International Journal of Physical Sciences Vol. 7(9), pp. 1353 - 1361, 23 February, 2012
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.1739
ISSN 1992 - 1950 ©2012 Academic Journals

Full Length Research Paper

An investigation of porosity and magnetohydrodynamic flow of non-Newtonian nanofluid in coaxial cylinders

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Accepted 31 January, 2012

The magnetohydrodynamic (MHD) flow of third grade nanofluid between coaxial porous cylinders was investigated in this work. Two types of series solutions were presented for constant and variable viscosity. The resulting nonlinear coupled equations were solved by employing homotopy analysis method (HAM). The convergence of the series solutions has been discussed explicitly. The recurrence formulae for finding the coefficients were also given. Comparison with the existing studies was made and the role of pertinent parameters was graphically illustrated at the end.

Key words: Non-Newtonian nanofluid, porous media, magnetohydrodynamic (MHD) flow, heat transfer, nonlinear equations, series solutions.

INTRODUCTION

In the past, much attention has been given to the non-Newtonian fluids (Malik et al., 2011; Mahomed and Hayat, 2007; Hameed and Nadeem, 2007; Dehghan and Shakeri, 2009; Ellahi, 2009). Recent advances in nanotechnology have led to the development of a new innovative class of heat transfer fluids, called nanofluids created by dispersing nanoparticles (10 to 50 nm) in traditional heat transfer fluids (Choi, 2009). Very little efforts were devoted to examine the non-Newtonian nanofluids. Since, the nanotechnology has been widely used in industrial cooling applications, nuclear reactors, transportation industry (automobiles, trucks and airplanes), micro-electromechanical systems, electronics and instrumentation and biomedical applications (nano-drug delivery, cancer therapeutics and cryopreservation) (Choi, 2009), therefore, many studies have focused on nanofluids these days. For instance, Khanafer et al. (2003) seems to be the first who have examined heat transfer performance of nanofluids inside an enclosure taking into account the solid particle dispersion. Bachok et al. (2010) examined the boundary layer of nanofluids over a moving surface when the plate is assumed to move in the same or opposite directions to free stream. The boundary layer flow induced in a nanofluid due to a linearly stretching sheet that was investigated by Makinde and Aziz (Makinde and Aziz, 2011). Kuznetsove and Nield (2010) examined the natural convection flow of nanofluid over a vertical plate. Some relevant work on nanofluids can be seen from the list of references (Choi et al., 2001; Khan and Pop, 2010; Lotfi et al., 2010; Ellahi et al., 2011). After these studies, nanotechnology was considered by many to be one of the significant forces that will drive the next major industrial revolution of this century. Nanotechnology represents the most relevant technological cutting edge currently being explored. It aims at manipulating the structure of the matter at the molecular level with the goal to innovate virtually every industry and public endeavor, including biological sciences, physical sciences, electronics cooling, transportation, the environment and national security as well.

Moreover, porous media is been used to transport and store energy in many industrial applications, such as heat pipe, solid matrix heat exchangers, electronic cooling and chemical reactors. An important characteristic for the combination of the fluid and the porous medium is the

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tortuosity which represents the hindrance to flow diffusion imposed by local boundaries or local viscosity (Vafai, 2011). Furthermore, magnetohydrodynamic (MHD) is effectively used in many applications including power generators, pumps, accelerators, electrostatic filters, droplet filters, the design of heat exchangers, the cooling of reactors, etc., (Sutton and Sherman, 1965). To the best of our knowledge, no work has been reported yet in the literature on MHD flow of non-Newtonian nanofluid in coaxial porous cylinders.

With this motivation, pertinent works (Hayat et al., 2007; Ellahi and Riaz, 2010; Ellahi and Afzal, 2009) were reviewed for MHD flow of non-Newtonian nanofluids in coaxial porous cylinders to fill the gap in the existing literature. The nonlinear coupled equations along with nanoparticle concentration have been solved by the homotopy analysis method which does not require any small or large parameters and has been proven to be successful in tackling nonlinear equations (Liao, 2004, 2003; Abbasbandy, 2006; Hayat et al., 2006; Ellahi, 2010). Convergences of the obtained series solutions are properly discussed. To the best of our knowledge, the series solutions for this particular model have not been presented before.

FORMULATION OF PROBLEM

For an incompressible viscous fluid, the equations of conservation of mass, momentum, thermal energy and nanoparticles are,

$$\rho_f\left(\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}.\nabla\mathbf{V}\right)=\operatorname{div}\mathbf{T}-\frac{\mu\varphi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}+\left[\phi\rho_p+(1-\phi)\rho_f\right]\left[1-\beta_T(\theta-\theta_w)\right]g, \quad (1)$$

$$(\rho c)_f\left(\frac{\partial \theta}{\partial t}+\mathbf{V}.\nabla\theta\right)=k\nabla^2\theta+(\rho c)_p\left[D_b\nabla\phi.\nabla\theta+\frac{D_T}{\theta_w}\nabla\theta.\nabla\theta\right], \quad (2)$$

$$\left(\frac{\partial \phi}{\partial t}+\mathbf{V}.\nabla\phi\right)=D_b\nabla^2\phi+\frac{D_T}{\theta_w}\nabla^2\theta, \quad (3)$$

along with the boundary conditions:

$$v(R_1)=v_0, v(R_2)=0; \theta(R_1)=\theta_w, \theta(R_2)=0; \phi(R_1)=\phi_w, \phi(R_2)=0. \quad (4)$$

Here, $\mathbf{V}=[0,0,u(r)]$ is the velocity, θ is the temperature, ϕ is the nanoparticle volume fraction, ρ_f is the density of the base fluid, ρ_p is the density of the nanoparticles, g is the gravitational acceleration and μ, k and β_T are the viscosity, thermal conductivity and volumetric thermal expansion coefficient of the nanofluid, respectively. The Brownian diffusion coefficient and the thermophoresis diffusion coefficient are, respectively denoted by D_b and D_T.

For third grade fluids, physical considerations were taken into account by Fosdick and Rajagopal [1980] in order to obtain the following form for the constitutive law:

$$\mathbf{T}=-p\,\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\beta_1\mathbf{A}_3+\beta_2(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1)+\beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1 \quad (5)$$

where p is the pressure, μ is the dynamic viscosity, $\mathbf{I}$ is the identity tensor and $\alpha_i\,(i=1,2)$ and $\beta_j\,(j=1-3)$ are material constants. Moreover, the coefficients μ,α_1,α_2 and β_3 must satisfy the following inequalities:

$$\mu\geq 0,\ \alpha_1>0,\ \beta_3\geq 0 \text{ and } |\alpha_1+\alpha_2|\leq\sqrt{24\mu\beta_3}. \quad (6)$$

Defining the dimensionless parameters:

$$\Lambda=\frac{2\beta_3 v_0^2}{\mu_0 R^2},\ c=\frac{\frac{\partial p}{\partial z}R^2}{\mu_0 v_0},\ r=\frac{\bar{r}}{R},\ v=\frac{\bar{v}}{v_0},\ \mu=\frac{\bar{\mu}}{\mu_0},\ \theta=\frac{\bar{\theta}-\theta_w}{\theta_m-\theta_w},$$
$$\phi=\frac{\bar{\phi}-\phi_w}{\phi_m-\phi_w},\Gamma=\frac{\mu_0^2 v_0}{k(\theta_m-\theta_w)},\ P=\frac{\varphi}{k_1R^2},\ M^2=\frac{\sigma B_0^2R^2}{\mu_0}, \quad (7)$$
$$B_r=\frac{(\rho_p-\rho_w)R^2(\phi_m-\phi_w)g}{\mu_0 v_0},\ G_r=\frac{(\theta_m-\theta_w)\rho_{fw}R^2(1-\phi_w)g}{\mu_0 v_0},$$

where P is porosity parameter, Λ is the third grade parameter, M is the MHD parameter, G_r is the thermophoresis diffusion constant, B_r is the Brownian diffusion constant, v_0 is the reference velocity, μ_0 is the reference viscosity, θ_w is the reference temperature, $\bar{\theta}$ is the pipe temperature, θ_m is the fluid temperature, R is the radius of pipe and ρ_{fw} is the density of the base fluid.

In view of constitutive law and governing equations from Equations 1 to 3, after dropping bars for simplicity, lead to the following non-dimensional coupled form:

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}\frac{dv}{dr}+\mu\frac{d^2v}{dr^2}+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3+3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}$$
$$=c+P\left[\mu+\Lambda\left(\frac{dv}{dr}\right)^2\right]v+M^2v-G_r\theta-B_r\phi, \quad (8)$$

$$(1+\alpha_1N_t)\left(\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}\right)+N_b\frac{d\theta}{dr}\frac{d\phi}{dr}=0, \quad (9)$$

$$N_b\left(\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}\right)+N_t\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=0. \quad (10)$$

The corresponding boundary conditions are,

$$v(1)=1,\ v(2)=0;\ \theta(1)=1,\ \theta(2)=0;\ \phi(1)=1,\ \phi(2)=0. \quad (11)$$

SOLUTION OF PROBLEM

Here, we find the series solutions of the nonlinear governing equations using homotopy analysis method for two cases, namely, constant and variable viscosity.

Case 1

For constant viscosity model, we take:

$$\mu = 1. \tag{12}$$

It is natural to choose,

$$v_0(r) = \frac{(8-r^3)}{7}, \quad \theta_0 = \frac{(8-r^3)}{7}, \quad \phi_0 = \frac{(8-r^3)}{7} \tag{13}$$

as an initial approximations of v, θ and ϕ, respectively, which satisfy the corresponding boundary conditions given in Equation 11. We use the method of higher order differential mapping (Van Gorder and Vajravelu, 2009) to define an auxiliary linear operator L by,

$$L = \frac{d^2}{dr^2}, \tag{14}$$

with the property,

$$L(c_1 + c_2 r) = 0, \tag{15}$$

where c_1 and c_2 are the arbitrary constants.

From Equations 8 to 10, we define the following nonlinear operators:

$$\begin{aligned}\mathbf{N}_1[v^*(r,p),\theta^*(r,p),\phi^*(r,p)] &= \frac{1}{r}\frac{dv^*}{dr} + \frac{d^2v^*}{dr^2} + \frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^3 - c \\ &+ 3\Lambda\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} + G_r\theta^* + B_r\phi^* \\ &- P\left[1+\Lambda\left(\frac{dv^*}{dr}\right)^2\right]v^* - M^2 v,\end{aligned} \tag{16}$$

$$\mathbf{N}_2[v^*(r,p),\theta^*(r,p),\phi^*(r,p)] = \frac{\alpha}{r}\frac{d\theta^*}{dr} + (\alpha + \alpha_1 N_t)\frac{d^2\theta^*}{dr^2} + N_b\frac{d\theta^*}{dr}\frac{d\phi^*}{dr}, \tag{17}$$

$$\mathbf{N}_3[v^*(r,p),\theta^*(r,p),\phi^*(r,p)] = \frac{1}{r}\frac{d\phi^*}{dr} + \frac{d^2\phi^*}{dr^2} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2}\right) \tag{18}$$

and then construct the homotopy:

$$\mathbf{H}_1[v^*(r,p)] = (1-p)L[v^*(r,p) - v_0(r)] - p\hbar\mathbf{N}_1[v^*(r,p),\theta^*(r,p),\phi^*(r,p)] \tag{19}$$

$$\mathbf{H}_2[\theta^*(r,p)] = (1-p)L[\theta^*(r,p) - v_0(r)] - p\hbar\mathbf{N}_2[v^*(r,p),\theta^*(r,p),\phi^*(r,p)] \tag{20}$$

$$\mathbf{H}_3[v^*(r,p)] = (1-p)L[\phi^*(r,p) - v_0(r)] - p\hbar\mathbf{N}_3[v^*(r,p),\theta^*(r,p),\phi^*(r,p)] \tag{21}$$

where $\hbar$ is a nonzero auxiliary parameter.

Setting $\mathbf{H}_1[v^*(r,p)] = \mathbf{H}_2[\theta^*(r,p)] = \mathbf{H}_3[\phi^*(r,p)] = 0,$ we have the zero — order deformation equations:

$$(1-p)L[v^*(r,p) - v_0(r)] = p\hbar\mathbf{N}_1[v^*(r,p),\theta^*(r,p),\phi^*(r,p)] \tag{22}$$

$$(1-p)L[\theta^*(r,p) - \theta_0(r)] = p\hbar\mathbf{N}_2[v^*(r,p),\theta^*(r,p),\phi^*(r,p)] \tag{23}$$

$$(1-p)L[\phi^*(r,p) - \phi_0(r)] = p\hbar\mathbf{N}_3[v^*(r,p),\theta^*(r,p),\phi^*(r,p)] \tag{24}$$

subject to the boundary conditions:

$$\left.\begin{aligned} v^*(r,0) = v_0(r),\ \theta^*(r,0) = \theta_0(r),\ \phi^*(r,0) = \phi_0(r) \\ v^*(r,1) = v(r), \theta^*(r,1) = \theta(r),\ \phi^*(r,1) = \phi(r)\end{aligned}\right\}, \tag{25}$$

where $p \in [0,1]$ is an embedding parameter. When p increases from 0 to 1, $v^*(r,p)$, $\theta^*(r,p)$, $\phi^*(r,p)$ vary from $v_0(r)$, $\theta_0(r)$, $\phi_0(r)$ to $v(r)$, $\theta(r)$, $\phi(r)$, respectively. Since in homotopy analysis method (HAM) solution, the convergence depends upon the choice of $\hbar$, therefore, at $p = 1,$ we obtain:

$$\left.\begin{aligned} v(r) &= v_0(r) + \sum_{m=1}^{\infty}\frac{1}{m!}\left.\frac{\partial^m v^*(r,p)}{\partial p^m}\right|_{p=0} \\ \theta(r) &= \theta_0(r) + \sum_{m=1}^{\infty}\frac{1}{m!}\left.\frac{\partial^m \theta^*(r,p)}{\partial p^m}\right|_{p=0} \\ \phi(r) &= \phi_0(r) + \sum_{m=1}^{\infty}\frac{1}{m!}\left.\frac{\partial^m \phi^*(r,p)}{\partial p^m}\right|_{p=0}\end{aligned}\right\}. \tag{26}$$

The m th order deformation problems with the corresponding boundary conditions are given by:

$$L[v_m(r) - \chi_m v_{m-1}(r)] = \hbar\Re 1_m(r), \tag{27}$$

$$L[\theta_m(r) - \chi_m \theta_{m-1}(r)] = \hbar\Re 2_m(r), \tag{28}$$

$$L[\phi_m(r) - \chi_m \phi_{m-1}(r)] = \hbar\Re 3_m(r), \tag{29}$$

$$v_m(2) = 0, \theta_m(2) = 0, \phi_m(2) = 0,\ v_m(1) = 1, \theta_m(1) = 1, \phi_m(1) = 1, \tag{30}$$

Where

$$\begin{aligned}\Re 1_m(r) &= \frac{1}{r}\frac{dv_{m-1}}{dr} + \frac{d^2v_{m-1}}{dr^2} + G_r\theta_m + B_r\phi_m - (1-\chi_m)c + \\ &\frac{\Lambda}{r}\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{dv_i}{dr} + 3\Lambda\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{d^2v_i}{dr^2} \\ &- Pv_{m-1} - P\Lambda\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}v_i - M^2 v_{m-1},\end{aligned} \tag{31}$$

and

$$\Re 2_m(r) = \frac{\alpha}{r}\frac{d\theta_{m-1}}{dr} + (1+\alpha+\alpha_1 N_t)\frac{d^2\theta_{m-1}}{dr^2} + N_b \sum_{k=0}^{m-1}\left(\frac{d\phi_{m-1-k}}{dr}\right)\frac{d\theta_k}{dr} \quad (32)$$

$$\Re 3_m(r) = \frac{1}{r}\frac{d\phi_{m-1}}{dr} + \frac{d^2\phi_{m-1}}{dr^2} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta_{m-1}}{dr} + \frac{d^2\theta_{m-1}}{dr^2}\right) \quad (33)$$

are recurrence formulae, in which

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \quad (34)$$

Case 2

For space dependent viscosity, we choose:

$$\mu = r \quad (35)$$

For HAM solution, we select the initial guess and linear operator given in Equations 13 and 14, respectively. With the same contrast, the respective $zeroth$ and m th order deformation problems are:

$$(1-p)L[v^*(r,p) - v_0(r)] = p\hbar \mathbf{N}_4[v^*(r,p), \theta^*(r,p), \phi^*(r,p)] \quad (36)$$

$$(1-p)L[\theta^*(r,p) - \theta_0(r)] = p\hbar \mathbf{N}_5[v^*(r,p), \theta^*(r,p), \phi^*(r,p)] \quad (37)$$

$$(1-p)L[\phi^*(r,p) - \phi_0(r)] = p\hbar \mathbf{N}_6[v^*(r,p), \theta^*(r,p), \phi^*(r,p)] \quad (38)$$

$$\left.\begin{aligned} v^*(r,p)\big|_{r=1} &= 1,\ v^*(r,p)\big|_{r=2} = 0 \\ \theta^*(r,p)\big|_{r=1} &= 1, \theta^*(r,p)\big|_{r=2} = 0 \\ \phi^*(r,p)\big|_{r=1} &= 1, \phi^*(r,p)\big|_{r=2} = 0 \end{aligned}\right\} 0, \quad (39)$$

$$L[v_m(r) - \chi_m v_{m-1}(r)] = \hbar \Re 4_m(r), \quad (40)$$

$$L[\theta_m(r) - \chi_m \theta_{m-1}(r)] = \hbar \Re 5_m(r), \quad (41)$$

$$L[\phi_m(r) - \chi_m \phi_{m-1}(r)] = \hbar \Re 6_m(r), \quad (42)$$

$$\left.\begin{aligned} v_m(2) = 0, \theta_m(2) = 0, \phi_m(2) = 0 \\ v_m(1) = 1, \theta_m(1) = 1, \phi_m(1) = 1 \end{aligned}\right\}, \quad (43)$$

Where

$$\mathbf{N}_4[v^*(r,p), \theta^*(r,p)] = \frac{2}{r}\frac{dv^*}{dr} + \frac{d^2v^*}{dr^2} + \frac{\Lambda}{r^2}\left(\frac{dv^*}{dr}\right)^3 + \frac{3\Lambda}{r}\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} + G_r\theta^*$$
$$+ B_r\phi^* - P\left[1 + \frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^2\right]v^* - \frac{M^2v^*}{r} - \frac{c}{r}, \quad (44)$$

$$\mathbf{N}_4[v^*(r,p), \theta^*(r,p), \phi^*(r,p)] = \frac{\alpha}{r}\frac{d\theta^*}{dr} + (1+\alpha+\alpha_1 N_t)\frac{d^2\theta^*}{dr^2} + N_b\frac{d\theta^*}{dr}\frac{d\phi^*}{dr}, \quad (45)$$

$$\mathbf{N}_5[v^*(r,p), \theta^*(r,p), \phi^*(r,p)] = \frac{1}{r}\frac{d\phi^*}{dr} + \frac{d^2\phi^*}{dr^2} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2}\right), \quad (46)$$

$$\begin{aligned} \Re 4_m(r) = {} & 2r\frac{dv_{m-1}}{dr} + r^2\frac{d^2v_{m-1}}{dr^2} + \Lambda\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{dv_i}{dr} + \\ & 3\Lambda r\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{d^2v_i}{dr^2} - (1-\chi_m)cr \\ & -M^2rv_{m-1} + G_r r\theta_m - pr^2v_{m-1} \\ & -P\Lambda r\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}v_i + B_r r\phi_m, \end{aligned} \quad (47)$$

$$\Re 5_m(r) = \frac{\alpha}{r}\frac{d\theta_{m-1}}{dr} + (1+\alpha+\alpha_1 N_t)\frac{d^2\theta_{m-1}}{dr^2} + N_b\sum_{k=0}^{m-1}\left(\frac{d\phi_{m-1-k}}{dr}\right)\frac{d\theta_k}{dr} \quad (48)$$

$$\Re 6_m(r) = \frac{1}{r}\frac{d\phi_{m-1}}{dr} + \frac{d^2\phi_{m-1}}{dr^2} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta_{m-1}}{dr} + \frac{d^2\theta_{m-1}}{dr^2}\right). \quad (49)$$

Finally, by using Taylor's theorem, the series solutions at $p = 1$ has the following structure:

$$\left.\begin{aligned} v(r) &= v_0(r) + \sum_{m=1}^{\infty}\frac{1}{m!}\left.\frac{\partial^m v^*(r,p)}{\partial p^m}\right|_{p=0} \\ \theta(r) &= \theta_0(r) + \sum_{m=1}^{\infty}\frac{1}{m!}\left.\frac{\partial^m \theta^*(r,p)}{\partial p^m}\right|_{p=0} \\ \phi(r) &= \phi_0(r) + \sum_{m=1}^{\infty}\frac{1}{m!}\left.\frac{\partial^m \phi^*(r,p)}{\partial p^m}\right|_{p=0} \end{aligned}\right\}. \quad (50)$$

Convergence of the solutions

Here, we discuss the convergence of the solutions. The solutions given in Equations 26 and 50 contain the auxiliary parameter $\hbar$, which gives the convergence region and rate of approximation for the HAM solutions. As pointed out by Liao (2003), the convergence region and rate of approximations given by the HAM are strongly dependent upon $\hbar$. Figures 1 to 6 are plotted for 20th order of approximations for the dimensionless velocity profile, temperature profile and nanoparticle concentration. These figures clearly elucidate that the range for the admissible values of $\hbar$ for all cases is approximately $-1.5 \le \hbar \le 0$.

RESULTS AND DISCUSSION

Here, the influence of some interesting parameters on the velocity, temperature and mass concentration were discussed. To see the effects of emerging parameters for

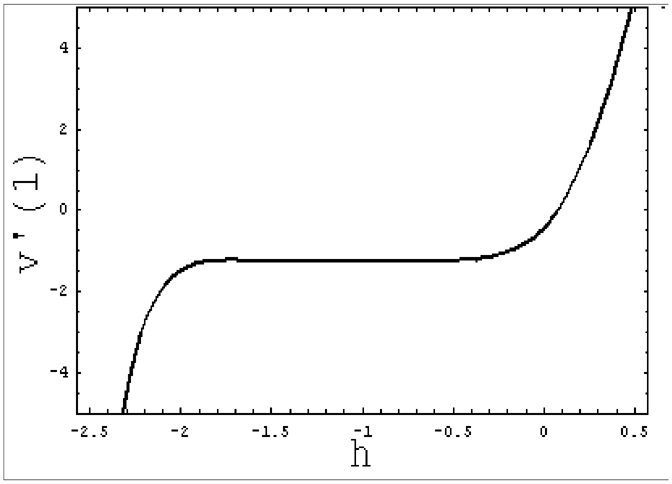

Figure 1. $\hbar-$ curve for velocity profile for constant viscosity.

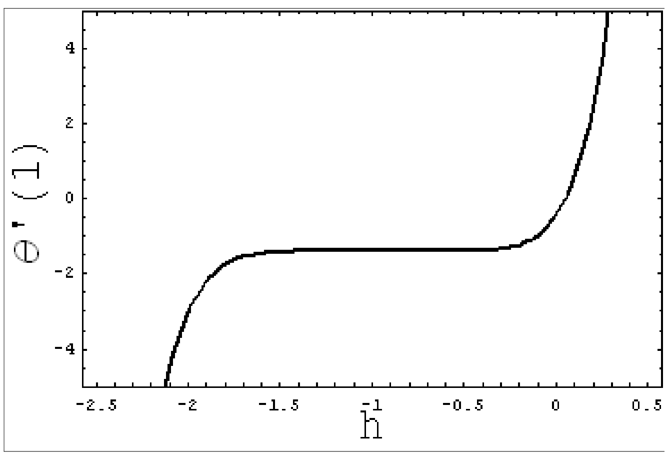

Figure 2. $\hbar-$ curve for temperature profile for constant viscosity.

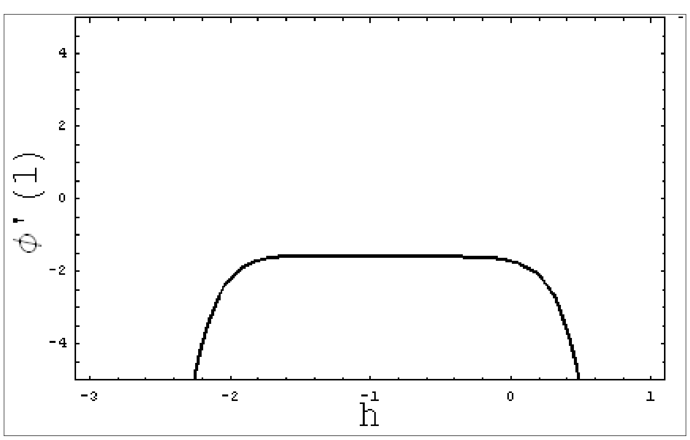

Figure 3. $\hbar-$ curve for nanoparticle concentration profile for constant viscosity.

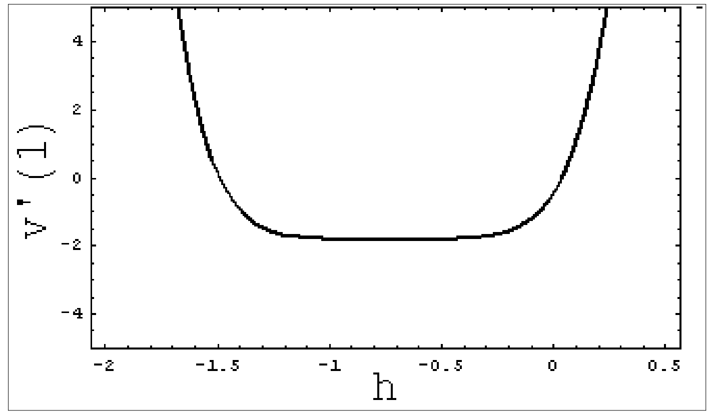

Figure 4. $\hbar-$ curve for velocity profile for variable viscosity.

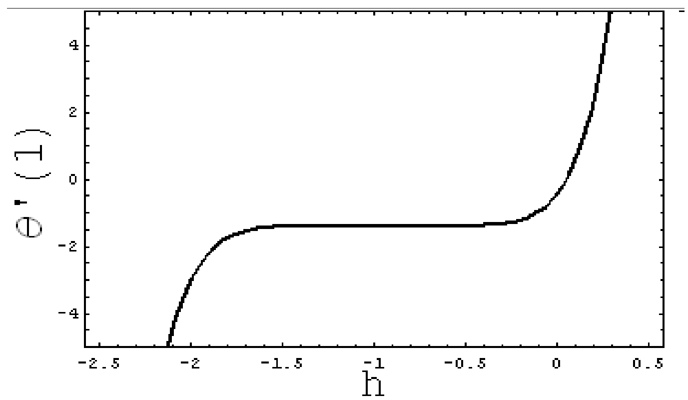

Figure 5. $\hbar-$ curve for temperature profile for variable viscosity.

constant and variable viscosity, Figures 7 to 22 have been displayed. The investigations of the effects of MHD parameter M and porosity parameter P on velocity are as shown in Figures 7 to 10. Figures 11 to 14 and 15 to 18 have been prepared to explain the effects of N_b and N_t on velocity and temperature profiles, respectively. The effects of N_b and N_t on mass concentration have been plotted in Figures 19 to 22.

In Figures 7 to 14, the velocity decreases by an increase in the MHD and porosity parameters. To see the effects of thermophoresis parameter and Brownian diffusion coefficient on temperature profile, Figures 15 to 18 have been displayed. These figures elucidate that temperature increases by increasing the Brownian diffusion coefficient when thermophoresis parameter coefficient is fixed and behave in an opposite manner when we vary thermophoresis parameter keeping Brownian diffusion coefficient fixed. This is in accordance with the fact that for thermal boundary, the effects of thermophoresis.

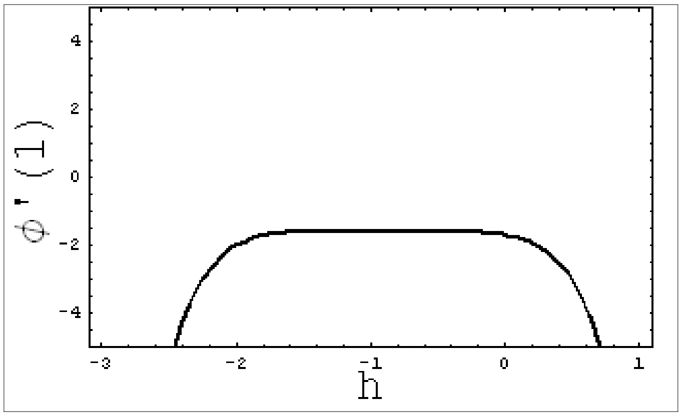

Figure 6. $\hbar-$ curve for nanoparticle concentration for variable viscosity.

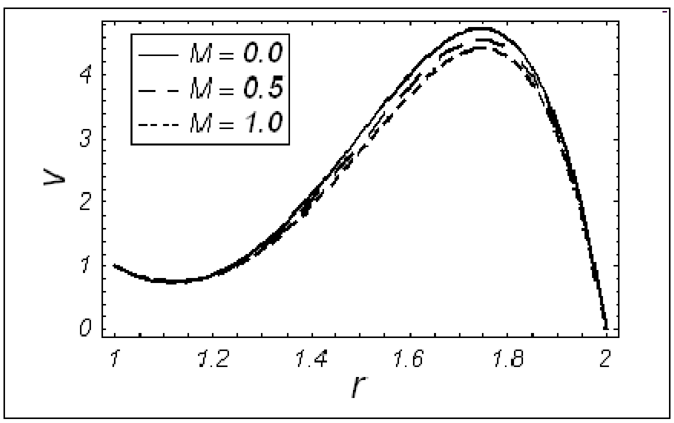

Figure 7. Effects of M on velocity profile when $N_b = 1, N_t = 1$ and $P = 0.5$ for constant viscosity.

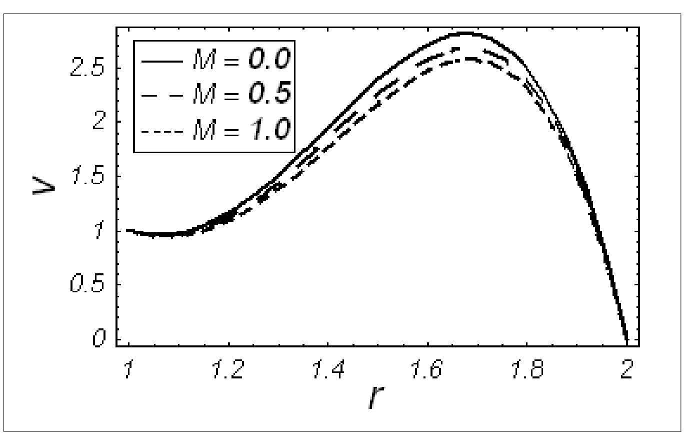

Figure 8. Effects of M on velocity profile when $N_b = 1, N_t = 1$ and $P = 0.5$ for variable viscosity.

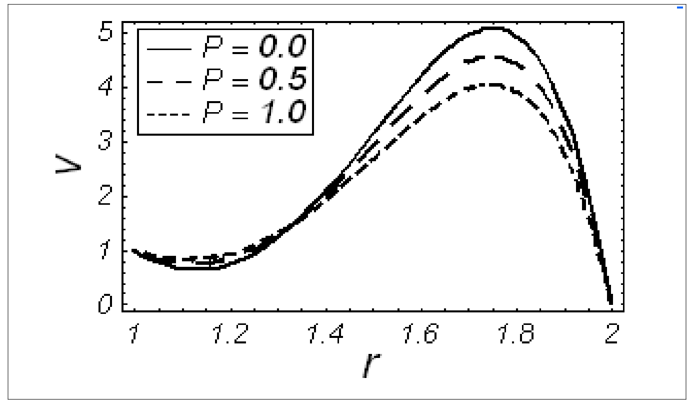

Figure 9. Effects of P on velocity profile when $N_b = 1, N_t = 1$ and $M = 0.5$ for constant viscosity.

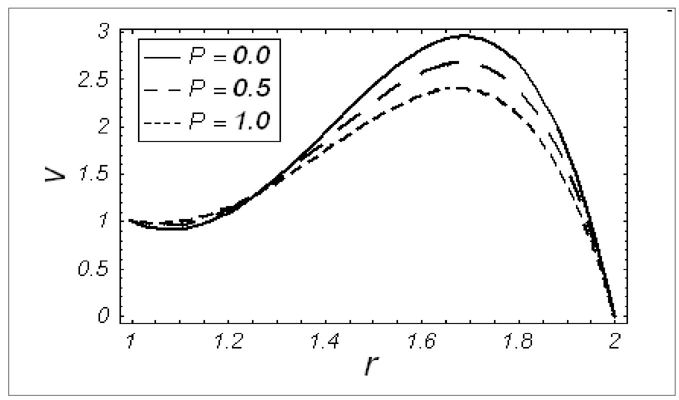

Figure 10. Effects of P on velocity profile when $N_b = 1, N_t = 1$ and $M = 0.5$ for variable viscosity.

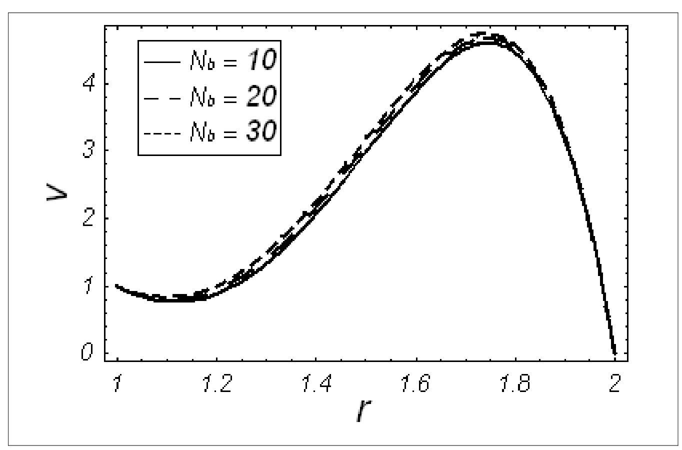

Figure 11. Effects of N_b on velocity profile when $N_t = 1, M = 0.5$ and $P = 0.5$ for constant viscosity.

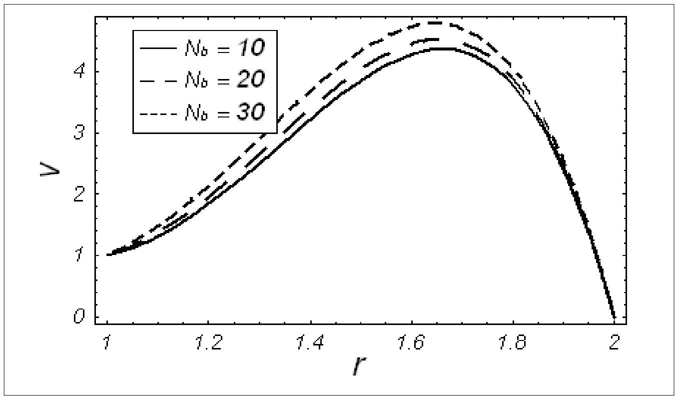

Figure 12. Effects of N_b on velocity profile when $N_t = 1, M = 0.5$ and $P = 0.5$ for variable viscosity.

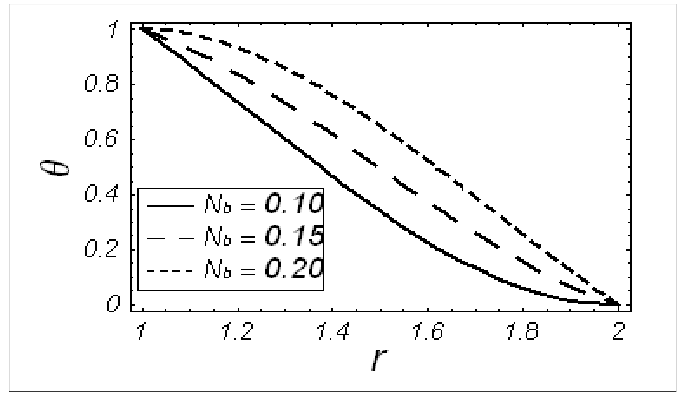

Figure 15. Effects of N_b on temperature distribution for constant viscosity when $P = M = 0.5$ and $N_t = 0.1$.

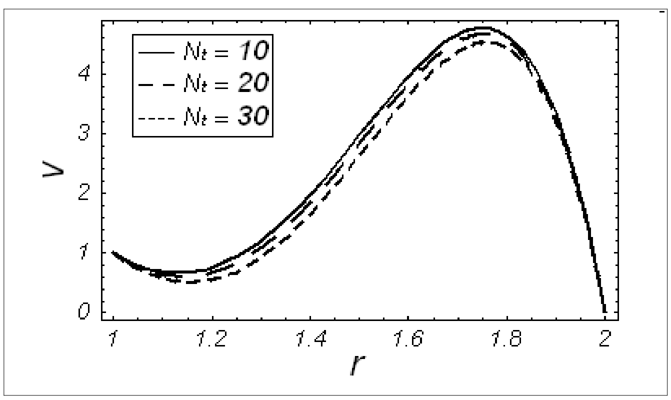

Figure 13. Effects of N_t on velocity profile when $N_b = 1, M = 0.5$ and $P = 0.5$ for constant viscosity.

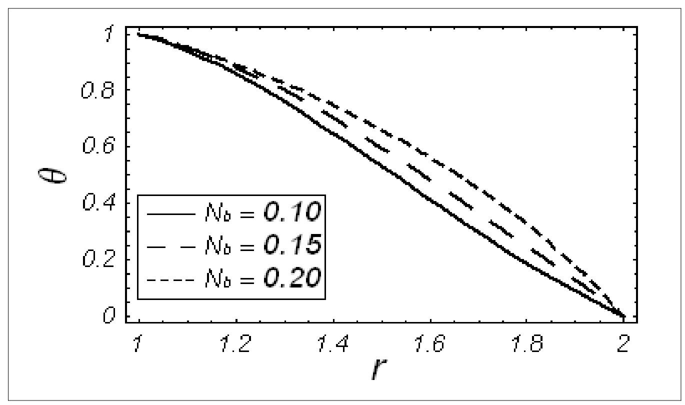

Figure 16. Effects of N_b on temperature distribution for variable viscosity when $P = M = 0.5$ and $N_t = 0.1$.

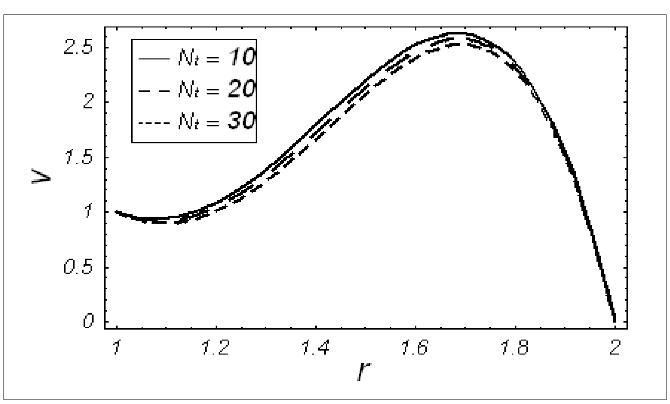

Figure 14. Effects of N_t on velocity profile when $N_b = 1, M = 0.5$ and $P = 0.5$ for variable viscosity.

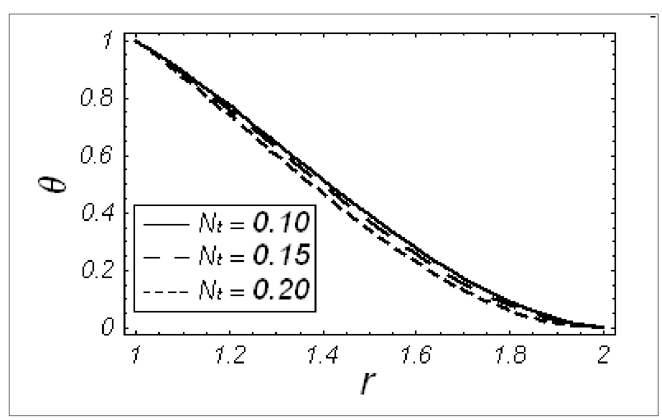

Figure 17. Effects of N_t on temperature distribution for constant viscosity when $P = M = 0.5$ and $N_b = 0.1$.

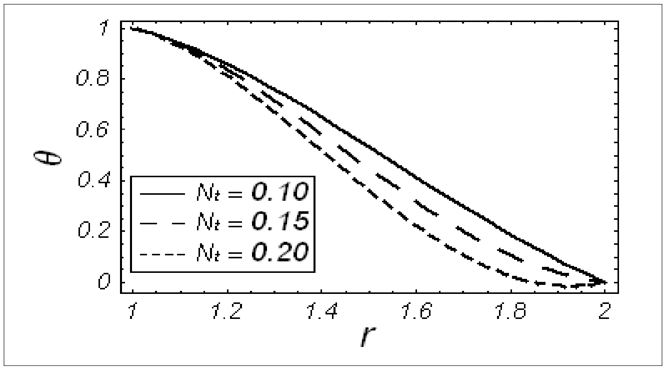

Figure 18. Effects of N_t on temperature distribution for variable viscosity when $P = M = 0.5$ and $N_b = 0.1$

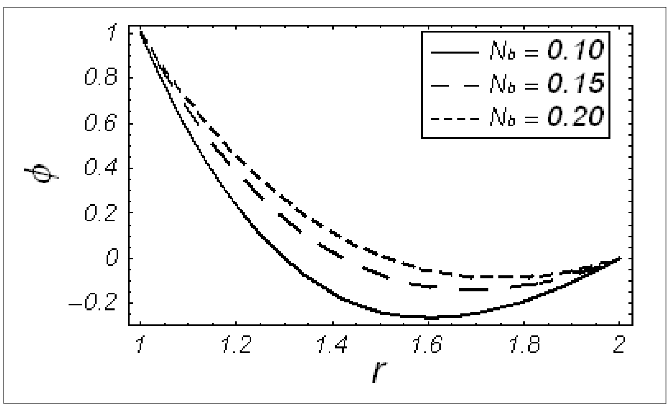

Figure 19. Effects of N_b on nanoparticle concentration for constant viscosity when $P = M = 0.5$ and $N_t = 0.1$

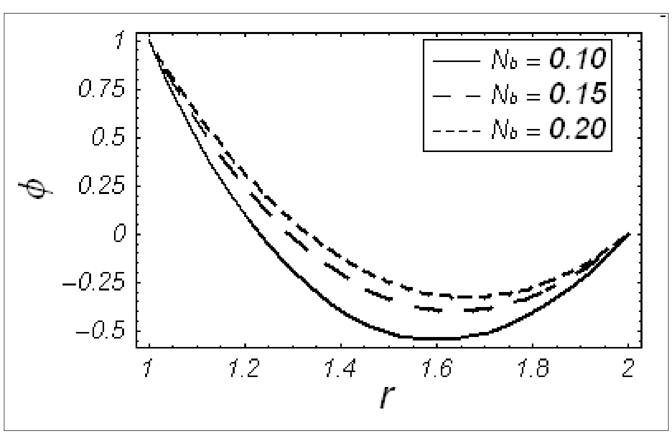

Figure 20. Effects of N_b on nanoparticle concentration for variable viscosity when $P = M = 0.5$ and $N_t = 0.1$.

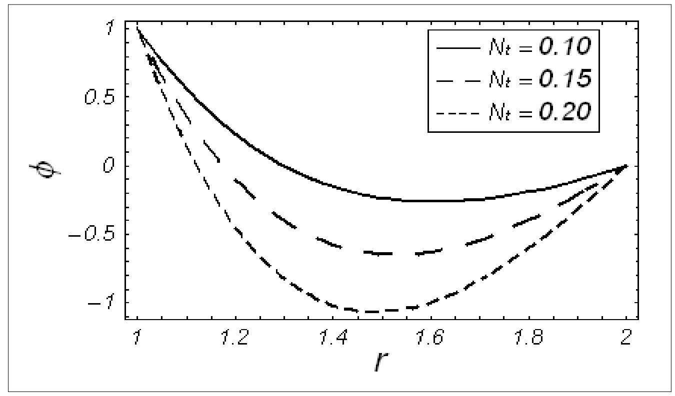

Figure 21. Effects of N_t on nanoparticle concentration for constant viscosity when $P = M = 0.5$ and $N_b = 0.1$

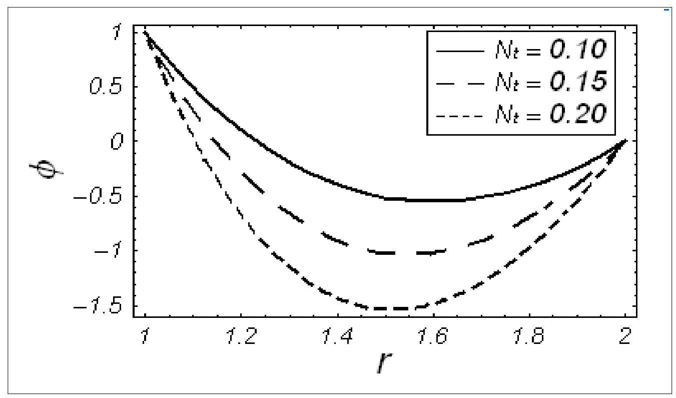

Figure 22. Effects of N_t on nanoparticle concentration for variable viscosity when $P = M = 0.5$ and $N_b = 0.1$

parameter and Brownian diffusion coefficient are different. Figures 19 to 22 bring out the influence of nanoparticles concentration. Here, it was revealed that the mass concentration is an increasing function of N_b and decreases by increasing N_t.

Conclusion

The effects of MHD flow of non-Newtonian nanofluid in coaxial porous cylinders were examined in this study. The resulting nonlinear differential system was solved by employing HAM. In this method, we control the convergence by an auxiliary parameter $\hbar$. Variations of MHD parameter M and porosity parameter P on velocity and temperature profiles was analyzed for constant and variable viscosity as shown in Figures 7 to

18. The effects of thermophoresis parameter and Brownian diffusion coefficient on temperature profile are as shown in Figures 19 to 22. It is interesting to note that when $P = M = G_r = B_r = 0$, then one recovers the case (Sutton and Sherman, 1965). The problem reduces to the case Sutton and Sherman (1965) for $P = G_r = B_r = 0$. The case of Ellahi and Riaz (2010) can also be recovered by $M = G_r = B_r = 0$. It is also worth mentioning that the presented solutions are valid for all values of sundry parameters. To the best of our knowledge, the series solutions by HAM for this particular model have not been published yet.

ACKNOWLEDGEMENTS

The authors will like to thank USEFP for honoring through Fulbright Fellowship and CIES which provided them financial support under OLF, thus, making it possible for them visit to the Pennsylvania State University, York Campus, USA. They also thank the Higher Education Commission of Pakistan and PCST for awarding them with NRPU and Productive Scientists of Pakistan Awards.

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