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Full Length Research Paper

Bayesian analysis of the two component mixture of inverted exponential distribution under quadratic loss function

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The inverted exponential distribution is studied as a prospective life distribution. A two component mixture of inverted exponential distribution is considered in this paper. The Bayes estimators and Bayes posterior risk for the unknown parameters θ_1, θ_2 and mixing weight p of the mixture model are derived under quadratic loss function. For comparative study of these Bayes estimates uniform, improper and informative priors are considered. The Bayes and maximum likelihood estimators and Bayes posterior risks are viewed as a function of the test termination time. As a special case, the limiting expressions for these estimates are derived under the condition of infinite test termination time. Finally, a mixture data is simulated and numerical study is given to illustrate the results.

Key words: Inverted exponential distribution, mixture models, Bayes estimates, quadratic loss function, fixed test termination time.

INTRODUCTION

In a statistical perspective for a given observation and its estimation beside what risk we can expect for it, we may be interested in which probability the corresponding loss is going to occur. Quadratic loss function is a simple and meaningful function for approximating the quality loss in most situations. Berger (1985) discussed a number of loss functions in the literature of statistical decision theory. To study a population that is supposed to comprise a number of subpopulations, a finite mixture of some suitable probability distributions mixed in an unknown proportion can be used. Everitt and Hand (1981) discussed finit mixture models for different probabiltiy distribution. Saleem and Aslam (2008a) worked out on prior selection for the mixture of Rayleigh distribution using predictive intervals. Saleem and Aslam (2008b) considered a two component mixture of Rayleigh distribution using uniform and jeffrey's priors. Saleem and Aslam (2009) also considered Bayesian analysis of Rayleigh survival time assuming random censor time. Singh et al. (2010) find out Bayesian estimator of inverse Gaussian parameters under general entropy loss function using Lindley's approximation. Gamma, lognormal and inverse Gaussian distributions are commonly used models in life testing in reliability studies. One of the mentioned distributions can be used in many applications if the failure is mainly due to aging or the wearing out process. Sanku Dey (2007) considered the inverted exponential distribution as a life distribution and studied it from a Bayesian viewpoint. We consider the two component mixture of inverted exponential distribution.

THE POPULATION AND MODEL

We consider a two component mixture of inverted exponential distributions with unknown parameters θ_1, θ_2 and unknown mixing weights p and q where $q = 1 - p$. Let $f_1(t) = \frac{1}{t^2\theta_1}\exp\left(-\frac{1}{t\theta_1}\right)$ and $f_2(t) = \frac{1}{t^2\theta_2}\exp\left(-\frac{1}{t\theta_2}\right)$;$\theta_1, \theta_1, t > 0$ be the density functions

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of two inverted exponential distributions with parameters θ_1 and θ_2, respectively, then the density function of two component mixture with mixing weights p and q can be written as:

$$f(t) = pf_1(t) + qf_2(t) \quad (1)$$

The corresponding distribution function of the mixture distribution is:

$$F(t) = pF_1(t) + qF_2(t) \quad (2)$$

Where $F_1(t) = \exp\left(-\frac{1}{t\theta_1}\right)$ and $F_2(t) = \exp\left(-\frac{1}{t\theta_2}\right)$ are the distribution functions of two inverted exponential distributions with parameters θ_1 and θ_2, respectively. The quadratic loss function can be defined as:

$$L_Q(\hat{\theta}, \theta) = \left(1 - \frac{\hat{\theta}}{\theta}\right)^2 \quad (3)$$

where $\hat{\theta}$ is the estimate of parameter θ

The Bayes estimate $\hat{\theta}_Q$ of θ under quadratic loss is given by:

$$\hat{\theta}_Q = \frac{E_\theta\left(1/_\theta\right)}{E_\theta\left(1/_{\theta^2}\right)} \quad (4)$$

And the corresponding posterior risk is given by:

$$\rho(\hat{\theta}_Q) = \left(1 - \frac{\left(E_\theta\left(1/_\theta\right)\right)^2}{E_\theta\left(1/_{\theta^2}\right)}\right) \quad (5)$$

SAMPLING

Let n units from the mixture model be employed to a life testing experiment with a test termination time T. Let the test be conducted and it is observed that out of n, r units have life time in the interval $[0, T]$ and $n - r$ units are still working when the test termination time is over. Suppose that r_1 and r_2 objects are identified as the members of subpopulation I and II, respectively such that $r = r_1 + r_2$. We define t_{ij}, the failure time of the jth unit belonging to the ith subgroup. Where $j = 1,2, \ldots, r_i$; $i = 1,2$; $0 < t_{1j}, t_{2j} < T$.

The likelihood function

The likelihood function for the given mixture distribution can be written as:

$$L(\theta_1, \theta_2, p; t) \propto \left\{\prod_{j=1}^{r_1} pf(t_{1j})\right\}\left\{\prod_{j=1}^{r_2} qf(t_{2j})\right\}\{1 - F(T)\}^{n-r} \quad (6)$$

Where $F(t) = p\exp\left(-\frac{1}{T\theta_1}\right) + q\exp\left(-\frac{1}{T\theta_2}\right)$

It becomes:

$$L(\theta_1, \theta_2, p; t) \propto \sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}p^{a_k}q^{b_l}\left(\frac{1}{\theta_1}\right)^{r_1}\left(\frac{1}{\theta_2}\right)^{r_2}\exp\left(-\frac{A_{1k}}{\theta_1}\right)\exp\left(-\frac{A_{2l}}{\theta_2}\right) \quad (7)$$

Where

$$a_k = r_1 + k,\ b_l = r_2 + l,\ A_{1k} = S_1 + \frac{k}{T}, \qquad A_{2l} = S_2 + \frac{l}{T},$$
$$S_1 = \sum_{j=1}^{r_1}\frac{1}{t_{1j}}\ and\ S_2 = \sum_{j=1}^{r_2}\frac{1}{t_{2j}}$$

It can be written as:

$$L(\theta_1, \theta_2, p; t) = K\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}p^{a_k}q^{b_l}\left(\frac{1}{\theta_1}\right)^{r_1}\left(\frac{1}{\theta_2}\right)^{r_2}\exp\left(-\frac{A_{1k}}{\theta_1}\right)\exp\left(-\frac{A_{2l}}{\theta_2}\right)) \quad (8)$$

Which is the likelihood function of the above mixture distribution where K is the normalizing constant.

MAXIMUM LIKELIHOOD ESTIMATES (MLEs) OF θ_1, θ_2 and p

Taking log on the both sides of Equation 6, we get:

$$L(\theta_1,\theta_2,p;t) = K\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}p^{a_k}q^{b_l}\left(\frac{1}{\theta_1}\right)^{r_1}\left(\frac{1}{\theta_2}\right)^{r_2}\exp\left(-\frac{A_{1k}}{\theta_1}\right)\exp\left(-\frac{A_{2l}}{\theta_2}\right) \quad (9)$$

Partially differentiating Equation 9 with respect to $\theta_1, \theta_2 and\ p$, respectively and equating to zero we get the following expressions:

Solving $\frac{\partial \ln L(\theta_1,\theta_2,p;t)}{\partial\theta_1} = 0$

$$\Rightarrow r_1 = \frac{S_1}{\theta_1} - \frac{p(n-r)\exp\left(-\frac{1}{T\theta_1}\right)}{T\theta_1\left\{1-p\exp\left(-\frac{1}{T\theta_1}\right)-q\exp\left(-\frac{1}{T\theta_2}\right)\right\}} \quad (10)$$

Similarly solving $\frac{\partial \ln L(\theta_1,\theta_2,p;t)}{\partial\theta_2} = 0$

$$\Rightarrow r_2 = \frac{S_2}{\theta_2} - \frac{q(n-r)\exp\left(-\frac{1}{T\theta_2}\right)}{T\theta_2\left\{1-p\exp\left(-\frac{1}{T\theta_1}\right)-q\exp\left(-\frac{1}{T\theta_2}\right)\right\}} \quad (11)$$

Similarly from $\frac{\partial \ln L(\theta_1,\theta_2,p;t)}{\partial p} = 0$

$$\Rightarrow r_2 = \frac{qr_1}{p} - \frac{q(n-r)\left\{\exp\left(-\frac{1}{T\theta_2}\right)-\exp\left(-\frac{1}{T\theta_1}\right)\right\}}{\left\{1-p\exp\left(-\frac{1}{T\theta_1}\right)-q\exp\left(-\frac{1}{T\theta_2}\right)\right\}} \quad (12)$$

The maximum likelihood estimates (MLEs) of $\theta_1, \theta_2 and\ p$ can be obtained by solving Equations 10, 11 and 12 simultaneously. It is not possible to solve the above system of equations analytically. However, they can be solved by numerical iterative procedures.

EXPRESSION FOR THE BAYES ESTIMATORS USING UNIFORM PRIOR

Let us assume that $\theta_1, \theta_2 and\ p$ are uniformly distributed over$(0,\infty)$. Thus, their priors are $g_1(\theta_1) \propto k_1$, $g_2(\theta_2) \propto k_2$ and $g_3(p) = 1$, respectively. Assuming the independence of $\theta_1, \theta_2 and\ p$, the joint priori can be written as $g(\theta_1,\theta_1,p) \propto k$. Using this joint prior and the likelihood function of Equation 8, the expression for the joint posterior distribution of $\theta_1, \theta_2 and\ p$ can be written as:

$$P(\theta_1,\theta_1,p|t) \propto L(\theta_1,\theta_1,p;t)g(\theta_1,\theta_1,p)$$

$$P(\theta_1,\theta_2,p|t) \propto \sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}p^{a_k}q^{b_l}\left(\frac{1}{\theta_1}\right)^{r_1}\left(\frac{1}{\theta_2}\right)^{r_2}\exp\left(-\frac{A_{1k}}{\theta_1}\right)\exp\left(-\frac{A_{2l}}{\theta_2}\right)$$

$$P(\theta_1,\theta_2,p|t) = K\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}p^{a_k}q^{b_l}\left(\frac{1}{\theta_1}\right)^{r_1}\left(\frac{1}{\theta_2}\right)^{r_2}\exp\left(-\frac{A_{1k}}{\theta_1}\right)\exp\left(-\frac{A_{2l}}{\theta_2}\right) \quad (13)$$

where, K is the normalizing constant.

Solving this expression for K we get:

$$K = \frac{1}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}B(a_k+1,b_l+1)\frac{\Gamma(r_1-1)\Gamma(r_2-1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}$$

So, Equation 13 will become:

$$P(\theta_1,\theta_2,p|t) = \frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}p^{a_k}q^{b_l}\left(\frac{1}{\theta_1}\right)^{r_1}\left(\frac{1}{\theta_2}\right)^{r_2}\exp\left(-\frac{A_{1k}}{\theta_1}\right)\exp\left(-\frac{A_{2l}}{\theta_2}\right)}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}B(a_k+1,b_l+1)\frac{\Gamma(r_1-1)\Gamma(r_2-1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}} \quad (14)$$

Where $B(.,.)$ and $\Gamma(.)$ are the beta and gamma functions. Using the marginal posterior distributions of $\theta_1, \theta_2 and\ p$, the expressions for the Bayes estimates and their corresponding posterior risks can be obtained.

Under the quadratic loss function, the Bayes estimates are as follows:

$$\hat{\theta}_1|t = \frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2-1}}}{(r_1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1+1}(A_{2l})^{r_2-1}}} \quad (15)$$

$$\hat{\theta}_2|t = \frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2}}}{(r_2)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2+1}}} \quad (16)$$

$$\hat{p}|t = \frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k-1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}} \quad (17)$$

The expressions for the posterior risks can be obtained from following expressions:

$$\rho(\hat{\theta}_1|t) = \left[1 - \frac{\left\{\frac{(r_1-1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2-1}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}\right\}^2}{\left\{\frac{r_1(r_1-1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1+1}(A_{2l})^{r_2-1}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}\right\}}\right] \quad (18)$$

$$\rho(\hat{\theta}_2|t) = \left[1 - \frac{\left\{\frac{(r_2-1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}\right\}^2}{\left\{\frac{r_2(r_2-1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2+1}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}\right\}}\right] \quad (19)$$

$$\rho(\hat{p}|t) = \left[1 - \frac{\left\{\frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}\right\}^2}{\left\{\frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k-1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}\right\}}\right] \quad (20)$$

EXPRESSION FOR THE BAYES ESTIMATORS AND THEIR POSTERIOR RISKS USING INFORMATIVE PRIORS

Let us assume that the prior distributions of θ_1 *and* θ_2 are inverse gamma with hyper parameters (a_1, b_1) *and* (a_2, b_2), respectively whereas priori of p is beta with hyper parameters (a, b). Under the assumption that θ_1, θ_2 *and* p are independently distributed, then, the joint prior can be written as:

$$g(\theta_1, \theta_2, p) \propto p^{a-1}\, q^{b-1} \left(\frac{1}{\theta_1}\right)^{a_1+1} \left(\frac{1}{\theta_2}\right)^{a_2+1} \exp\left(-\frac{b_1}{\theta_1}\right) \exp\left(-\frac{b_2}{\theta_2}\right) \tag{21}$$

Using this joint informative priori and the likelihood function of Equation 8, the joint posterior distribution of θ_1, θ_2 *and* p is as follows:

$$P(\theta_1, \theta_2, p|t) = \frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} p^{\alpha_k - 1}\, q^{\beta_l - 1} \left(\frac{1}{\theta_1}\right)^{\alpha_1+1}\left(\frac{1}{\theta_2}\right)^{\alpha_2+1} \exp\left(-\frac{\beta_{1k}}{\theta_1}\right)\exp\left(-\frac{\beta_{2l}}{\theta_2}\right)}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \mathrm{B}(\alpha_k, \beta_l) \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}} \tag{22}$$

Where $\alpha_k = a_k + a$, $\beta_l = b_l + b, \alpha_1 = r_1 + a_1, \alpha_2 = r_2 + a_2, \beta_{1k} = A_{1k} + b_1$ and $\beta_{2l} = A_{2l} + b_2$.

Using the marginal posterior distributions of θ_1, θ_2 *and* p, the expressions for the Bayes estimates and their corresponding posterior risks can be obtained.

Under the quadratic loss function, the Bayes estimates are as follows:

$$\hat{\theta}_1|t = \frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \frac{\mathrm{B}(\alpha_k, \beta_l)}{(\beta_{1k})^{\alpha_1+1}(\beta_{2l})^{\alpha_2}}}{(\alpha_1+1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \frac{\mathrm{B}(\alpha_k, \beta_l)}{(\beta_{1k})^{\alpha_1+2}(\beta_{2l})^{\alpha_2}}} \tag{23}$$

$$\hat{\theta}_2|t = \frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \frac{\mathrm{B}(\alpha_k, \beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2+1}}}{(\alpha_2+1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \frac{\mathrm{B}(\alpha_k, \beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2+2}}} \tag{24}$$

$$\hat{p}|t = \frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \frac{\mathrm{B}(\alpha_k - 1, \beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \frac{\mathrm{B}(\alpha_k - 2, \beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}} \tag{25}$$

Expressions for the posterior risks are:

$$\rho(\hat{\theta}_1|t) = \left[1 - \frac{\left\{\dfrac{\alpha_1 \sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \frac{\mathrm{B}(\alpha_k, \beta_l)}{(\beta_{1k})^{\alpha_1+1}(\beta_{2l})^{\alpha_2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \frac{\mathrm{B}(\alpha_k, \beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}}\right\}^2}{\left\{\dfrac{\alpha_1(\alpha_1+1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \frac{\mathrm{B}(\alpha_k, \beta_l)}{(\beta_{1k})^{\alpha_1+2}(\beta_{2l})^{\alpha_2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k \binom{n-r}{k}\binom{k}{l} \frac{\mathrm{B}(\alpha_k, \beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}}\right\}}\right] \tag{26}$$

$$\rho\left(\hat{\theta}_2|t\right)=\left[1-\frac{\left\{\frac{\alpha_2\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(\alpha_k,\beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2+1}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(\alpha_k,\beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}}\right\}^2}{\left\{\frac{\alpha_2(\alpha_2+1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(\alpha_k,\beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2+2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(\alpha_k,\beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}}\right\}}\right] \quad (27)$$

$$\rho\left(\hat{p}|t\right)=\left[1-\frac{\left\{\frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(\alpha_k-1,\beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(\alpha_k,\beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}}\right\}^2}{\left\{\frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(\alpha_k-2,\beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(\alpha_k,\beta_l)}{(\beta_{1k})^{\alpha_1}(\beta_{2l})^{\alpha_2}}}\right\}}\right] \quad (28)$$

EXPRESSION FOR THE BAYES ESTIMATORS AND THEIR POSTERIOR RISKS USING IMPROPER PRIORS

Assuming the improper priors for θ_1, θ_2 and p such that $g_1 \propto \frac{1}{\theta_1}$, $g_2(\theta_2) \propto \frac{1}{\theta_2}$ and $g_3(p) = 1$. Using the independence of these parameters, the joint priori can be written as $g(\theta_1, \theta_2, p) \propto \frac{1}{\theta_1\theta_2}$. Combining this joint prior with the likelihood function of Equation 8, the joint posterior distribution will be as follows:

$$P(\theta_1,\theta_2,p|t)=\frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}p^{a_k}q^{b_l}\left(\frac{1}{\theta_1}\right)^{r_1+1}\left(\frac{1}{\theta_2}\right)^{r_2+1}\exp\left(-\frac{A_{1k}}{\theta_1}\right)\exp\left(-\frac{A_{2l}}{\theta_2}\right)}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}B(a_k+1,b_l+1)\frac{\Gamma(r_1)\Gamma(r_2)}{(A_{1k})^{r_1}(A_{2l})^{r_2}}} \quad (29)$$

Using the marginal posterior distributions of θ_1, θ_2 and p, the expressions for the Bayes estimates and their corresponding posterior risks under quadratic loss function are as follows:

$$\hat{\theta}_1|t=\frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1+1}(A_{2l})^{r_2}}}{(r_1+1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1+2}(A_{2l})^{r_2}}} \quad (30)$$

$$\hat{\theta}_2|t=\frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2+1}}}{(r_2+1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2+2}}} \quad (31)$$

$$\hat{p}|t = \frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k-1,b_l+1)}{(A_{1k})^{r_1-1}(A_{2l})^{r_2-1}}} \quad (32)$$

The expressions for the posterior risks can be obtained from following expressions:

$$\rho(\hat{\theta}_1|t) = \left[1 - \frac{\left\{\frac{(r_1+1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1+1}(A_{2l})^{r_2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2}}}\right\}^2}{\left\{\frac{r_1(r_1+1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1+2}(A_{2l})^{r_2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2}}}\right\}}\right] \quad (33)$$

$$\rho(\hat{\theta}_2|t) = \left[1 - \frac{\left\{\frac{(r_2+1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2+1}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2}}}\right\}^2}{\left\{\frac{r_2(r_2+1)\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2+2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2}}}\right\}}\right] \quad (34)$$

$$\rho(\hat{p}|t) = \left[1 - \frac{\left\{\frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2}}}\right\}^2}{\left\{\frac{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k-1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2}}}{\sum_{k=0}^{n-r}\sum_{l=0}^{k}(-1)^k\binom{n-r}{k}\binom{k}{l}\frac{B(a_k+1,b_l+1)}{(A_{1k})^{r_1}(A_{2l})^{r_2}}}\right\}}\right] \quad (35)$$

THE LIMITING EXPRESSIONS

When the sample is uncensored, $T \to \infty$, $r_1 \to n_1, r_2 \to n_2$ and $r \to n$ consequently all the observations are incorporated in the sample thus, we get maximum information for analysis. In this case, the likelihood function of the mixture model become:

$$L(\theta_1,\theta_2,p;t) \propto p^{n_1} q^{n_2}\left(\frac{1}{\theta_1}\right)^{n_1}\left(\frac{1}{\theta_2}\right)^{n_2}\exp\left(-\frac{S_1}{\theta_1}\right)\exp\left(-\frac{S_2}{\theta_2}\right) \quad (36)$$

Expression for maximum likelihood estimates

Using the likelihood function of Equation 36, the limiting expression for the MLEs of θ_1, θ_2 and p, respectively becomes as follows:

$$\hat{\theta}_1 = \frac{S_1}{n_1} \quad (37)$$

$$\hat{\theta}_2 = \frac{S_2}{n_2} \quad 38)$$

$$\hat{p} = \frac{n_1}{n_1+n_2} \quad (39)$$

$$where\ S_1 = \sum_{j=1}^{n_1} \frac{1}{t_{1j}}\ and\ S_2 = \sum_{j=1}^{n_2} \frac{1}{t_{2j}}$$

And their corresponding variances are:

$$var(\hat{\theta}_1) = \frac{S_1^2}{n_1^3} \quad (40)$$

$$var(\hat{\theta}_2) = \frac{S_2^2}{n_2^3} \quad (41)$$

$$var(\hat{p}) = \frac{n_1 n_2}{(n_1+n_2)^3} \quad (42)$$

Expression for the Bayes estimates using uniform prior

Assuming that $\theta_1, \theta_2\ and\ p$ are uniformly distributed over $(0,\infty)$, the joint prior distribution of these parameters can be written as $g(\theta_1, \theta_1, p) \propto c$, where c is some constant. Using this prior with likelihood of Equation 36, the joint posterior distribution of $\theta_1, \theta_2\ and\ p$ can be written as follows:

$$P(\theta_1, \theta_2, p; t) = Kp^{n_1} q^{n_2} \left(\frac{1}{\theta_1}\right)^{n_1} \left(\frac{1}{\theta_2}\right)^{n_2} \exp\left(-\frac{S_1}{\theta_1}\right) \exp\left(-\frac{S_2}{\theta_2}\right) \quad (43)$$

Where K is the normalizing constant. Thus, the limiting expressions for the Bayes estimates of $\theta_1, \theta_2\ and\ p$ are:

$$\hat{\theta}_1 = \frac{S_1}{n_1} \quad (44)$$

$$\hat{\theta}_2 = \frac{S_2}{n_2} \quad (45)$$

$$\hat{p} = \frac{n_1-1}{n_1+n_2} \quad (46)$$

And their corresponding posterior risks are:

$$\rho(\hat{\theta}_1) = \frac{1}{n_1} \quad (47)$$

$$\rho(\hat{\theta}_2) = \frac{1}{n_2} \quad (48)$$

$$\rho(\hat{p}) = 1 - \frac{(n_1-1)(n_1+n_2+1)}{n_1(n_1+n_2)} \quad (49)$$

respectively.

Expression for the Bayes estimates using informative prior

Assuming the informative priors for $\theta_1, \theta_2\ and\ p$ such that $\theta_1\ and\ \theta_2$ are independently distributed as inverse gamma with hyper parameters $(a_1, b_1)\ and\ (a_2, b_2)$, respectively and p is distributed as beta with hyper parameters(a, b), then, using the joint prior distribution of $\theta_1, \theta_2\ and\ p$ from Equation 21 and the likelihood of Equation 36, the joint posterior distribution can be written as:

$$P(\theta_1, \theta_2, p; t) = Kp^{\alpha-1}\ q^{\beta-1} \left(\frac{1}{\theta_1}\right)^{\alpha_1+1} \left(\frac{1}{\theta_2}\right)^{\alpha_2+1} \exp\left(-\frac{\beta_1}{\theta_1}\right) \exp\left(-\frac{\beta_2}{\theta_2}\right) \quad (50)$$

$$where\ \alpha = n_1 + a\ , \beta = n_2 + b, \alpha_1 = n_1 + a_1, \alpha_2 = n_2 + a_2, \beta_1 = S_1 + b_1 \text{and } \beta_2 = S_2 + b_2$$

The value of normalizing constant is $K = \frac{1}{\mathrm{B}(\alpha,\beta)\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(\beta_1)^{\alpha_1}(\beta_2)^{\alpha_2}}}$

Thus, the limiting expressions for the Bayes estimates of $\theta_1, \theta_2\ and\ p$ are, respectively as follows:

$$\hat{\theta}_1 = \frac{\beta_1}{\alpha_1+1} \quad (51)$$

$$\hat{\theta}_2 = \frac{\beta_2}{\alpha_2+1} \quad (52)$$

$$\hat{p} = \frac{\alpha-2}{(\alpha+\beta-2)} \quad (53)$$

And their corresponding posterior risks are:

$$\rho(\hat{\theta}_1) = \frac{1}{\alpha_1+1} \quad (54)$$

$$\rho(\hat{\theta}_2) = \frac{1}{\alpha_2+1} \quad (55)$$

Table 1. Bayes estimates of 〖 θ〗_1 and their posterior risks (in brackets) with different sample size.

Prior / *n*	Uniform	Informative	Improper
100	11.7468 (0.02777)	10.33069 (0.02381)	11.4294 (0.02703)
200	11.10802 (0.01351)	10.41245 (0.01250)	10.95991 (0.01333)
300	10.63034 (0.00901)	10.17922 (0.00855)	10.53543 (0.00893)
400	10.64432 (0.00680)	10.29881 (0.00654)	10.57240 (0.00676)

Table 2. Bayes estimates of 〖 θ〗_2 and their posterior risks (in brackets) with different sample size.

Prior / *n*	Uniform	Informative	Improper
100	13.7567 (0.01613)	12.49151 (0.01449)	13.5383 (0.01587)
200	14.04008 (0.00813)	13.35332 (0.00769)	13.92686 (0.00806)
300	13.44101 (0.00535)	13.00243 (0.00515)	13.36953 (0.00532)
400	13.26822 (0.00403)	12.93930 (0.00392)	13.21494 (0.00402)

$$\rho(\hat{p}) = 1 - \frac{(\alpha-2)(\alpha+\beta-1)}{(\alpha-1)(\alpha+\beta-2)} \quad (56)$$

Expression for the Bayes estimates using improper prior

Assuming the independent improper priors for θ_1, θ_2 *and* p such that $g_1 \propto \frac{1}{\theta_1}$, $g_2(\theta_2) \propto \frac{1}{\theta_2}$ and $g_3(p) = 1$, the joint priori can be written as $g(\theta_1, \theta_2, p) \propto \frac{1}{\theta_1 \theta_2}$.

Combining this joint prior with the likelihood of Equation 36, the joint posterior distribution of θ_1, θ_2 *and* p given data can be written as:

$$P(\theta_1, \theta_2, p; t) = \frac{p^{n_1} q^{n_2} \left(\frac{1}{\theta_1}\right)^{n_1+1} \left(\frac{1}{\theta_2}\right)^{n_2+1} \exp\left(-\frac{S_1}{\theta_1}\right) \exp\left(-\frac{S_2}{\theta_2}\right)}{B(n_1+1, n_2+1) \frac{\Gamma(n_1)\Gamma(n_2)}{(S_1)^{n_1}(S_2)^{n_2}}} \quad (57)$$

Thus, the limiting expressions for the Bayes estimates of θ_1, θ_2 *and* p using Equation 57 are, respectively as follows:

$$\hat{\theta}_1 = \frac{S_1}{n_1+1} \quad (58$$

$$\hat{\theta}_2 = \frac{S_2}{n_2+1} \quad (59)$$

$$\hat{p} = \frac{n_1-1}{n_1+n_2} \quad (60)$$

And their corresponding posterior risks are:

$$\rho(\hat{\theta}_1) = \frac{1}{n_1+1} \quad (61)$$

$$\rho(\hat{\theta}_2) = \frac{1}{n_2+1} \quad (62)$$

$$\rho(\hat{p}) = 1 - \frac{(n_1-1)(n_1+n_2+1)}{n_1(n_1+n_2)} \quad (63)$$

NUMERICAL EXAMPLE

We take a random sample of size $n = 100$ from the mixture of two component inverted exponential distribution truncated at $T = 10$. To generate a mixture data, we make use of probabilistic mixing with probability p and $1-p$ taking $p = 0.375$. A uniform number $'u'$ is generated 1000 times and if $u \leq p$, the observation is taken from F_1 (the inverted exponential distribution with parameter $\theta_1 = 10$) and from F_2 (the inverted exponential distribution with parameter $\theta_2 = 13$) otherwise. As one data set does not help to clarify the performance of method, we simulate this procedure 1000 times. Also, different sample sizes are considered and the values of Bayes estimates and their posterior risks are obtained. A comparison of the estimates for $n = 100, 200, 300$ *and* 400 are shown in Tables 1 to 3.

Bayes estimators with different truncation time

Different truncation times are considered as well and the

Table 3. Bayes estimates of p and their posterior risks (in brackets) with different sample size.

Prior / *n*	Uniform	Informative	Improper
100	0.34347 (0.01825)	0.34295 (0.01731)	0.34348 (0.01825)
200	0.35989 (0.00866)	0.35915 (0.00844)	0.35990 (0.00866)
300	0.36438 (0.00573)	0.36378 (0.00563)	0.36438 (0.00573)
400	0.36074 (0.00435)	0.36035 (0.00430)	0.36075 (0.00435)

Table 4. Bayes estimates of θ_1 and their posterior risks (in brackets) with different Truncation time T.

Prior / T	Uniform	Informative	Improper
10	10.63034 (0.00901)	10.17922 (0.00855)	10.53543 (0.00893)
20	10.63165 (0.00901)	10.18046 (0.00855)	10.53673 (0.00893)
30	10.63210 (0.00901)	10.18088 (0.00855)	10.53717 (0.00893)
40	10.63232 (0.00901)	10.18109 (0.00855)	10.53739 (0.00893)
50	10.63246 (0.00901)	10.18122 (0.00855)	10.53752 (0.00893)
∞	10.4629 (0.00893)	10.0241 (0.00847)	10.3703 (0.00885)

values of Bayes estimates and their posterior risks are obtained. A comparison of the estimates for $T = 10, 20, 30, 40 \text{ and } 50$ are shown in Tables 4 to 6.

Limiting expressions for Bayes and ml estimators

Numerical results for limiting expressions (that is, as $T \to \infty$) of Bayes and MLEs are shown in Tables 7 to 9.

Table 5. Bayes estimates of θ_2 and their posterior risks (in brackets) with different truncation time T.

Prior / T	Uniform	Informative	Improper
10	13.44102 (0.00535)	13.00243 (0.00516)	13.36952 (0.00532)
20	13.44132 (0.00535)	13.00271 (0.00515)	13.36982 (0.00532)
30	13.44141 (0.00535)	13.00281 (0.00515)	13.36992 (0.00532)
40	13.44147 (0.00535)	13.00286 (0.00515)	13.36997 (0.00532)
50	13.44150 (0.00535)	13.00289 (0.00515)	13.37000 (0.00532)
∞	13.3021 (0.00532)	12.8707 (0.00513)	13.2317 (0.00529)

Table 6. Bayes estimates of p and their posterior risks (in brackets) with different truncation time T.

Prior / T	Uniform	Informative	Improper
10	0.36438 (0.00573)	0.36378 (0.00563)	0.36438 (0.00573)
20	0.36431 (0.00573)	0.36371 (0.00563)	0.36431 (0.00573)
30	0.36428 (0.00573)	0.36368 (0.00563)	0.36428 (0.00573)
40	0.36427 (0.00573)	0.36367 (0.00563)	0.36427 (0.00573)
50	0.36427 (0.00573)	0.36367 (0.00563)	0.36427 (0.00573)
∞	0.3700 (0.00562)	0.36928 (0.00553)	0.3700 (0.00562)

CONCLUSION

In real life phenomena, the importance of mixture models is un-deniable. In addition to the advantage of additional

Table 7. Bayes and ML estimate of θ_1 and their posterior risks (in brackets) with different sample size (limiting case that is, T→∞).

Prior / *n*	Uniform	Informative	Improper	MLE
100	9.30329 (0.02632)	8.28466 (0.02273)	9.06474 (0.02564)	9.30329 (2.27766)
200	10.8184 (0.01333)	10.1528 (0.01234)	10.676 (0.01316)	10.8184 (1.5605)
300	10.4629 (0.00893)	10.0241 (0.00847)	10.3703 (0.00885)	10.4629 (0.97744)
400	10.5127 (0.00667)	10.1789 (0.00641)	10.4431 (0.00662)	10.5127 (0.73677)

Table 8. Bayes and ML estimate of θ_2 and their posterior risks (in brackets) with different sample size (limiting case that is, T→∞).

Prior / *n*	Uniform	Informative	Improper	MLE
100	13.5314 (0.01613)	12.2891 (0.01449)	13.31660 (0.01587)	13.5314 (2.95319)
200	13.3904 (0.0080)	12.7485 (0.00758)	13.2841 (0.00794)	13.3904 (1.43442)
300	13.3021 (0.00532)	12.8707 (0.00513)	13.2317 (0.00529)	13.3021 (0.94119)
400	13.1532 (0.0040)	12.8299 (0.00389)	13.1008 (0.00398)	13.1532 (0.69202)

Table 9. Bayes and ML estimate of p and their posterior risks with different sample size (limiting case that is, T→∞).

Prior / *n*	Uniform	Informative	Improper	MLE
100	0.3700 (0.01658)	0.36792 (0.01580)	0.3700 (0.01658)	0.38000 (0.00236)
200	0.3700 (0.0084)	0.36893 (0.00820)	0.3700 (0.0084)	0.3750 (0.00117)
300	0.3700 (0.00562)	0.36928 (0.00553)	0.3700 (0.00562)	0.37333 (0.00078)
400	0.3725 (0.00418)	0.37192 (0.00413)	0.3725 (0.00418)	0.3750 (0.00059)

information provided by prior distributions in Bayesian statistics, another advantage of Bayes estimates over the MLEs is that they can be easily evaluated for the mixture models. Whereas, MLE can only be obtained by some iterative procedures for mixture models. Furthermore from Tables 1, 2 and 3, it is clear that the Informative priors provide more accurate and efficient Bayes estimates than those of the uniform and improper priors. Also, an increase in sample size provides us improved estimates. Tables 4, 5 and 6 show that as the test termination time is increased, estimates become closer to the real parametric value with an increase in efficiency as well. Which also supports the theory that as test termination time is increased more observations are incorporated in the sample and thus, more information sample contains. Finally, it is also clear from Tables 7, 8 and 9 that the limiting expressions for the Bayes estimates using Informative prior outperformed the MLEs as well.

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