and for each $W_\alpha \cap Y \in \mu_1$ there exists some $V_\lambda \in \mu$ such that $W_\alpha \cap Y \subset V_\lambda$. Hence $(Y, T|_Y)$ is F- Finitistic.

Theorem 6

Let (X, T) be any topological space. Let I and F be respectively Ideal and Filter on X such that $I \cup F = P(X)$ and $I \cap F = \emptyset$. Then (X, T) is I-Finitistic if and only if (X, T) is F-Finitistic.

Proof

Let (X, T) be I-Finitistic. Let μ be any open cover of (X, T). Since (X, T) is I-Finitistic, there exists a finite order subfamily say $\nu = \{V_\alpha : \alpha \in \Delta_1\}$ of T such that $X - (\cup_{\alpha\in\Delta 1} V_\alpha) \in I$. Also for each $V_\alpha \in \nu$ there exists some $U_\lambda \in \mu$ such that $V_\alpha \subset U_\lambda$. Since $I \cup F = P(X)$ and $I \cap F = \emptyset$, therefore $X - (\cup_{\alpha\in\Delta 1} V_\alpha) \in I \Rightarrow X - (\cup_{\alpha\in\Delta 1} V_\alpha) \notin F$. Hence (X, T) is F-Finitistic.

Converse

Suppose (X, T) is F-Finitistic. Let μ be any open cover of (X, T). Since (X, T) is F-Finitistic, there exists a finite order subfamily say $\nu = \{V_\alpha : \alpha \in \Delta_1\}$ of T such that $X - (\cup_{\alpha\in\Delta 1} V_\alpha) \notin F$. Also for each $V_\alpha \in \nu$ there exists some $U_\lambda \in \mu$ such that $V_\alpha \subset U_\lambda$. Since $I \cup F = P(X)$ and $I \cap F = \emptyset$, therefore $X - (\cup_{\alpha\in\Delta 1} V_\alpha) \notin F \Rightarrow X - (\cup_{\alpha\in\Delta 1} V_\alpha) \in I$. Hence (X, T) is I-Finitistic.

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