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Full Length Research Paper

On nonstandard finite difference schemes for initial value problems in ordinary differential equations

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In this paper, we present the theory of nonstandard finite difference schemes which can be used to solve some initial value problems in ordinary differential equations. Methods of construction and mode of implementation of these methods are also discussed. We also examine the stability properties of the constructed schemes.

Keywords: Nonstandard, stability, nonlocal approximation, initial value problem, ordinary differential equations.

INTRODUCTION

It is a well known fact that finite difference scheme is one of the oldest and popular techniques for numerical solution of ordinary differential equations. In most of the equations in mathematical physics, engineering and in some physical sciences, finite difference schemes have been designed and investigated both from the theoretical point view, which is the convergence aspect, and the practical point of view which is the consistency and stability point of view (Anguelov and Lubuma, 2000). One may be forced to ask the question; why do we need non standard methods when we have numerous standard methods used to solve ordinary differential equations? One of the shortcomings of standard methods is that qualitative properties of the exact solution are not usually transferred to the numerical solution. In the consideration of the step-size, in practice, the limit of the step-size is not reached. What we obtain is the numerical solution for one or several values of the step-size. This shortcoming may create a lot of problems which may affect the stability properties of the standard approach.

In the nonstandard approach, preservation of some essential properties of exact solution is guaranteed. The source of motivation for this work is that of Anguelov and Lubuma (2003). A systematic procedure based on nonlocal approximation is used to construct a qualitative stable nonstandard finite difference methods for the solution of a differential equations.

We shall consider the initial value problem of the form;

$$\frac{dy}{dt} = f(y), \quad y(t_o) = y_o , \tag{1}$$

where $y(t): [t_o, T) \rightarrow R$ is unknown and the function $f: \mathbf{IR} \rightarrow \mathbf{IR}$ is given. We shall assume that equation (1) satisfy the popular Lipschitz condition. For the numerical approximation of (1), we shall replace the continuous interval $[t_o, T)$ by the mesh of discrete point $\{t_{n+1} = t_o + nh\}$, $n \geq 0$ and $h > 0$ is the step-size. We shall use y_n to represent an approximation to the solution $y(t_n)$ at the point t_n such that $y_n \simeq y(t_n)$. The numerical solution y_n is generated by a finite difference scheme of the form

$$y_{n+1} = F(h,y_n) \tag{2}$$

Nonstandard difference equations of the form (2) were first introduced by Mickens (2000) as a powerful numerical method that preserve significant properties of exact solution of the differential equations (Auguelov and Lubuma, 2003). Nonstandard finite difference schemes were defined as follows using two of Micken's rules.

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Definition 1

The difference equation (2) is called a nonstandard finite difference method if at least one of the following conditions is met.

(a) In the first order discrete derivative that occurs in (2), the conventional
denominator, h, is replaced by a non-negative function $\phi(h)$ such that

$$\phi(h) = h + 0(h^2) \text{ as } h \to 0 \quad (3)$$

(b) Nonlinear terms f(y) are approximated in a nonlocal form, that is by a suitable function of several points of mesh (Anguelov and Lubuma 2003). The advantages of nonstandard methods over the traditional methods can be seen in Anguelov and Lubuma (2000, 2001 and 2003).

Definition 2

Assume that equation (1) satisfy some property P. The numerical scheme is called qualitatively stable with respect to property P (or P-stable) if for every value of h > 0 the set of solutions of (1) satisfy property P. We shall now state some theorems that will be of good use throughout this work.. For the purpose of this work, we shall assume that function F(h, y) in (2) has continuous derivative with respect to both variables for h > o, $y \in R$ and that

$$F(o,y) = y \text{ and } \frac{\partial F}{\partial h}(o,y) = f(y) \quad (4)$$

It is necessary to note that consistency implies (4) if y is the solution of differential equation.

Theorem 1

The difference scheme (2) is stable with respect to monotone dependence on initial
Value if;

$$\frac{\partial F}{\partial y}(h,y) \geq o, \; y \in \mathbf{IR}, \; h > o \quad (5)$$

The proof to this theorem can be seen in Anguelov and Lubuma (2003)

Definition 3

A set $G(\Omega)$ of real-valued functions defined on a subset Ω of $[t_o, \infty)$ is said to be monotonically dependent on the initial value t_o if for every two functions , $y,z \in \Omega$, we have

$$y(t_o) \leq z(t_o) \Rightarrow y(t) \leq z(t), \; t \in \Omega \quad (6)$$

It is necessary to state at the stage that since equation (1) is assumed to satisfy Lipschitz condition, the set of solutions for equation (1) is monotonically dependent on the initial value at t_o.

Definition 4

The finite deference scheme (2) is stable with respect to monotonicity of solutions if for every $y_o \in \mathbf{IR}$, the solution y_n of (2) is an increasing or decreasing sequence.

Theorem 2

Assume that the difference scheme (2) is stable with respect to monotone dependence on initial value. Assume also that for every h > 0, the equations

$$y = F(h,y), \text{ and } f(y) = o \quad (7)$$

have the same roots considered with their multiplicity, then the difference scheme (2) is stable with respect to monotonicity of solutions. The proof can also be found in any of the standard book on nonstandard difference schemes. It must be mentioned here that if the condition in (7) is satisfied, then the difference scheme (2) is elementary stable.

Nonstandard finite difference schemes for $y^1 = y^2$, y(o) = 1

We shall approximate the nonlinear terms in the right hand side of equation

$$y^1 = y^2, \; y(o) = 1 \quad (8)$$

in three different ways. Each of the nonlocal approximation will produce a nonstandard scheme which will be implemented.

$$y^2 \simeq a y_n^2 + (1 - a) y_n y_{n+1}, \; a \in \mathbf{IR}$$

$$y^2 \simeq y_n y_{n+1}$$

$$y^2 \simeq y_n \frac{(y_{n-1} + y_{n+1})}{2}$$

In a general form, any linear combination of the expressions listed above (i) – (iii) with the sum of the coefficients equals 1, approximate y^2, the error of order

O(h) for sufficiently y(t). It must be noted that the function f(y) in equation (1) may be approximated by an expression, which contains certain number of free parameters. These parameters are determined in such a way that the scheme satisfies qualitative stable property.

Nonstandard scheme one

Here

$$y^2 \simeq ay_n^2 + (1-a)y_n y_{n+1}, \quad a\in \mathbf{IR} \qquad (9)$$

Using (8) and (9), we have

$$\frac{y_{n+1} - y_n}{\varphi(h)} = ay_n^2 + (1-a)y_n y_{n+1}, \quad a\in \mathbf{IR}$$

$$y_{n+1} - y_n = a\phi(h)y_n^2 + \phi(h) - a\,\phi(h)y_n y_{n+1}$$

$$y_{n+1}\,(1-(1-a)y_n\,\phi(h)) = y_n + a\,\phi(h)y_n^2$$

$$y_{n+1} = \frac{y_n + a\,\phi(h)y_n^2}{(1-(1-a)y_n\,\phi(h))} \qquad (10)$$

Equation (10) is the nonstandard scheme which is a one-step nonstandard scheme.

Nonstandard scheme two

Here we approximate f(y) in (8) by

$$y^2 \simeq y_n y_{n+1} \qquad (11)$$

Using (8) and (11) we obtain

$$\frac{y_{n+1} - y_n}{\phi(h)} = y_n y_{n+1}$$

$$\Rightarrow \quad y_{n+1} - y_n = \phi(h)\, y_n y_{n+1}$$

that is, $y_{n+1}\,(1-\phi(h)\,y_n) = y_n$

This produce a nonstandard method;

$$y_{n+1} \quad \frac{y_n}{1-\phi(h)y_n} \qquad (12)$$

Nonstandard scheme three

The nonlinear term y^2 is approximated non locally by

$$y^2 \simeq y_n\,\frac{(y_{n-1} + y_{n+1})}{2} \qquad (13)$$

We use (8) and (13) to obtain

$$\frac{y_{n+1} - y_n}{\phi(h)} = y_n\,\frac{(y_{n-1} + y_{n+1})}{2}$$

$$\Rightarrow\; 2y_{n+1} - 2y_n = y_n\,(y_{n-1} + y_{n+1})\;\phi(h)$$

$$\Rightarrow\; 2y_{n+1} - y_n y_{n+1}\,\phi(h) = \phi(h)\,y_n y_{n-1} + 2y_n$$

that is,

$$(2-y_n\,\phi(h))\,y_{n+1} = \phi(h)\,y_n\,y_{n-1} + 2y_n$$

and this produce a nonstandard scheme

$$y_{n+1} = \frac{\phi(h)\,y_n\,y_{n-1} + 2y_n}{2-y_n\,\phi(h)} \qquad (14)$$

Equation (14) can be written compactly as

$$y_{n+2} = \frac{\phi(h)y_{n+1}\;y_n + 2y_{n+1}}{2-y_n\,\phi(h)} \qquad (15)$$

This is a two-step numerical scheme that will need two starting points. We have been able to construct three robust nonstandard finite difference scheme for the solution of initial value problem (8)

Some qualitative stability properties of the schemes

Let us consider the first scheme (10). Equation (10) can be written as

$$F(h, y_n) = \frac{y_n + a\phi(h)y_n^2}{1-(1-a)y_n\,\phi(h)}, \qquad (16)$$

With :

$$F(h, y) = \frac{y + a\phi(h)y^2,}{1-(1-a)y\,\phi(h)} \qquad (17)$$

Here we determine $\frac{\partial F(h,y)}{\partial y} > 0$ and obtain the followings

$$\frac{\partial F(h,y)}{\partial y} = \frac{1-(1-a)y\,\phi(h)[1+2a\phi(h)y] + [y+a\phi(h)y^2]\,[(1-a)\,\phi(h)]}{[1-(1-a)y\,\phi(h)]^2} \geq 0 \qquad (18)$$

Inequality (18) is true if;

$$[1-(1-a)y\,\phi(h)]\,[1+2a\phi(h)y] + [y+a\phi(h)y^2]\,[(1-a)\,\phi(h)] \geq 0 \qquad (19)$$

for this to happen, $a < 0$.

Theorem 3

The nonstandard scheme (10) is stable with respect to monotone dependence on initial value and monotone of solutions and this, the scheme is elementary stable.

Proof

Let us consider

$$F(h, y) = y + \frac{\phi(h)y^2}{1-(1-a)y\,\phi(h)}$$

On simplification, we obtain

$$F(h, y) = \frac{y[1-(1-a)y\,\phi(h)] + \phi(h)y^2}{1-(1-a)y\,\phi(h)}$$

$$= \frac{y - \phi(h)y^2 + a\phi(h)y^2 + \phi(h)y^2}{1-(1-a)\,\phi(h)y}$$

$$= \frac{y + a\phi(h)y^2}{1-(1-a)\,\phi(h)y}$$

We have shown that

$$F(h,y) = y + \frac{a\phi(h)y^2}{1-(1-a)y\,\phi(h)} = \frac{y + a\phi(h)y^2}{1-(1-a)\phi(h)y} = \frac{y + f(y)\phi(h)}{1-(1-a)\phi(h)}$$

The above relationship implies that if $a \leq 0$,

$F(h,y) = y \Leftrightarrow f(y) = 0.$ for every $h > 0$.

This follow that scheme (10) is stable with respect to monotonicity dependence on initial value and monotonicity of solutions and hence the scheme is elementary stable.

Remark

It must be stated here that the function $\phi(h)$ in the scheme (10) remains unspecified. It can be determined to guarantee additional properties of the scheme. We have specified the value of $\phi(h)$ used in our numerical experiments.

We shall now proceed to show that the scheme (12) (14) are stable with respect to monotone dependence on initial value. To show this, we must be able to prove that

$$\frac{\partial F(h,y)}{\partial y} \geq 0,$$

$y \in R$, $h > 0$, when $F(h,y)$ is given as;

$$F(h, y) = \frac{y}{1-\phi(h)y} \tag{20}$$

and

$$F(h, y) = \frac{\phi(h)y^2 + 2y}{2 - y\phi(h)} \tag{21}$$

For equation (21) we have

$$\frac{\partial F}{\partial y} = \frac{[1-\phi(h)y]\,\frac{\partial y}{\partial y} - y\,\frac{\partial[1-\phi(h)y]}{\partial y}}{[1-\phi(h)y]^2}$$

$$= \frac{1-\phi(h)y + \phi(h)y}{[1-\phi(h)y]^2} = \frac{1}{[1-\phi(h)y]^2} > 0$$

Hence $\frac{\partial F}{\partial y} > 0$ since $\phi(h)$ is positive and $h > 0$.

This implies that scheme (12) is stable with respect to monotone dependence on initial value.

Let us consider equation (22)

$$\frac{\partial F(h,y),}{\partial y} = \frac{[2 - y\phi(h)y]\,\frac{\partial}{\partial y}\,[\phi(h)y^2 + 2y] - [\phi(h)y^2 + 2y]\,\frac{\partial}{\partial y}\,[2 - y\,\phi(h)]}{[2 - y\phi(h)]^2}$$

$$= \frac{[2 - y\phi(h)y]\,[2\phi(h)y + 2] - [\phi(h)y^2 + 2y]\,[-\phi(h)]}{[2 - y\phi(h)]^2}$$

$$= \frac{4\phi(h)y + 4 - 2\phi^2(h)y^2 - 2\phi(h)y + \phi^2(h)y^2 + 2\phi(h)y}{[2 - y\phi(h)]^2}$$

$$= \frac{4\phi(h)y + 4 - \phi^2(h)y^2}{[2 - y\phi(h)]^2} \geq 0$$

The above inequality holds if $4\,\phi(h)y + y + 4 - \phi^2(h)y^2 \geq 0$

$$4\phi(h)y + 4 - \phi^2(h)y^2 \geq 0 \tag{22}$$

Inequality (23) is quadratic in y and inquality (23) will hold if

$y = \phi(h) \geq {}^1/_2$.

Therefore scheme (14) will be stable with respect to monotone dependence on initial value and therefore elementary stable if and only if $y = \phi(h) \leq {}^1/_2$.

Numerical implementation

A Fortran programme was written to implement the three schemes and the results are presented in a graphical

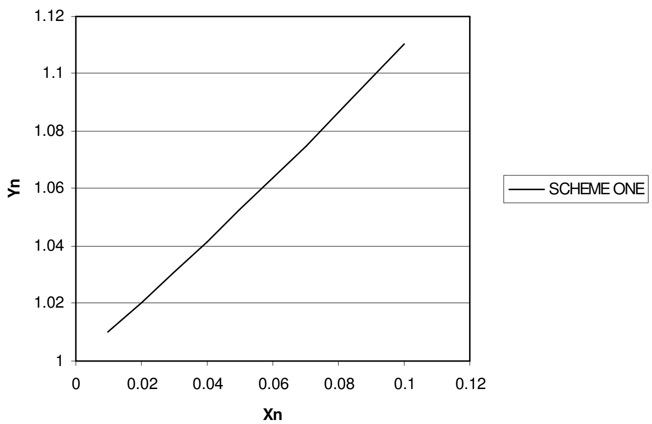

Figure 1. Nonstandard Method One, 10 ITERATIONS, y(0)=1, h=0.01, a=-0.1.

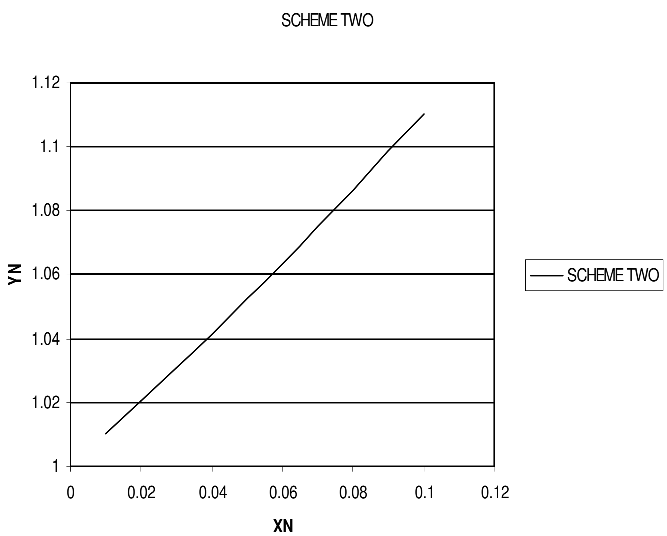

Figure 2. Nonstandard Method Two. 10 ITERATIONS, y(0) = 1, phile(h)=1-exp(-h), h = 0.01

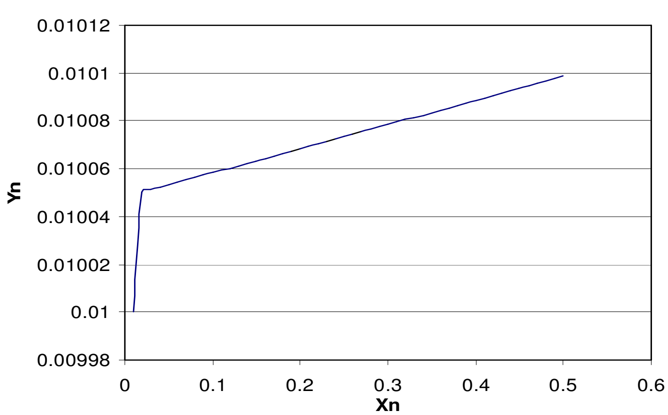

Figure 3. Nonstandard Method Three, 50 ITERATIONS, y(0)=1, y(1)=h*y(0) h = 0.01, phile(h) = 1-exp(-h)

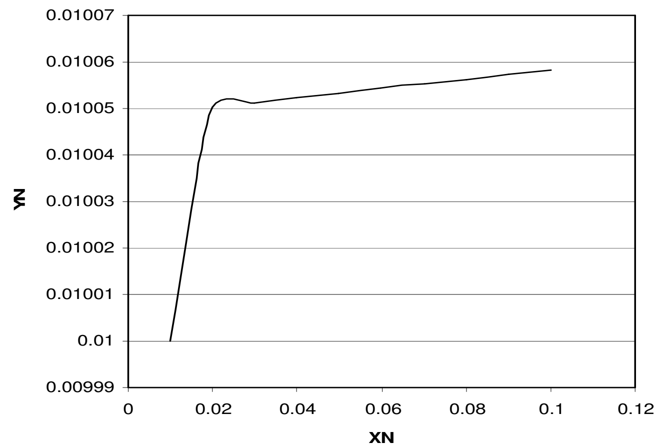

Figure 4. Nonstandard Method three 10 ITERATIONS, y(0)=1, y(1)=h*y(0) H=0.01, phile(h)=1-exp(-h).

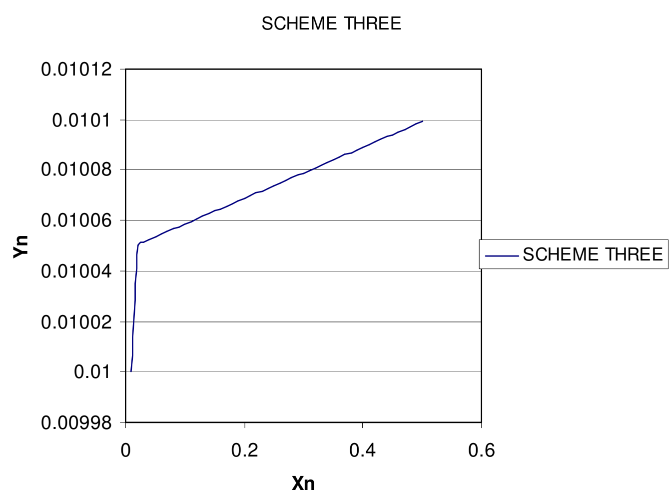

Figure 5. Nonstandard Method Three, 50 ITERATIONS, y(0)=1, h=0.01, Phile(h)=h

form which are presented below. Various values of h, $\phi(h)$ and 'a' are used and one can see the consistency of the schemes. Scheme three present a very interesting result. We made use of two starting values because the scheme is a two-step numerical method. Applications of the definitions and the theorems stated in section two pointed to the fact that the three schemes are stable with respect to monotone dependence on initial value and therefore they are elementary stable. We also use the standard fourth stage Runge-Kutta method to solve the ordinary differential equation in question. The result of Figures 1 – 5 shows a sense of stability in comparison with the result presented in Figures 6 and 7.

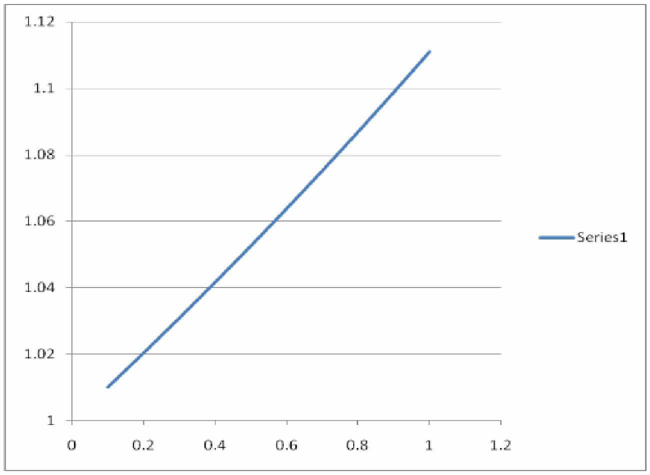

Figure 6. The solution of the problem using fourth stage classical Runge Kutta method h = 0.01 a = 0 and b = 1,with 10 iterations.

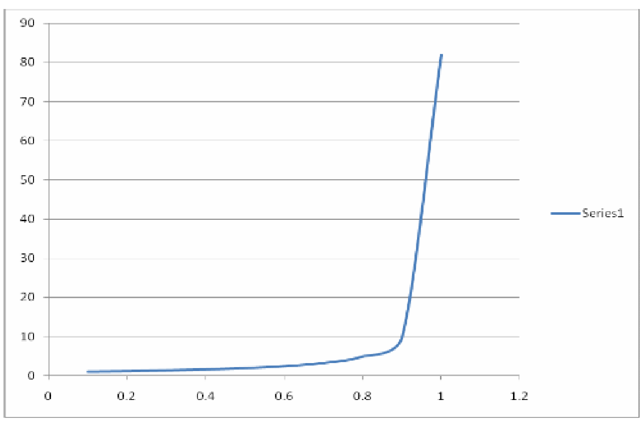

Figure 7. The problem was solved with h = 0.1; a = 1; b = 1; 10 iterations.

CONCLUSION AND ACKNOWLEDGEMENT

We have discussed three numerical schemes. Nonstandard scheme for the solution of initial value problem in ordinary differential equations had been discussed and implemented. We have introduced free parameter that control the stability properties of the first scheme. The first author is very grateful to Prof. W. Ogana, the director, of AMMSI for the generous fellowship granted him by AMMSI, with the headquarter at the Department of Mathematics, University of Nairobi, Kenya. Part of this work was completed when the first author visited the Department of Mathematics University of Pretoria, South Africa. We appreciate the hospitality of Prof Jean Lubuma of the department of Mathematics and Applied Mathematics, University of Pretoria for providing accommodation during the first author's visit to South Africa.

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