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Verified bounds for nonlinear systems via Hansen-Sengupta method

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Efficient numerical validation of solution to nonlinear systems of equations using the Hansen-Sengupta method is the main focus of our studies. An extremely fast method due to (Uwamusi, 2007) is used to accelerate basic characteristic convergence of the method. This is not surprising because the midpoints and radii of the convergent interval vectors obtained are coupled sequence via the inclusion isotonicity of interval arithmetic which satisfied the filter net condition of a fixed point operator. We compare our results with those obtained using traditional real floating point of Newton method. The emphasis is on the rigor of the bounds.

Key words: Nonlinear systems, Hansen-Sengupta method, Newton method, circular complex arithmetic MSC (2000) 65G(20), 65G(30).

INTRODUCTION

Using the notation

$$X = \left((x_1, x_2, ..., x_n)^T \in IR^n \,\middle|\, \underline{x_i} \le x_i \le \overline{x_i},\, 1 \le i \le n \right),$$

It is a common problem that given

$$F: ID \to IR^n,\, ID \subset IR^n \quad,\text{ ID bounded,}$$

Where

$[\underline{x_i}, \overline{x_i}]$ is an interval.

$x \in [X]$ and $[X] = [\underline{x_1}, \overline{x_1}] \times [\underline{x_2}, \overline{x_2}] \times ... \times [\underline{x_n}, \overline{x_n}]$
are a box, we rigorously can verify that there exists a unique $X^* \in X$ such that

$$F(X^*) = 0, \tag{1.1}$$

with a high yield of mathematical certainty.

The basic mathematical tools used are the contraction mapping theorem, Brouwer fixed point theorem, and the Miranda's theorem; see e.g., (Kearfott, 1998).

To describe this we review the well known Miranda's theorem: Miranda's theorem asserts that if $X \in IR^n$, and that the faces of X be represented by

$$\underline{X_i} = (x_1, ..., x_{i-1}, \underline{x_i}, x_{i+1}, ..., x_n)^n$$

$$\overline{X_i} = (x_1, ..., x_{i-1}, \overline{x_i}, \mathbf{x}_{i+1}, ..., x_n)^T.$$

Assuming $\mathbf{F} = (f_1, f_2, ..., f_n)^T$ is a continuous interval extension for f defined on $\mathbf{X} \subseteq ID$. Given that

$$f_i(\underline{X_i}) f_i(\overline{X_i}) \le 0,$$

For each $1 \le i \le n$, then there is an $X \in [X] \ni F(X) = 0$.

Central to the above, we review the following theorem due to (Kearfott, 1998):

Theorem 1.1(Kearfortt, 1998)

Assuming **f** is any natural extension of f, and $\mathbf{x} \in IR$ is contained within the domain of f, then f(x) contains the range $f^{u}(x)$ of f over **x.**

Usually Newton's method is often used to provide functional iterative steps to bound all solutions in a given domain of a mathematical problem even in the presence of nonlinearities, round off errors and uncertainty in the data.

Interval Newton's method as a gate way to many other numerical methods for nonlinear system abstractly can be written in the general form;

$$F'(x^{(k)})(\hat{X}^{(k)} - x^{(k)}) = -F(x^{(k)}) \text{ , (k=0, 1,....,.)} \qquad (1.2)$$

Where

$\hat{X}^{(k)}$ and $x^{(k)}$ are input values of x and $\bar{x}$ at the k^{-th} iteration;

$\hat{X}^{(}$ is the newly estimated bound of x at the k-th iteration;

The term $F'(x^{(k)})$ is an interval extension of the Jacobian matrix $\dfrac{\partial f_i(x)}{\partial x_i}$.

Setting A= $F^{/}(x)$ where $A = \left[\underline{A}, \bar{A}\right] \in IR^{n \times n}$ and

b$\in F(\bar{x})$, we define $[\underline{A}, \bar{A}] = [A' - A'', A' + A'']$ as an interval matrix where $A' = \dfrac{1}{2}(\underline{A} + \bar{A})$ is the centre matrix

and $A'' = (\bar{A} - \underline{A})$ is the radius. The interval vector b can similarly be defined. We characterize the solution set of resulting linear interval system (1.2) by the interval hull.

$$\sum(A,b) = \left(\bar{A}^{-1} b \in R^n \middle| \bar{A} \in A, \bar{b} \in b\right) \qquad (1.3)$$

Assuming that A is a regular interval matrix.

The set $\sum(A,b)$ is generally not often an interval vector and need not even be convex, see e.g., (Neumaier, 1990). Thus $\sum(A,b)$ possesses a very

complicated structure. Regularity of interval matrix A implies that every matrix $A' \in A$ has rank n.

The terms A' and b' vary over the bounded intervals such that $A' \in [\underline{A}, \bar{A}]$ and $b' \in [\underline{b}, \bar{b}]$ respectively.

Let us note that computation of system (1.3) is always very expensive. Frequently used method is the Hansen – Sengupta operator due to (Hansen and Gupta, 1981). The above expository discussion formed the basis of our research.

The remaining sections in the paper are arranged as follows: In the section below, we give a review of circular interval arithmetic.

THE CIRCULAR INTERVAL ARITHMETIC AND THE HANSEN-SENGUPTA METHOD

We aim to accelerate the basic convergence characteristic of Hansen and Sengupta method earlier mentioned at the beginning of this paper. The idea is based on a fast interval method due to Uwamusi (2007).

Here we shall adopt the approach to solve the Hansen-Sengupta method. The improvement of this method is also our main focus of study.

We consider two intervals $[a] = [a', a'']$ and $[b] = [b', b'']$ where a' is the centre of [a] and a'' is the radius of [a]. The same explanation goes for interval [b]. The basic arithmetic operations for intervals are as follow:

$$[a] \oplus [b] = \{a' + b', a'' + b''\}$$
$$[a](-)[b] = \{a' - b', a'' + b''\}$$
$$[a] \otimes [b] = \{a' \cdot b', |a'| |b''| + |b'| |a''| + a''b''\}$$
$$[a](/)[b] = \{b' / D, b'' / D\}, D = b'^2 - b''^2$$

We assumed that $0 \notin D$.

The inclusion isotonicity of $F([a])$ is given as $f([a]) \supseteq \{F([a]) : a \in A.\}$ given that $F([a])$ is a circular interval extension of a closed complex function over a disk $A \ni |midA| - radA \leq |F([a])| \leq |midF([A])| + radF([a])$ for all $a \in A$ and that $f([a]) \in F([a])$. The disk to be inverted in the form of circular interval matrix $[a]_{ij} = \{a', r\}$, where $|a'| > r$ is defined in an analogous manner due to Carstensen and Petkovic (1994):

$$[a]_{ij}^{-1} = \{a', r\} = \begin{cases} \dfrac{1}{a'\left(1 - \dfrac{r^2}{|a'|^2}\right)} & , \dfrac{r}{|a'|^2 - r^2} \end{cases} \qquad (2.1)$$

The inverse operation above is bounded by the equations

$$\left| Mid(INV_i([a]_{ij})) \right| \le |a'| \, |a'|^2 - r^2,$$

$$rad(INV_i([a]_{ij})) \le \frac{2r}{|a'|^2 - r^2} \qquad (2.2)$$

We introduce the Hansen-Sengupta operator in a manner analogous to Hansen and Sengupta (1981) see also, (Goldsztejn, 2007) as follows:

$$[H]([x], \bar{x}) = \bar{x} + \Gamma(H[V], -HF(x), [x], \bar{x}), \qquad (2.3)$$

Where for instance

$$\Gamma([A], [b], [x]) = [y] \qquad (2.4)$$

is defined by

$$[y_i] = \Gamma([a_{ii}], [b_i] - \sum_{j>i} [a_{ij}][x_j], [x_i]). \qquad (2.5)$$

The hull of interval (Gauss-Siedel) is given by

$$S = \{ x \in [x] \mid (A \in [A])(\exists b \in [b])(Ax=b) \} \subseteq \Gamma([A], [b], [x]) \qquad (2.6)$$

This has wide ranging applications for instance in electrical Engineering, see e.g. (Nakaya et al., 2006)

CONVERGENCE

In what follows we state a theorem showing how Hansen-Sengupta method can be used to improve the enclosure and existence of solutions.

Theorem 3.1 (Goldsztejn, 2008)

Let [x], [y], [z] $\in IR^n$, $\bar{x} \in [x]$ and $[A] \in IR^{n \times n}$ such that

$$[x] \subseteq [z], f(\bar{x}) \in [y] \qquad \text{and}$$

$[A] \supseteq \{ \frac{\partial f}{\partial x}(x) \in IR^{n \times n} \mid x \in [x] \}$. If $[x']$ denotes equation (2.5), then

(1) $x \in [x]$, and f(x) = 0 implies $x \in [x']$.

(2) If $\phi \ne [x'] \subseteq \text{int}[x]$ then f has a unique zero in $[x']$.

Definition 3.1

A sequence $x^{(k)}$ $(k = 0, 1, 2, ...)$ satisfying the inclusion

$$x = x^{(0)} \supseteq x^{(1)} \supseteq ... \supseteq x^{(k)} \supseteq x^{(k+1)} \supseteq ... \quad \text{such that}$$

$x^* \in x, f(x^*) = 0$, implies that $x^{(*)} \in x^{(k)} \, \forall k \ge 0$ is called strongly convergent if $x^{(\infty)} \ne 0$ and this implies that $rad(x^{(\infty)}) = 0$ and $f(x^{(\infty)}) = 0$.

Following Neumaier (1990) the convergence analysis of Hansen-Sengupta is implied by the following consideration; we first note that regularity of HA is implied by the identity $HA = B = I + [-1,1]|H| rad(A) = i + [-1,1]V$ centered about an identity matrix I, where V is defined by $V = |H| rad(A) \ge 0$. The matrix $B = [I-V, I+V]$ is not only diagonally dominant but inverse positive and bounded below by a null matrix. Furthermore, the spectral radius is given as $\rho(V) < 1$. There exists a vector $x \ge 0$ for which $rad(B)x \ge x$ for some x>0. It follows that $I - (I + [-1,1]V|x = |B'-B| = Rad(B)x < x$.

In addition, HA is an M-matrix which is regular since all M- matrices are regular.

It follows from the system (2.3), that $0 \in H(x^{(\infty)}, \bar{x}^{\infty}) - \bar{x}^{\infty} = \Gamma(HA, -HF(\bar{x}^{\infty}), z)$. Besides the aforementioned facts, it is also valid that $0 \in z \subseteq (HA)^T (-HF(\bar{x}))$ since the matrix $(HA)^T$ is thin and regular too.

We have to show that the sequence generated by Hansen-Sengupta method satisfies the Cauchy condition. Let $T : X \rightarrow X$ be a contraction map. There exist a constant η such that $0 < \eta < 1$. Let any positive number $k \ge 1$ be given and for any generated intervals:

$x_k, x_{k+1} \ni x_k \le x_{k+1}$, the Hausdorff metric is defined by $d(x_k, x_{k+1}) = d(T(x_{k-1}), T(x_k)) \le \eta d(x_{k-1}, x_k)$.

Mathematical induction verifies that:

$$d(x_k, x_{k+1}) \le \eta^k d(x_0, x_1).$$

Now for any $0 \le k < t$, we have that;

$$d(x_k, x_t) \le \left(\eta^k + \eta^{k+1} + ... + \eta^{t-1} \right) d(x_0, x_1) \le \frac{\eta^k}{1-\eta} d(x_0, x_1)$$

Moreover the generated interval sequence from Hansen-Sengupta operator is also a completion for the sequence of interval vectors and in addition sum ably bounded by a

$$\text{Metric} \sum_{k=0}^{\infty} d(x_k, x_{k+1}) < \infty .$$

Table 1. Showing results for Hansen-Sengupta method with error bounds.

Iterations	Results Mid(X_k), Rad(X_k)	$\left\|F(\bar{X}_k)\right\|_\infty$
1	0.358937022, 1.7095978×10^{-2} 0.600208287, $1.63819889 \times 10^{-2}$ 0.808328540, 8.809388×10^{-3}	2.6257369×10^{-2}
2	0.336461223, 3.350353×10^{-3} 0.585636548, 2.417562×10^{-3} 0.801816087, 8.112458×10^{-3}	3.351703×10^{-3}
3	0.337953381, 3.4978×10^{-5} 0.585355878, 3.4033×10^{-5} 0.801709854, 1.5718×10^{-5}	2.18688×10^{-4}
4	0.337917117, 3.6264×10^{-5} 0.585289640, 6.6708×10^{-5} 0.801634504, 4.15×10^{-7}	3.1×10^{-8}
5	0.337917117, 4×10^{-9} 0.585289640, 7.305×10^{-12} 0.801634504, 1.9341×10^{-12}	0
6	0.337917117, 0 0.585289640, 0 0.801634504, 0	0

Table 2. Showing results obtained by Neumaier (2001) when Newton method in point arithmetic is used.

Iteration	Results X_k	$\left\|F(X_k)\right\|_\infty$
1	0.358333 0.60000000 0.8083333	3.2×10^{-2}
2	0.3386243 0.5856949 0.8018015	1.09×10^{-3}
3	0.3379180 0.5852901 0.8016347	1.57×10^{-06}
4	0.3379171 0.5852896 0.8016345	2.84×10^{-12}
5	0.3379171 0.5852896 0.8016345	5.55×10^{-16}
6	0.3379171 0.5852896 0.8016345	5.00×10^{-16}

Therefore we have that $\lim\limits_{k \to \infty} HF(\bar{x}_k) = 0$ and,

filter net condition of Cauchy sequence is satisfied.

PRACTICAL EXAMPLE

Consider the nonlinear system of equations:

$$F(x) = \begin{cases} x_2^2 - 3x_1^2 \\ x_1^2 + x_1 x_3 + x_3^2 - 3x_2^2 \\ x_2^2 + x_2 + 1 - 3x_3^2 \end{cases}$$

$$X^{(0)} = (\frac{1}{4}, \frac{1}{2}, \frac{3}{4})^T, \quad \varepsilon = 0.01 .$$

Results are presented in Table 1 showing computed values for Hansen-Sengupta method.

Conclusion

We have obtained a method to accelerate the basic convergence behavior of Hansen-Sengupta method. We achieved this using the method of Uwamusi (2007). The results computed formed a coupled sequence converging to a fixed point as the width of the radii tends to zero. This convergence behavior is due to inclusion isotonicity enjoyed by interval arithmetic. This is ably represented in Table 1. In Table 2, we showed results obtained by Neumaier (2001) where in, he used ordinary floating point arithmetic of Newton method. Our results showed valida-ted worst case error bounds for the solution of the nonlinear system of equations.

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