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Full Length Research Paper

A new operational matrix for Legendre wavelets and its applications for solving fractional order boundary values problems

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In this article, we generalize the Legendre wavelets operational matrix of derivatives to fractional order derivatives in the Caputo sense. Legendre wavelets and their properties are employed for deriving Legendre wavelets operational matrix of fractional derivatives and a general procedure for forming this matrix is introduced. Then truncated Legendre wavelets expansions together with these matrices are used for numerical solution of Bagley–Torvik fractional order boundary value problems. Several examples are included to demonstrate accuracy and applicability of the proposed method.

Key words: Shifted Legendre polynomials, Legendre wavelets, Caputo derivative, fractional order boundary value problems.

INTRODUCTION

The idea of derivatives of non integer order initially appeared in a letter from Leibniz to L'Hospital in 1695. For three centuries, studies on the theory of fractional order were mainly constraint to the field of pure theoretical mathematics, which were only useful for mathematicians. In the last several decades, many researchers found that derivatives of non-integer order are very suitable for the description of various physical phenolmena such as damping laws, diffusion process, etc. These findings evoked the growing interest of studies of fractional calculus in various fields such as physics, chemistry and engineering. For these reasons, we need reliable and efficient techniques for the solution of fractional differential equations (Li and Zhao, 2010; Miller and Ross, 1993; Oldham and Spanier, 1974; Podlubny, 1999; Saadatmandia and Dehghan, 2010).

The existence and uniqueness of solutions for fractional differential equations have been investigated by many authors such as Podlubny (1999). Most fractional differential equations do not have exact analytic solution, therefore approximation and numerical techniques must be used. In the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian's decomposition method (Gejji and Jafari, 2007; Momani and Noor, 2006; Momani and Shawagfeh, 2006; Ray et al., 2006); He's variational iteration method (Momani and Odibat, 2006; Odibat and Momani, 2006); homotopy perturbation method (Momani and Odibat, 2007; Sweilam et al., 2007); homotopy analysis method (Hashim et al., 2009); collocation method (Al-Mdallal et al., 2010; Rawashdeh, 2006); Galerkin method (Ervin and Roop, 2005) and other methods (Kumar and Agrawal, 2006; Yuste, 2006).

Wavelets theory is a relatively new and emerging area in mathematical research. As a powerful tool, wavelets have been extensively used in signal processing, numerical analysis and many other areas. Wavelets permit the accurate representation of a variety of functions and operators. In this paper, The Legendre Wavelets are first introduced, then by using shifted Legendre polynomial and their properties, the operational matrix of derivative and fractional derivative are derived. Then, applications of these matrices for solving Bagley–Torvik fractional order boundary value problems are described. Illustrative

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examples are given to demonstrate the efficiency and capability of the proposed method.

The article is organized as follows: Subsequently, this study introduces some necessary definitions and mathematical preliminaries of the fractional calculus theory and Legendre wavelets which are required for establishing the results of this study. This was followed by establishing the Legendre operational matrix of derivatives, and the fractional derivatives are derived and the general procedures for forming these matrices are introduced, after which Bagley–Torvik fractional order boundary value problems are introduced and then a method based on Legendre wavelet and its operational matrices are established for solving these fractional boundary value problems. Finally, the study is concluded.

PRELIMINARIES AND NOTATIONS

In this section we give some necessary definitions and mathematical preliminaries of the fractional calculus theory, Legendre polynomials and wavelets which are required for establishing our results (Canuto et al., 1988; Li and Zhao, 2010; Oldham and Spanier, 1974; Podlubny, 1999; Saadatmandia and Dehghan, 2010).

The fractional derivative

Here, we introduce a fractional differential operator D^{α} proposed by Caputo in his work on the theory of viscoelasticity.

Definition 1

A real function $f(x), x>0$, is said to be in the space $C_{\mu}, \mu \in R$ if there exists a real number $p>\mu$ such that $f(x)=x^{p} f_{1}(x)$, where $f_{1}(x) \in C[0, \infty)$, and it is said to be in the space C_{μ}^{m} iff $f^{m}(x) \in C_{\mu}, m \in N$.

Definition 2

The fractional derivative of $f(x)$, in the Caputo sense is defined as:

$$D^{\alpha} f(x)=J^{m-\alpha} f^{(m)}(x)=\frac{1}{\Gamma(m-\alpha)} \int_{0}^{x}(x-t)^{m-\alpha-1} f^{(m)}(t) d t, \quad (1)$$

for $m-1<\alpha \leq m, m \in N, x>0, f \in C_{-1}^{m}$.

For the Caputo derivative we have:

$$D^{\alpha} x^{\beta}=\begin{cases}0 & \beta \leq[\alpha] \text { and } \beta \in \mathbf{Z}^{+} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} & \beta>[\alpha] \text { and } \beta \in \mathbf{Z}^{+}\end{cases} \quad (2)$$

In which $[\alpha]$ denotes the smallest integer greater than or equal to α.

Lemma 1

if $m-1<\alpha \leq m, m \in N$, and $f \in C_{\mu}, \mu \geq-1$, then

$$D^{\alpha} J^{\alpha} f(x)=f(x) \quad (3)$$

and

$$J^{\alpha} D^{\alpha} f(x)=f(x)-\sum_{k=0}^{m-1} f^{(k)}\left(0^{+}\right) \frac{x^{k}}{k!}, \quad x>0. \quad (4)$$

Miller and Ross (1993), Oldham and Spanier (1974), Podlubny (1999) and Saadatmandia and Dehghan (2010) show more details on the mathematical properties of fractional derivatives and integrals.

Shifted Legendre polynomials and their properties

The well-known Legendre polynomials are defined on the interval [-1, 1] and can be determined with the aid of the following recurrence formulae (Canuto et al., 1988):

$$(m+1) L_{m+1}(t)=(2 m+1) t L_{m}(t)-m L_{m-1}(t), \quad m=1,2,3, \ldots, \quad (5)$$

where $L_{0}(t)=1, \ L_{1}(t)=t$.

In order to use Legendre polynomials on the interval [0,1] we define the so-called shifted Legendre polynomials by introducing the change of variable $t = 2x - 1$. Let the shifted Legendre polynomials $L_m(2x-1)$ be denoted by $P_m(x)$. Then $P_m(x)$ can be obtained as follows:

$$(m+1) P_{m+1}(x)=(2 m+1)(2 x-1) P_{m}(x)-m P_{m-1}(x), \quad m=1,2,3, \ldots,$$

where $P_{0}(x)=1$ and $P_{1}(x)=2 x-1$.

The analytic form of the shifted Legendre polynomial $P_{m}(x)$ of degree m can be expressed as

$$P_{m}(x)=\sum_{k=0}^{m}(-1)^{m+k} \frac{(m+k)!}{(m-k)!} \frac{x^{k}}{(k!)^{2}} \quad (6)$$

The orthogonality condition for these polynomials is

$$\int_{0}^{1} P_{m}(x) P_{n}(x) d x=\begin{cases}\dfrac{1}{2 m+1} & \text{for } m=n, \\ 0 & \text{for } m \neq n.\end{cases} \quad (7)$$

In the next theorem we derived a relation between shifted Legendre wavelets and their derivatives that is very important for deriving the operational matrix of the derivative for the Legendre wavelet.

Theorem 1

Let $P_m(x)$ be the shifted Legendre polynomials into [0, 1] and $P_m^{'}(x)$ be derivative of $P_m(x)$ with respect to x. Then we have

$$P_m^{'}(x) = 2\sum_{\substack{k=0 \\ k+m\ odd}}^{m-1} (2k+1)P_k(x) \quad (8)$$

Proof

Suppose the Legendre expansion of function $u(x)$ be

$$u(x) = \sum_{k=0}^{\infty}\hat{u}_k L_k(x), \quad (9)$$

then $u^{'}(x)$ can be represented as (Canuto et al., 1988)

$$u^{'}(x) = \sum_{k=0}^{\infty}\hat{u}_k^{(1)} L_k(x) \quad (10)$$

where

$$\hat{u}_k^{(1)} = (2k+1)\sum_{\substack{p=k+1 \\ p+k\ odd}}^{\infty} \hat{u}_p, \qquad k \geq 0 \quad (11)$$

Now, by taking $u(x) = L_m(x)$ in Equation 10 we have $\hat{u}_m = 1$ and $\hat{u}_i = 0$ for $i \neq m$. Consequently

$$\hat{u}_k^{(1)} = \begin{cases} 2k+1 & m+k \quad is\, odd\,, \ \ k \leq m-1 \\ 0 & otherwise \end{cases} \quad (12)$$

Now Equation 6 implies that

$$L_m^{'}(x) = \sum_{\substack{k=0 \\ m+k\ odd}}^{m-1} \left(2k+1\right)L_k(x). \quad (13)$$

By substituting $x = 2t-1$ in Equation 8 we have

$$P_m^{'}(t) = 2\sum_{\substack{k=0 \\ m+k\ odd}}^{m-1} \left(2k+1\right)P_k(t). \quad (14)$$

This proves the desired result.

Wavelets and Legendre wavelets

In recent years, wavelets have found their way into many different fields of science and engineering (Hosseini et al., 2011; Mohammadi and Hosseini, 2010, 2011; Razzaghi and Yousefi, 2001). Wavelets constitute a family of functions constructed from the dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right) \ a, b \in R, a \neq 0 \quad (15)$$

Legendre wavelets $\psi_{nm}(t) = \psi(k, n, m, t)$ have four arguments; n, k can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval $[0,1)$ by;

$$\psi_{nm}(t) = \begin{cases} \sqrt{(m+1/2)}\,2^{\frac{k+1}{2}} L_m(2^{k+1}t - (2n+1)) & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0 & otherwise \end{cases} \quad (16)$$

where $m = 0,1,\ldots,M$ and $n = 0,1,\ldots,2^k - 1.$

The coefficient $\sqrt{(m+1/2)}$ is for orthonormality. Here $L_m(t)$ are the well-known Legendre polynomials of order m, which have been previously described. A function $f(t)$ defined over [0, 1) can be expanded in the terms of Legendre wavelets as;

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t) \quad (17)$$

where $c_{nm} = (f(t), \psi_{nm}(t))$, and $(.,.)$ denotes the inner product.

If the infinite series in Equation 17 is truncated, then it can be written as

$$f(t) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t) \quad (18)$$

where C and $\Psi(t)$ are $2^k(M+1)\times 1$ matrices given by

$$C = [c_{0,0}, c_{0,1}, \ldots, c_{0,M}, \ldots, c_{2,M}, \ldots, c_{(2^k-1),0}, c_{(2^k-1),1}, \ldots, c_{(2^k-1),M}]^T$$
$$\Psi(t) = [\psi_{0,0}, \psi_{0,1}, \ldots, \psi_{0,M}, \ldots, \psi_{(2^k-1),M}, \ldots, \psi_{(2^k-1),0}, \psi_{(2^k-1),1}, \ldots, \psi_{(2^k-1),M}]^T$$
(19)

OPERATIONAL MATRICES OF DERIVATIVE AND FRACTIONAL DERIVATIVE

In the following study we introduce a new method for deriving Legendre wavelet operational matrix of derivative and fractional derivative.

Theorem 2

Let $\Psi(t)$ be the Legendre wavelets vector defined in Equation

19. The derivative of the $\Psi(t)$ can be expressed by

$$\frac{d\Psi(t)}{dt} = D\Psi(t) \quad (20)$$

where D is the $2^k(M+1)$ operational matrix of derivative defined thus:

$$D = \begin{pmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & F \end{pmatrix}, \quad (21)$$

where F is $(M+1)\times(M+1)$ matrix and its (r,s)th element is thus:

$$F_{r,s} = \begin{cases} 2^{k+1}\sqrt{(2r-1)(2s-1)} & r = 2,...,(M+1),\ s = 1,...,r-1 \ and\ (r+s) odd \\ 0 & othewise \end{cases}$$
(22)

Proof

By using shifted Legendre polynomial into $[0,1]$ the rth element of vector $\Psi(t)$ in Equation 19 can be written as;

$$\Psi_r(t) = \psi_{n,m}(t) = 2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\,P_m(2^k t - n)\,\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]},\ i = 1,2,...,2^k(M+1) \quad (23)$$

where $r = n(M+1)+(m+1)$, $m = 0,1,...,M$, $n = 0,1,...,(2^k-1)$ and $\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}$ is the characteristic function defined as $\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}(t) = \begin{cases} 1 & t \in [\frac{n}{2^k},\frac{n+1}{2^k}] \\ 0 & otherwise \end{cases}$.

By differentiation with respect to t in Equation 23 we have

$$\frac{d\Psi_r(t)}{dt} = 2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\,2^k P'_m(2^k t - n)\,\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]} \quad (24)$$

This function is zero outside the interval $[\frac{n}{2^k},\frac{n+1}{2^k}]$, hence its Legendre wavelets expansion only have those elements of basis Legendre wavelets in $\Psi(t)$ that are non-zero in the interval $[\frac{n}{2^k},\frac{n+1}{2^k}]$ that is, $\Psi_i(t),\ i = n(M+1)+1, n(M+1)+2,...,(n+1)(M+1)$. Therefore its Legendre wavelet expansion has the following form:

$$\frac{d\Psi_r(t)}{dt} = \sum_{i=n(M+1)+1}^{(n+1)(M+1)} a_i \Psi_i(t)$$

This implies that the operational matrix D is a block matrix as defined in Equation 21. Moreover we have $\frac{d}{dt}P_0(t) = 0$ This results to $\frac{d\Psi_r(t)}{dt} = 0$ for $r = 1,(M+1)+1,...,(2^k-1)(M+1)+1$. Consequently, the first row of thevmatrix F defined in Equation 21 is zero. Now by substituting $P'_m(2^k t - n)$ from Equation 9 into Equation 19 we have;

$$\frac{d\Psi_r(t)}{dt} = 2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\,2^k \sum_{\substack{j=0 \\ j+m\ odd}}^{m-1} 2(2j+1)P_j(2^k t - n)\,\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]} \quad (25)$$

Expanding this equation in Legendre wavelets basis, we have

$$\frac{d\Psi_r(t)}{dt} = 2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\,2^k \sum_{\substack{j=0 \\ j+m\ odd}}^{m-1} 2(2j+1)P_j(2^k t - n)\,\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]} =$$
$$2^{k+1}\sum_{\substack{s=1 \\ s+r\ odd}}^{r-1}\sqrt{(2r-1)(2s-1)}\,\Psi_{n(M+1)+s}(t), \quad (26)$$

So if we choose $F_{r,s}$ as

$$F_{r,s} = \begin{cases} 2^{k+1}\sqrt{(2r-1)(2s-1)} & r = 2,...,(M+1),\ s = 1,...,r-1 \ and\ (r+s) odd \\ 0 & othewise \end{cases}$$

then Equation 20 holds and proves the desired results.

Corollary

By using Equation 20, the operational matrix for the nth derivative can be derived as

$$\frac{d^n\Psi(t)}{dt^n} = D^n\Psi(t) \quad (27)$$

where D^n is the nth power of the matrix D .

In the next theorem we generalize the operational matrix of the derivative of Legendre wavelets to fractional order derivative.

Theorem 3

Let $\Psi(t)$ be the Legendre wavelets vector defined in Equation 19. Suppose that $\alpha > 0$ then

$$D^\alpha\Psi(t) = D^{(\alpha)}\Psi(t) \quad (28)$$

where $D^{(\alpha)}$ is the $(2^k(M+1))\times(2^k(M+1))$ operational matrix of the fractional derivative of the order $\alpha>0$, $N-1<\alpha\le N$ in the Caputo sense and its (p,q)th elements is

$$\left(D^{(\alpha)}\right)_{pq}\begin{cases}0 & 1\le p\le[\alpha]\\ 2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\left(\sum_{i=0}^{m}\sum_{j=0}^{i}b_{jq}\binom{i}{j}\frac{(-1)^{m+i}(m+i)!\;2^{kj}\;n^{i-j}}{(m+i)!(i!)^2}\right) & [\alpha]+1\le p\le 2^k(M+1)\end{cases} \tag{29}$$

In which b_{jq} are the qth coefficients of the Legendre wavelet expansion of functions $f_j(t)=t^j\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}$, $j=0,...i$.

Proof

Suppose that $\Psi_p(t)$, $p=nM+(m+1)$, $m=0,1,...,M$, $n=0,1,...,(2^k-1)$ be the pth element of the vector $\Psi(t)$ defined in Equation 19. By using the shifted Legendre polynomial, $\Psi_p(t)$ can be written as;

$$\Psi_p(t)=\psi_{m,n}(t)=2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}P_m(2^kt-n)\;\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]} \tag{30}$$

By using Equation 6 we have

$$\Psi_p(t)=2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\sum_{i=0}^{m}\frac{(-1)^{m+i}(m+i)!}{(m+i)!}\frac{(2^kt-n)^i}{(i!)^2}\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]} \tag{31}$$

Expanding $(2^kt-n)^i$, Equation 31 can be written as

$$\Psi_p(t)=2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\sum_{i=0}^{m}\sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{m+i}(m+i)!\;2^{kj}\;n^{i-j}}{(m+i)!(i!)^2}t^j\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]} \tag{32}$$

Let D^{α} be a fractional order derivative, then for Equation 32 we have;

$$D^{\alpha}\Psi_p(t)=2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\sum_{i=0}^{m}\sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{m+i}(m+i)!\;2^{kj}\;n^{i-j}}{(m+i)!(i!)^2}D^{\alpha}(t^j\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}) \tag{33}$$

This equation can be rewritten as

$$D^{\alpha}\Psi_p(t)=2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\sum_{i=0}^{m}\sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{m+i}(m+i)!\;2^{kj}\;n^{i-j}}{(m+i)!(i!)^2}f_j(t) \tag{34}$$

where $f_j(t)=t^j\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}$, $j=0,...i$ and can be derived as follows:

$$f_j(t)=D^{\alpha}\left(t^j\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}\right)=\frac{1}{\Gamma(N-\alpha)}\int_{\frac{n}{2^k}}^{t}\left(\frac{\frac{d^N(t^j)}{dt^N}}{(t-x)^{\alpha+1-N}}\right)dx\,\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}+$$
$$\frac{1}{\Gamma(N-\alpha)}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}\left(\frac{\frac{d^Nt^j}{dt^N}}{(t-x)^{\alpha+1-N}}\right)dx\,\chi_{[\frac{n}{2^k},1]},\quad j=0,...,i \tag{35}$$

Now, we approximate functions $f_j(t)$, $j=0,...i$ in terms of Legendre wavelets as

$$f_j(t)=\sum_{q=1}^{2^k(M+1)}b_{jq}\Psi_q\;,\; j=0,...,i, \tag{36}$$

where $b_{jq}=\left(f_j(t),\Psi_q(t)\right)$.

Substituting Equation 36 into Equation 34 and by changing the order of series we have,

$$D^{\alpha}\Psi_p(t)=2^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\sum_{q=1}^{(2^kM+1)}\left(\sum_{i=0}^{m}\sum_{j=0}^{i-1}b_{jq}\binom{i}{j}\frac{(-1)^{m+i}(m+i)!\;2^{kj}\;n^{i-j}}{(m+i)!(i!)^2}\right)\Psi_q(t) \tag{37}$$

This leads to the desired results.

It is useful to note that the functions $f_j(t)$, $j=0,...i$ defined in Equation 35 can be calculated easily and their Legendre wavelet coefficients derived straightforwardly.

APPLICATION AND RESULTS

In this study, in order to show the high importance of operational matrix of derivative, we apply it to solve boundary value fractional problems. These problems are considered because closed form solutions are available for them. This allows one to compare the results obtained using this scheme with the analytical solution.

Bagley–Torvik boundary value problems

The general Bagley–Torvik boundary value problems of Order 2 have the form (Al-Mdallal et al., 2010)

$$A_0D^2y(t)+A_1D^{\frac{3}{2}}y(t)+A_2y(t)=f(t),\quad t\in[0,T], \tag{38}$$

subject to boundary conditions

$$y(0)=\alpha_0,\quad y(T)=\alpha_1, \tag{39}$$

where A_0, A_1, A_2, α_0 and α_1 are constants with $A_0 \neq 0$, and $y \in L_1[0,T]$.

In Equation 38, D^{α} denotes the fractional derivatives in the Caputo sense. The existence and uniqueness of the exact solution for these problems are discussed by Podlubny (1999). In this section, we introduce a new method based on Legendre wavelets expansion and its operational matrices of derivatives and fractional derivatives. For solving the Bagley–Torvik boundary value problems of the form (Equation 38) subject to the boundary conditions (Equation 39), we approximate the $y(t)$ and $f(t)$ by the Legendre wavelets as

$$\begin{aligned} y(t) &= C^T \Psi(t) \\ f(t) &= F^T \Psi(t) \end{aligned} \tag{40}$$

where vector $C = [c_{0,0}, c_{0,1}, \ldots, c_{0,M}, \ldots, c_{2,M}, \ldots, c_{(2^k-1),0}, c_{(2^k-1),1}, \ldots, c_{(2^k-1),M}]^T$ is an unknown vector and F is a known vector. By using Equations 27 and 28 we have

$$\begin{aligned} D^2 y(x) &= C^T D^{(2)} \Psi(x), \\ D^{\frac{3}{2}} y(x) &= C^T D^{(\frac{3}{2})} \Psi(x). \end{aligned} \tag{41}$$

Substituting Equations 40 into 41, the residual $R_m(x)$ can be derived as

$$R_m(x) = \left(A_0 C^T D^{(2)} + A_1 C^T D^{(\frac{3}{2})} + A_2 C^T \right) \Psi(x). \tag{42}$$

By using the typical Tau method (Canuto et al., 1988); we generate $2^k(M+1)-2$ linear equations by applying

$$\left\langle R_{2^k(M+1)}(x), \Psi_j(x) \right\rangle = 0,\ j = 0,1,\ldots,2^k(M+1)-2 \tag{43}$$

Also, by considering boundary conditions we have

$$\begin{aligned} y(0) &= C^T \Psi(0) = \alpha_0 \\ y(1) &= C^T \Psi(1) = \alpha_1 \end{aligned} \tag{44}$$

Together, Equations 43 and 44 generate $2^k(M+1)$ set of linear equations. These linear equations can be solved for unknown coefficients of the vector C. By substituting C in Equation (33), an approximation solution $y(x)$ can be obtained.

Numerical results

In this study, we will consider the three fractional order Bagley–Torvik boundary value problems. We used the method described in the "Bagley–Torvik boundary value problems" for solving these problems. The algorithms are performed by Maple 12 with 16 digits precision.

Example 1

Consider the following boundary value problem in the case of the inhomogeneous Bagley-Torvik Equation (Al-Mdallal et al., 2010);

$$\begin{cases} D^2 y(t) + D^{\frac{3}{2}} y(t) + y(t) = t^2 + 4\sqrt{\dfrac{t}{\pi}} + 2 \\ y(0) = 0, \quad y(5) = 25. \end{cases}$$

where the exact solution is $y(t) = t^2$.

We solve this fractional boundary value problem by applying the method described in the "Bagley–Torvik boundary value problems" Section using Legendre wavelets expansion and its operational matrices of derivatives with M=2, k=1. Using Equation 43, we obtain four linear equations and by applying boundary condition, we have two linear equations. By solving this linear system, we get the unknown vector C. By substituting this vector in Equation 40, we obtain the exact solution.

Example 2

Consider the boundary value problem

$$\begin{cases} D^{\frac{3}{2}} y(t) + y(t) = t^5 - t^4 + \dfrac{128}{7\sqrt{\pi}} t^{3.5} - \dfrac{64}{5\sqrt{\pi}} t^{2.5} \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

The exact solution is $y(t) = t^4(1-t)$. Here, we solve this problem using the Legendre wavelets method with M = 6, k = 2. Similar to Example 1 by solving the linear system derived for this problem, we obtain the exact solution.

Example 3

Consider the Bagley–Torvik boundary value problem

$$\begin{cases} D^2 y(x) + \dfrac{1}{2} D^{\frac{3}{2}} y(x) + \dfrac{1}{2} y(x) = 8v(t) - 8v(t-1) \\ y(0) = 0,\ y(20) = -1.48433, \end{cases}$$

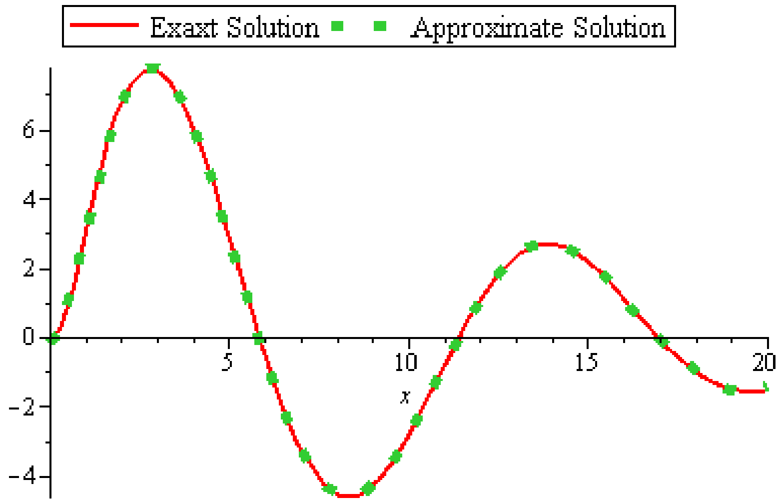

Figure 1. Approximate and exact solutions for Example 3 in the interval [0, 20].

Table 1. The absolute errors for different values of M and k.

x	M=10, k=1	M=10, k=2	M=12, k=1	M=12, k=2
4	3.6×10^{-5}	1.3×10^{-5}	1.9×10^{-5}	9.8×10^{-6}
8	3.8×10^{-5}	1.5×10^{-5}	7.6×10^{-6}	3.4×10^{-7}
12	4.0×10^{-5}	9.4×10^{-6}	8.8×10^{-7}	5.5×10^{-7}
16	2.7×10^{-5}	6.5×10^{-7}	1.1×10^{-7}	5.4×10^{-8}

in which $\nu(t)$ is the Heaviside function. This example was solved theoretically by Podlubny (1999); and the compact form of the solution was given by:

$$y(x)=\int_0^x 8G(x-t)(\nu(t)-\nu(t-1))d\nu \,, \quad (45)$$

where G is the fractional Green's function defined as

$$G(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!}\sum_{j=0}^{\infty}\frac{(j+k)!\left(-t^{\frac{1}{2}}\right)^j}{j!2^j\Gamma\left(\frac{1}{2}j+\frac{1}{2}k+\frac{3}{2}k+2\right)} \quad (46)$$

In Equation 46, Γ is the gamma function. Here, we solve this problem using the Legendre wavelets method as has previously been described with (M, k) = (10, 1), (M, k) = (10, 2), (M, k) = (12, 1) and (M, k) = (12, 2). Figure 1 shows the theoretical solution derived by Podlubny (1999) and the approximate solution with (M, k) = (12, 2). The absolute errors for different values of x in the interval $[0,20]$ are shown in Table 1. From Table 1, we see that we can achieve a good approximation with the exact solution by using a few terms of Legendre wavelets.

Conclusion

In this article, a general formulation for deriving the Legendre wavelets operational matrices of fractional derivative has been derived. This matrix was used to approximate a numerical solution of Bagley–Torvik fractional boundary value problems. Our approach was based on the truncated Legendre wavelets expansion and the Tau method. The solution obtained for these problems show that the proposed method can effectively solve them effectively and it is very simple and easy to implement.

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