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Full Length Research Paper

Fuzzy ideals in partially ordered pseudoeffect algebras

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Pseudoeffect algebras are non-commutative generalizations of effect algebras, which can serve as models of both quantum structures and non-commutative logics. The main contribution of this study is twofold. Firstly, we initiate an order-theoretic extension of pseudoeffect algebras, called partially ordered pseudoeffect algebras (abbreviated po-PEAs). Secondly, we investigate the fuzzy ideal theory of po-PEAs. In particular, we show that a fuzzy ideal in a po-PEA is finitely generated if and only if it is finitely valued, and every fuzzy ideal in a Noetherian po-PEA is finitely generated.

Key words: Pseudoeffect algebra, fuzzy set, fuzzy ideal, fuzzy logic.

INTRODUCTION

In recent years, modelling of uncertainty has received much attention and become an increasingly active research area. In fact, researchers working in economics, engineering, environmental science, sociology, medical science and many other fields need to deal with the complexity of uncertain data almost everyday. The nature of the uncertainties appearing in diverse domains could be very different. In addition to probability theory, researchers have developed some other mathematical tools for dealing with various kinds of uncertainties. These tools include fuzzy sets (Zadeh, 1965), rough sets (Pawlak, 1982) and the newly-emerging theory of Molodtsov’s soft sets (Molodtsov, 1999). The study on comparison and combinations of soft sets, fuzzy sets and rough sets can be found in (Feng et al., 2011, 2010a). It is worth noting that soft set theory is also related to description logic (Jiang et al., 2010, 2011b) and has proved to be very useful in decision making under uncertainty (Anisseh and Yusuff, 2011; Cagman and Enginoglu, 2010; Feng et al., 2010b, c; Jiang et al., 2011a). Moreover, many authors have discussed the application of soft sets in various algebraic structures (Feng et al., 2008; Atagun and Sezgin, 2011; Sezgin et al., 2011; Sezgin and Atagun, 2011a, b).

The concept of fuzzy sets introduced by Zadeh (1965) is regarded as a fundamental approach to vagueness. Rosenfeld (1971) applied fuzzy sets to the study of algebraic structures and initiated the notion of fuzzy groups. He started a burst of papers on the topic of fuzzy algebras (Davvaz and Corsini, 2007; Jun and Song, 2006; Shabir et al., 2010a, b; Shabir and Khan, 2010; Zhan and Dudek, 2007). Fuzzy structures may give rise to more useful models in some practical applications. For instance, fuzzy lattices have proved to be useful in neural computing (Kaburlasos and Petridis, 1998, 1999, 2000).

The logic underlying classical theory of computation is Boolean (two-valued) logic. With the developing of theories modelling uncertainty, it is natural and necessary to establish some rational logic systems as the logical foundation for uncertain modelling and reasoning. Various non-classical logic systems and their algebraic counterparts have thus been proposed and extensively studied by many researchers. For instance, MV-algebras are algebraic counterparts of the Lukasiewicz infinite many valued propositional logic; while BL-algebras are algebraic models of Hajek’s basic fuzzy logic (Hajek, 1998). Note that BCK/BCI-algebras are two important classes of algebras of logic, since most of the algebras related to the t-norm based logic, such as BL-algebras, MV-algebras, MTL-algebras and Boolean algebras are subclasses of BCK-algebras.

To establish the mathematical foundation for quantum mechanics, many researchers have contributed to the area of quantum logics and related quantum structures. Pseudoeffect algebras (Dvurečenskij and Vetterlein, 2001a, b) are non-commutative generalizations of effect algebras (Foulis and Bennett, 1994). They were proposed by Dvurečenskij and Vetterlein for modelling unsharp measurements in quantum mechanical systems. It is interesting to see that several attempts have been

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made to discover the connections between pseudoeffect algebras and the aforementioned algebraic models of fuzzy logic (Vetterlein, 2004, 2005). Moreover, Jun and Walendziak (2006) applied the concept of fuzzy set to pseudo-MV algebras. They introduced the notions of fuzzy ideals and fuzzy implicative ideals in pseudo-MV algebras, gave characterizations of them and provided conditions for a fuzzy set to be a fuzzy ideal and a fuzzy implicative ideal.

The present paper can be seen as an attempt toward the study of pseudoeffect algebras by combining algebraic approach with fuzzy set theory. Based on the observation that a pseudoeffect algebra can be viewed as a poset equipped with a partial algebraic operation, we introduce partially ordered pseudoeffect algebras (abbreviated po-PEAs), which can be viewed as a new type of quantum structures. Generally, po-PEAs extend (standard) pseudoeffect algebras by substituting an arbitrary compatible order for the natural order derived from the partial addition. In contrast to existing studies which focus essentially on crisp ideals of pseudoeffect algebras, our study is mainly devoted to the discussion of fuzzy ideal in po-PEAs. The main purpose of this study is to establish a theory of great generality, in view of the fact that every crisp ideal of pseudoeffect algebra can be seen as a fuzzy ideal in the corresponding (naturally ordered) po-PEA.

PSEUDOEFFECT ALGEBRAS AND po-PEAS

Here, we first recall some basic notions and results in the theory of pseudoeffect algebras. Then, we shall introduce an order-theoretical extension of pseudoeffect algebras, called partially ordered pseudoeffect algebras. Birkhoff and Neumann (1936) realized that quantum mechanical systems are not governed by classical logical laws. Their pioneering work motivated more research works concerning the logic foundation of quantum mechanics. With the development of quantum logics, a number of algebraic structures have been proposed as their models (Miklós, 1998). Foulis and Bennett (1994) introduced effect algebras for modelling unsharp measurements in quantum mechanical systems. Dvurečenskij and Vetterlein (2001) introduced the following non-commutative generalizations of effect algebras.

Definition 1

A structure $(E;+,0,1)$, where + is a partial binary operation and 0, 1 are constants, is called a pseudoeffect algebra if for all $a,b,c\in E$, the following hold (Dvurečenskij and Vetterlein, 2001a):

(P1): $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case, $(a+b)+c=a+(b+c)$.
(P2): There is exactly one $d\in E$ and exactly one $e\in E$, such that $a+d=e+a=1$.
(P3): If $a+b$ exists, there are elements $d,e\in E$, such that $a+b=d+a=b+e$.
(P4): If $1+a$ or $a+1$ exists, then $a=0$.

Note that we write $a+b+c$ for the element $(a+b)+c=a+(b+c)$ if the hypothesis of (P1) is satisfied. Indeed, we may denote arbitrary finite sums of elements of a pseudoeffect algebra without brackets since (P1) holds. For convenience, we write $a\perp b$ if $a+b$ exists. In view of (P2), we may define two unary operations $^{\sim}$ and $^{-}$ on E by requiring $a+a^{\sim}=a^{-}+a=1$ for any $a\in E$.

A pseudoeffect algebra is said to be commutative if for all $a,b\in E$, $a\perp b$ if and only if $b\perp a$, in which case a + b = b + a. As mentioned in the earlier, a pseudoeffect algebra is a non-commutative generalization of an effect algebra by dropping commutativity. Thus, it is clear that the notion of effect algebras coincides with the notion of commutative pseudoeffect algebras. Darnel (1995) gave an example of a pseudoeffect algebra which is non-commutative and thus, is not an effect algebra. This shows that the class of effect algebras forms a proper subclass of the class of pseudoeffect algebras. The following basic results on pseudoeffect algebras were established (Dvurečenskij and Vetterlein, 2001a). Note that for any equation to hold, we mean that all sums that occur in it exist, and it holds.

Lemma 1

Let $(E;+,0,1)$ be a pseudoeffect algebra (Dvurečenskij and Vetterlein, 2001a). Then, the following hold in E for all $a,b,c\in E$:

(1) $a+0=0+a=a.$
(2) $a+b=0$, implies $a=b=0.$
(3) $0^{\sim}=0^{-}=1, 1^{\sim}=1^{-}=0.$
(4) $a^{\sim -}=a^{-\sim}=a.$
(5 $a+b=a+c$ implies $b=c,$ and $b+a=c+a$ implies $b=c.$
(6) $a+b=c$ iff $a=(b+c^{\sim})^{-}$ iff $b=(c^{-}+a)^{\sim}$.

In any pseudoeffect algebra E, one can define a binary relation $\leq_N$ on E by $a\leq_N b \Leftrightarrow (\exists c\in E) \quad a+c=b$. For all a, b ∈ E. As shown by Dvurečenskij and Vetterlein (2001a), $\leq_N$ is a partial order on E, which will be called the natural order on E in the sequel. Clearly, 0 is the least element and 1 is the greatest element in E, with respect to the natural order. Moreover, from (P3), it is easy to see that for all a, b ∈ E, a $\leq_N$ b, if and only if d + a = b for some d

∈ E. This says that $\leq_N$ is two-sided, which is in fact the main motivation for choosing the axiom (P3).

Lemma 2

Let $(E;+,0,1)$ be a pseudoeffect algebra (Dvurečenskij and Vetterlein, 2001a). Then, the following hold in E for all $a,a_1,b,b_1,c \in E$:

(1) $\leq_N$ is a partial order on E.
(2) $a \leq_N b$ iff $b^- \leq_N a^-$ iff $b^\sim \leq_N a^\sim$.
(3) If a ⊥ b, $a_1 \leq_N a$ and $b_1 \leq_N b$, then $a_1 + b_1$ exists.
(4) a ⊥ b iff $a \leq_N b^-$ iff $b \leq_N a^\sim$.
(5) If b ⊥ c, then a $\leq_N$ b iff a ⊥ c and a + c $\leq_N$ b + c. If c ⊥ b, then a $\leq_N$ b iff c ⊥ a and c + a $\leq_N$ c + b.

By Lemma 2, we know that the natural order on a pseudoeffect algebra E is compatible with the partial binary operation + on E. In other words, E equipped with the natural order on it, becomes an ordered partial algebraic structure. This observation motivates us to introduce the following order-theoretical extensions of pseudoeffect algebras.

Definition 2

A partially ordered pseudoeffect algebra (abbreviated po-PEA) is a structure $(E;+,0,1,\leq)$, such that:

(O1): (E; +, 0, 1) is a pseudoeffect algebra.
(O2): (E; ≤) is a poset.
(O3): ≤ is compatible with the partial binary operation + on E. That is, for all a, b, c ∈ E with a ≤ b, if a ⊥ c and b ⊥ c then a + c ≤ b + c; If c ⊥ a, and c ⊥ b, then c + a ≤ c + b.

In what follows, (E; +, 0, 1, ≤) is also denoted by (E; +, ≤) or simply by E. A po-PEA (E; +, ≤) is said to be naturally ordered if ≤ = $\leq_N$, that is, the partial order ≤ on E coincides with the natural one. Now it is easy to see how pseudoeffect algebras are characterized among po-PEAs; that is, every pseudoeffect algebra may be considered as a po-PEA equipped with the natural order. Moreover, a pseudoeffect algebra E is also a po-PEA with respect to the dual natural order, namely $\geq_N = \leq_N^{-1}$.

The following example gives a class of po-PEAs arising from intervals in partially ordered groups.

Example 1

Let (G; +, ≤) be a po-group and u a positive element of G (Dvurečenskij and Vetterlein, 2001a). We denote by (G, u) the structure (G; +, ≤, u), obtained by adding the element u as a constant. The po-group (G, u) is said to be unital if u is a strong unit of G, that is, if for all g ∈ G, there is an n ∈ N such that −nu ≤ g ≤ nu. The set Γ(G, u) = {g ∈ G | 0 ≤ g ≤ u} is called the unit interval of (G, u). The structure denoted (Γ(G, u); +, 0, u, ≤) is constituted of the unit interval of (G, u), the partial binary operation + which is the restriction of the group addition to those pairs of elements of Γ(G, u) whose sum lies again in Γ(G, u), the neutral element 0, the positive element u and the order of the po-group G restricted to Γ(G, u). It is easy to verify that (Γ(G, u); +, 0, u, ≤) is a naturally ordered po-PEA.

It is worth noting that there exist po-PEAs which are not naturally ordered. For illustration, we provide two examples as follows: the first one is a simple variation (Dvurečenskij and Vetterlein, 2001a).

Example 2

Let G = Z × Z × Z. Define for every two elements of G:

$$(a_1,b_1,c_1)+(a_2,b_2,c_2)=\begin{cases}V_1 & \text{if } a_2 \text{ is even,}\\ V_2 & \text{otherwise,}\end{cases}$$

where V_1 = (a1 + a2, b1 + b2, c1 + c2), V_2 = (a1 + a2, b2 + c1, b1 + c2), and define (a1, b1, c1) ≲ (a2, b2, c2) to hold if a2 < a1 or a2 = a1, b2 ≤ b1 and c2 ≤ c1. Then, as pointed out by Dvurečenskij and Vetterlein (2001a), (G; +, ≳) is a lattice ordered group. Here, ≳ denotes the dual of the order ≲ defined earlier. Taking u = (1, 0, 0), the unit interval Γ(G, u) = {(0, b, c) | b, c ≥ 0} ∪ {(1, b, c) | b, c ≤ 0} equipped with the restriction of the order ≳, becomes a naturally ordered po-PEA with the sum and constants defined according to Example 1. On the other hand, it is clear that ≳ is not the trivial order. Hence, one easily sees that the structure (Γ(G, u); +, (0, 0, 0), u, ≲) is a po-PEA which is not naturally ordered. Moreover, one can observe that (0, 1, 2) + (1, −2, −2) = (1, 0, −1), but (1, −2, −2) + (0, 1, 2) = (1, −1, 0). This indicates that Γ(G, u) is not an effect algebra since it is noncommutative.

The second example gives a small concrete po-PEA consisting of four elements which is linearly ordered, but not naturally ordered.

Example 3

Let E_4 = {0, a, b, 1} be an effect algebra with its Cayley table given as follows:

+	0	a	b	1
0	0	a	b	1
a	a	*	1	*
b	b	1	*	*
1	1	*	*	*

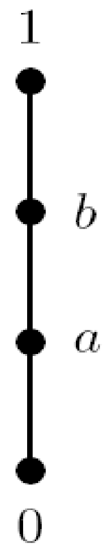

Figure 1. Hasse diagram of (E_4; ≤) (necklace).

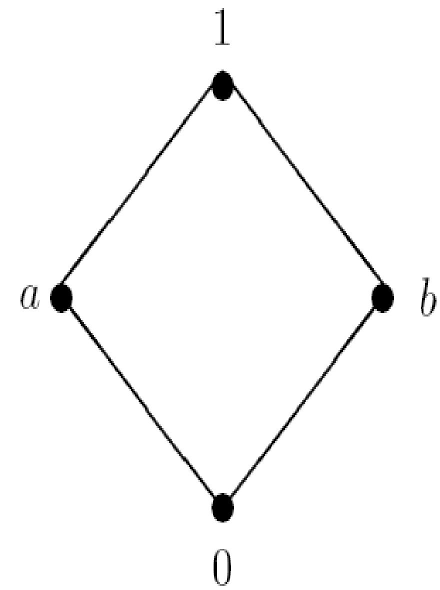

Figure 2. Hasse diagram of (E_4; $\leq_N$) (diamond).

Let (E_4; ≤) be a linear poset with the Hasse diagram (Figure 1).

By definition, one can verify that (E_4; +, 0, 1, ≤) is a po-PEA which is linearly ordered. On the other hand, it is easy to see that equipped with the natural order $\leq_N$, E_4 becomes a poset whose Hasse diagram is as shown in Figure 2.

Evidently, $\leq_N$ and ≤ are different partial orders on E_4. Hence, we deduce that (E_4; +, 0, 1, ≤) is a po-PEA that is not naturally ordered.

Definition 3

A nonempty subset I of a pseudoeffect algebra E is said to be an ideal in E, if it satisfies the following (Dvurečenskij and Vetterlein, (2001c):

(D1): If x ∈ I, y ∈ E and $y \leq_N x$, then y ∈ I.

(D2): If x, y ∈ I and y ⊥ x (equivalently, $x \leq_N y^{\sim}$), then y + x ∈ I. Note that $\leq_N$ denotes the natural order on E.

It is clear that 0 ∈ I for any ideal I in a pseudoeffect algebra E. Wu (2004) gave the following equivalent characterization of ideals in pseudoeffect algebras.

Proposition 1

Let I be a nonempty subset of a pseudoeffect algebra E (Wu, 2004). Then, I is an ideal in E if and only if it satisfies the following:

(D1): If a ∈ I, b ∈ E and $b \leq_N a$, then b ∈ I.

(D2’): If a ∈ I, $a \leq_N b$ and $\{(b^- + a)^{\sim}, (a + b^{\sim})^-\} \cap I \neq \emptyset$, then b ∈ I.

As one might suspect from the analogy with the case of pseudoeffect algebras, the notion of an ideal in a po-PEA can be defined as follows, which extends ideals in pseudoeffect algebras in a natural way.

Definition 4

Let (E; +, ≤) be a po-PEA. A nonempty subset I of E is said to be an ideal in E if it satisfies the following:

(I1): If x ∈ I, y ∈ E and y ≤ x, then y ∈ I.

(I2): If x, y ∈ I and y ⊥ x, then y + x ∈ I.

Note that ≤ is the partial order on E (may not be the natural order $\leq_N$).

FUZZY IDEALS IN po-PEAS

In recent years, various types of ideals in quantum structures have been actively studied by many authors (Shang and Li, 2003, 2007; Wu, 2004). An ideal of a pseudoeffect algebra can be interpreted as a series of measurement outcomes (satisfying certain conditions) of a physical experiment (Busch et al., 1995). Here, we shall initiate the concept of fuzzy ideals in po-PEAs and focus on the ideal theory of po-PEAs in a fuzzy setting.

Definition 5

Let (E; +, ≤) be a po-PEA and μ be a fuzzy set in E. Then μ is called a fuzzy ideal in E if it satisfies the following:

(F1): If x ≤ y, then μ(x) ≥ μ(y).
(F2): If y ⊥ x, then μ(y + x) ≥ min{μ(x), μ(y)}.

For illustration of the aforementioned definition, we revisit the pseudoeffect algebra E_4 = {0, a, b, 1} in Example 3. Let μ be a fuzzy set in E_4 given by μ(0) = 1, μ(b) = 0.5 and μ(a) = μ(1) = 0. Then, one easily verifies that μ is a fuzzy ideal in both the linearly ordered po-PEA (E_4; +, 0, 1, ≤) and the naturally ordered po-PEA (E_4; +, 0, 1, $\leq_N$).

The following result gives an equivalent characterization of fuzzy ideals in naturally ordered po-PEAs.

Proposition 2

Let (E; +, ≤) be a naturally ordered po-PEA and μ be a fuzzy set in E. Then μ is a fuzzy ideal in E if and only if it satisfies the following:

(F1): If a ≤ b, then $\mu(a) \geq \mu(b)$.
(F2'): If a ≤ b, then $\mu(b) \geq \min\{\mu(a), \max\{\mu((a+b^{\sim})^{-}), \mu((b^{-}+a)^{\sim})\}\}$.

Proof: We only need to show the equivalence between (F2) and (F2'). Note, also that $\leq = \leq_N$ since by the hypothesis, E is a naturally ordered po-PEA. First assume that (F2) holds and a ≤ b in E. Let $x = a$ and $y = (b^{-}+a)^{\sim}$. Then $y^{-} = b^{-} + a = b^{-} + x$, and so $x \leq y^{-}$. Hence, $y \leq x^{\sim}$ by Lemma 2 (2). Thus, by (F2), we have μ(x + y) ≥ min{μ(x), μ(y)}. But $x + y = a + (b^{-}+a)^{\sim} = b$, it follows that $\mu(b) \geq \min\{\mu(a), \mu((b^{-}+a)^{\sim})\}$.

On the other hand, by taking $x = a$ and $y = (a+b^{\sim})^{-}$, we obtain $\mu(b) \geq \min\{\mu(a), \mu((a+b^{\sim})^{-})\}$ in the same way. Hence, we have $\mu(b) \geq \min\{\mu(a), \max\{\mu((a+b^{\sim})^{-}), \mu((b^{-}+a)^{\sim})\}\}$, showing that (F2') holds.

Conversely, suppose that (F2') holds and y ⊥ x in E. Taking a = x and b = y + x, we have y + a = b. Then a ≤ b and $y = (a+b^{\sim})^{-}$ by Lemma 1 (6). Hence, by (F2'), we have $\mu(b) \geq \min\{\mu(a), \max\{\mu(y), \mu((b^{-}+a)^{\sim})\}\} \geq \min\{\mu(a), \mu(y)\}$.

Therefore $\mu(y+x) \geq \min\{\mu(x), \mu(y)\}$ and thus (F2) holds. This completes the proof.

Theorem 1

Let (E; +, ≤) be a po-PEA and A be a nonempty subset of E. Let μ be a fuzzy set in E defined by:

$$\mu(x) = \begin{cases} s & \text{if } x \in A, \\ t & \text{otherwise,} \end{cases}$$

for all x ∈ E, where s > t in [0, 1]. Then μ is a fuzzy ideal in E if and only if A is an ideal in E.

Proof: Suppose that μ is a fuzzy ideal in E. Let x ∈ A, y ∈ E and y ≤ x. Then μ(y) ≥ μ(x) = s, which gives μ(y) = s. Thus, y ∈ A, and so A satisfies (I1). Moreover, let x, y ∈ A and y ⊥ x. Then μ(x) = μ(y) = s and by (F2), it follows that μ(y + x) ≥ min{μ(x), μ(y)} = s. Thus, μ(y + x) = s, and so y + x ∈ A. This shows that A also satisfies (I2). Therefore we conclude that A is an ideal in E.

Conversely, assume that A is an ideal in E. Let x ≤ y in E. If y ∈ A, then x ∈ A. Hence μ(x) = s = μ(y), and so μ(x) ≥ μ(y). On the other hand, if y ∈ A, then μ(y) = t and clearly we deduce μ(x) ≥ μ(y). Thus we have that μ satisfies (F1). Furthermore, let x, y ∈ E and y ⊥ x. If x, y ∈ A, then y + x ∈ A by (I2). Hence, μ(x) = μ(y) = μ(y + x) = s and it is clear that μ(y + x) ≥ min{μ(x), μ(y)}. In addition, if either one of x, y does not belong to A, then we have μ(y + x) ≥ t = min{μ(x), μ(y)}. This shows that μ also satisfies (F2), and so μ is a fuzzy ideal in E as required.

The aforementioned assertion shows that (crisp) ideals and fuzzy ideals in po-PEAs are closely related. In particular, if the characteristic function is chosen as the fuzzy set μ, then one easily deduces the following result which states that an ideal in a po-PEA is a fuzzy ideal when it is identified with its characteristic function. As mentioned in the previously, every pseudoeffect algebra may be considered as a po-PEA equipped with the natural order. Hence, all results obtained, in the present paper, on fuzzy ideals in po-PEAs can also applied to ideals in pseudoeffect algebras.

Corollary 1

Let (E; +, ≤) be a po-PEA and A be a nonempty subset of E. Then, A is an ideal in E if and only if χ_A is a fuzzy ideal in E, where χ_A is the characteristic function of the set A.

Let μ be a fuzzy set over a universe E. For any t ∈ [0, 1], U (μ; t) = {x ∈ E | μ(x) ≥ t} is called a t-level set, or a t-cut of the fuzzy set μ. If t is not clearly specified, we simply say level set of μ. In fuzzy set theory, it is known that a fuzzy set can be related to a family of crisp sets through the notion of level sets. In fact, level sets of a fuzzy set μ over E can be used to define a nested family of subsets of E. Conversely, a fuzzy set μ can be reconstructed from its level sets by means of the formula μ(x) = sup{t | x ∈ U (μ; t)}.

This observation is commonly summarized by a representation theorem of fuzzy sets, which states that there is a one-to-one correspondence between a family of crisp sets satisfying certain conditions and a fuzzy set. This linkage indicates the inherent structure of a fuzzy set. Motivated by the representation theorem of fuzzy sets, we established as follow a connection between fuzzy ideals and (crisp) ideals in po-PEAs by using level sets.

Theorem 2

Let (E; +, ≤) be a po-PEA and μ be a fuzzy set in E. Then, μ is a fuzzy ideal in E if and only if the level set U (μ; t) of μ is an ideal in E for all t ∈ [0, 1], whenever it is nonempty.

Proof: Suppose that μ is a fuzzy ideal in E and U (μ; t) is nonempty for t ∈ [0, 1]. Let x ∈ U (μ; t), y ∈ E and y ≤ x. Then μ(y) ≥ μ(x) ≥ t, and so y ∈ U (μ; t). Hence, U (μ; t) satisfies (I1). Moreover, let x, y ∈ U (μ; t) and y ⊥ x. Then, we have μ(x) ≥ t and μ(y) ≥ t. By (F2), it follows that μ(y + x) ≥ min{μ(x), μ(y)} ≥ t, and so y + x ∈ U (μ; t). This shows that U (μ; t) satisfies (I2). Hence, U (μ; t) is an ideal in E, as required.

Conversely, assume that the level set U (μ; t) is an ideal in E whenever it is nonempty. Let x, y ∈ E and x ≤ y. If we choose t_0 = μ(y), then U (μ; t_0) is nonempty since y ∈ U (μ; t_0). Thus, by hypothesis, U (μ; t_0) is an ideal in E and x ∈ U (μ; t_0). It follows that μ(x) ≥ t_0 = μ(y), and so μ satisfies (F1). It remains to show that μ also satisfies (F2). Let x, y ∈ E and y ⊥ x. Taking t_0 = min{μ(x), μ(y)}, then it is clear that x, y ∈ U (μ; t_0) and U (μ; t_0) is an ideal in E. Hence, y + x ∈ U (μ; t_0) by (I2), and so we have μ(y + x) ≥ t_0 = min{μ(x), μ(y)}. This completes the proof.

The following argument shows that the intersection of a collection of fuzzy ideals in a po-PEA is also a fuzzy ideal in it.

Proposition 3

Let (E; +, ≤) be a po-PEA and {μ_t | t ∈ Λ} be a collection of fuzzy ideals in E. Let μ be a fuzzy set in E defined by μ(x) = inf {μ_t(x) | t ∈ Λ} for all x ∈ E. Then μ is a fuzzy ideal in E.

Proof: Let x, y ∈ E and x ≤ y. Then for any t ∈ Λ, μ_t(x) ≥ μ_t(y) since μ_t is a fuzzy ideal in E. Hence, we have μ(x) = inf{μ_t(x) | t ∈ Λ} ≥ inf{μ_t(y) | t ∈ Λ} = μ(y), which shows that μ satisfies (F1). It remains to show that μ satisfies (F2). In fact, let x, y ∈ E and y ⊥ x. Then, by hypothesis, we deduce that μ_t(y + x) ≥ min{μ_t(x), μ_t(y)} ≥ min{μ(x), μ(y)} for all t ∈ Λ. Therefore, μ(y + x) ≥ min{μ(x), μ(y)} as required.

By combining Corollary 1 and Proposition 3, we have the following immediate consequence.

Corollary 2

Let {A_t | t ∈ Λ} be a collection of ideals in a po-PEA E. Then $\bigcap\{A_t| \ t \in \Lambda\}$ is an ideal in E.

Proposition 4

Let (E; +, ≤) be a po-PEA and Λ be a linear poset. Let {A_t | t ∈ Λ} be a collection of ideals in E such that t < s if and only if $A_s \subset A_t$ for all s, t ∈ Λ. Then $\bigcup\{A_t| \ t \in \Lambda\}$ is an ideal in E.

Proof: Denote by A is the union of ideals {A_t | t ∈ Λ}. Let x ∈ A, y ∈ E and y ≤ x. Then $x \in A_{t_0}$ for some t_0 ∈ Λ. Since A_{t_0} is an ideal in E, we have $y \in A_{t_0}$ and so y ∈ A. This shows that A satisfies (I1). Furthermore, let x, y ∈ A and y ⊥ x. Then, there exist $t_1, t_2 \in \Lambda$ such that $x \in A_{t_1}$ and $y \in A_{t_2}$. If $t_1 = t_2$, then $y + x \in A_{t_1} \subseteq A$. Otherwise, we may assume that $t_1 < t_2$ without loss of generality. Then, by hypothesis, $A_{t_2} \subset A_{t_1}$ and it follows that $y + x \in A_{t_1} \subseteq A$. Hence, we conclude that A also satisfies (I2), which completes the proof.

Theorem 3

Let (E; +, ≤) be a po-PEA and Λ ⊂ [0, 1]. Let {A_t | t ∈ Λ} be a collection of ideals in E satisfying the following:

(1) $E = \bigcup\{A_t| \ t \in \Lambda\}$.
(2) t < s if and only if $A_s \subset A_t$ for all s, t ∈ Λ.

If we define a fuzzy set μ in E by μ(x) = sup{t ∈ Λ | x ∈ At} for all x ∈ E, then μ is a fuzzy ideal in E.

Proof: By Theorem 2, it suffices to show that the level set U (μ; s) of μ is an ideal in E for any s ∈ [0, 1] such that $U(\mu; s) \neq \varnothing$. To do this, note first that if $\{t \in \Lambda| \ t < s\} = \varnothing$, then one easily verifies that U (μ; s) = E, which is evidently an ideal in E. Otherwise, we shall consider the following two cases:

(1) $s = \sup\{t \in \Lambda \mid t < s\}$;

(2) $s \neq \sup\{t \in \Lambda \mid t < s\}$.

For the first case, we claim that $U(\mu; s) = \bigcap\{A_t \mid t < s\}$, which is an ideal in E by Corollary 2. In fact, let $x \in \bigcap\{A_t \mid t < s\}$. Then, for any t ∈ Λ with t < s, we have x ∈ A_t and so μ(x) = sup{t ∈ Λ | x ∈ A_t} ≥ t. It follows that μ(x) ≥ sup{t ∈ Λ | t < s} = s, showing that x ∈ U (μ; s). Conversely, let x ∈ U (μ; s). Then μ(x) = sup{t ∈ Λ | x ∈ A_t} ≥ s. For any t ∈ Λ with t < s, we claim that there exists some t_1 ∈ Λ such that $x \in A_{t_1}$ and t < t_1, otherwise we shall deduce that s ≤ t, which leads to a contradiction. Thus, $x \in A_{t_1} \subset A_t$ and so $x \in \bigcap\{A_t \mid t < s\}$ as required.

For the second case, note first that there exists ε > 0 such that $(s-\varepsilon, s) \cap \Lambda = \varnothing$. Then, we claim that $U(\mu; s) = \bigcup\{A_t \mid t \geq s\}$, and so by Proposition 4, U (μ; s) is an ideal in E. To see this, let $x \in \bigcup\{A_t \mid t \geq s\}$. Then $x \in A_t$ for some t ≥ s, and thus we have μ(x) ≥ t ≥ s. Therefore x ∈ U (μ; s). Conversely, let x ∈ U (μ; s). Then we have μ(x) ≥ s. If s > t for all t ∈ Λ such that x ∈ A_t, then by hypothesis, we easily deduce t < s −ε for all t ∈ Λ such that x ∈ A_t. But it follows that $\mu(x) = \sup\{t \in \Lambda \mid x \in A_t\} \leq s - \varepsilon < s$, which leads to a contradiction. Hence, there exists t_0 ≥ s such that $x \in A_{t_0}$, and so $x \in A_{t_0} \subseteq \bigcup\{A_t \mid t \geq s\}$. This completes the proof.

Theorem 4

Let (E; +, ≤) be a po-PEA and $\{A_n \mid n \in \mathbb{N}\}$ be a family of ideals in E which is nested, that is, $E = A_1 \supset A_2 \supset \cdots$. Let μ be a fuzzy set in E define by

$$\mu(x) = \begin{cases} \dfrac{n}{n+1} & \text{if } x \in A_n,\ A_{n+1} \text{ for some } n \in \mathbb{N}, \\ 1 & \text{otherwise,} \end{cases}$$

for all x ∈ E. Then μ is a fuzzy ideal in E.

Proof: Let x, y ∈ E and x ≤ y. Note first that if $y \in \bigcap_{n\in\mathbb{N}} A_n$, then clearly $x \in \bigcap_{n\in\mathbb{N}} A_n$ since $\bigcap_{n\in\mathbb{N}} A_n$ is an ideal in E by Corollary 3.7. Hence by definition of μ, we have μ(x) = μ(y) = 1, and so μ(x) ≥ μ(y). On the other hand, assume that y ∈ $A_k \setminus A_{k+1}$ for some $k \in \mathbb{N}$. If $x \in \bigcap_{n\in\mathbb{N}} A_n$, then μ(x) = 1 ≥ μ(y). Suppose that x ∈ $A_r \setminus A_{r+1}$ for some $r \in \mathbb{N}$. But by hypothesis, we also have x ∈ A_k. Therefore k ≤ r and it follows that $\mu(x) = \frac{r}{r+1} \geq \frac{k}{k+1} = \mu(y)$. Thus we conclude that μ satisfies (F1).

To show that μ also satisfies (F2), let x, y ∈ E and y ⊥ x. If $x, y \in \bigcap_{n\in\mathbb{N}} A_n$, then clearly $y + x \in \bigcap_{n\in\mathbb{N}} A_n$. Hence μ(x) = μ(y) = μ(y + x) = 1, and so μ(y + x) ≥ min{μ(x), μ(y)}. Now let us assume that x ∈ $A_k \setminus A_{k+1}$ and y ∈ $A_r \setminus A_{r+1}$ for $k\ r \in \mathbb{N}$. Without any loss of generality, let k ≤ r. Then y + x ∈ A_k and it follows that $\mu(y+x) \geq \frac{k}{k+1} = \min\{\mu(x), \mu(y)\}$.

Furthermore, suppose that $x \in \bigcap_{n\in\mathbb{N}} A_n$ and y ∈ $A_k \setminus A_{k+1}$ for some $k \in \mathbb{N}$. Then y + x ∈ A_k, and so we also have $\mu(y+x) \geq \frac{k}{k+1} = \min\{\mu(x), \mu(y)\}$. Finally, if $y \in \bigcap_{n\in\mathbb{N}} A_n$ and x ∈ $A_k \setminus A_{k+1}$ for some $k \in \mathbb{N}$, we deduce that μ(y + x) ≥ min{μ(x), μ(y)} in much the same way. Hence μ is a fuzzy ideal in E as required.

FINITELY GENERATED AND FINITELY VALUED FUZZY IDEALS

For any subset A of a po-PEA E, denote by 〈A〉 the intersection of all ideals in E containing A, which we call the ideal generated by A. It is easy to see that 〈A〉 is the smallest ideal in E containing A. Similarly, if μ is a fuzzy set in E, then the smallest fuzzy ideal in E containing μ is called the fuzzy ideal generated by μ, and is denoted by $\langle\mu\rangle_f$. A fuzzy set μ in E is said to be n-valued if μ(E) is a finite set of n elements. Note that μ is called finitely valued if no specific n is intended. Moreover, we say that a fuzzy ideal v in E is finitely generated if $v = \langle\mu\rangle_f$ for some finitely valued fuzzy set μ in E.

Theorem 5

Let (E; +, ≤) be a po-PEA and μ be a fuzzy set in E. Define a fuzzy set μ^* in E by $\mu^*(x) = \sup\{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle\}$ for all x ∈ E. Then μ^* is the fuzzy ideal in E generated by μ.

Proof: By Theorem 3.5, it suffices to show that the level set U (μ^*; s) of μ^* is an ideal in E for any s ∈ [0, 1] such that $U(\mu^*; s) \neq \varnothing$. In fact, it is easy to see that if s = 0, then U (μ^*; s) = E, clearly an ideal in E. If s = 0, we define a sequence {s_n} ⊆ [0, 1] as follows:

$$s_n = \begin{cases} s - \dfrac{1}{n} & \text{if} \quad s - \dfrac{1}{n} \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $U(\mu^*; s) = \bigcap_{n \in \mathbb{N}} \langle U(\mu; s_n) \rangle$. To see this, note first that if $x \in U(\mu^*; s)$, then $\mu^*(x) \geq s$, and so $\mu^*(x) > s_n$ for all $n \in \mathbb{N}$. This implies that for any $n \in \mathbb{N}$, there exists $t^* \in [0,1]$ such that $x \in \langle U(\mu; t^*) \rangle$ and $t^* > s_n$. It follows that $U(\mu; t^*) \subseteq U(\mu; s_n)$ and so $x \in \langle U(\mu; t^*) \rangle \subseteq \langle U(\mu; s_n) \rangle$. Hence we have $x \in \bigcap_{n \in \mathbb{N}} \langle U(\mu; s_n) \rangle$. Conversely, let $x \in \bigcap_{n \in \mathbb{N}} \langle U(\mu; s_n) \rangle$. Then it is clear that $s_n \in \{t \in [0,1] \| \, x \in \langle U(\mu; t) \rangle\}$ for all $n \in \mathbb{N}$. Thus we have $s_n \leq \sup\{t \in [0,1] \mid x \in \langle U(\mu; t) \rangle\} = \mu^*(x)$ for all $n \in \mathbb{N}$. Consequently, we have $s \leq \mu^*(x)$ and hence $x \in U(\mu^*; s)$ as required.

To show that μ^* contains μ, note first that for any x ∈ E, we have $\{t \in [0,1] \| \, x \in U(\mu; t)\} \subseteq \{t \in [0,1] \| \, x \in \langle U(\mu; t) \rangle\}$, since $U(\mu; t) \subseteq \langle U(\mu; t) \rangle$. Then it follows that

$$\mu(x) = \sup\{t \in [0,1] \mid x \in U(\mu; t)\} \leq \sup\{t \in [0,1] \| \, x \in \langle U(\mu; t) \rangle\} = \mu^*(x).$$

Therefore μ^* contains μ as required.

It remains to show that μ^* is the least fuzzy ideal containing μ. Assume that ν is a fuzzy ideal in E that contains μ. For any x ∈ E, if $\mu^*(x) = 0$, then evidently we have $\mu^*(x) \leq \nu(x)$. Let $\mu^*(x) = s$ and s = 0. Then we have $x \in U(\mu^*; s) = \bigcap_{n \in \mathbb{N}} \langle U(\mu; s_n) \rangle$. It follows that x ∈ 〈U(μ; s_n)〉, and so ν(x) ≥ μ(x) ≥ s_n for all $r \in \mathbb{N}$. Hence ν(x) ≥ s = $\mu^*(x)$, which says that ν contains μ^*. This completes the proof.

Theorem 6

Let ν be a fuzzy ideal in a po-PEA E. Then ν is finitely valued if and only if it is finitely generated.

Proof: Note first that if ν is a finitely valued fuzzy ideal in E, then it is clear that ν is finitely generated since $\nu = \langle \nu \rangle_f$. Conversely, let μ be an n-valued fuzzy set in E with n distinct values $t_1 > t_2 > \cdots > t_n$. Denote by G^i the inverse image of t_i under μ. That is, $G^i = \{x \in E \mid \mu(x) = t_i\}$.

Let $A^j = \langle \bigcup_{i=1}^{j} G^i \rangle$. We obtain the following chain of ideals in E: $A^1 \subseteq A^2 \subseteq \cdots \subseteq A^n = E.$ Define a fuzzy set ν in E by

$$\nu(x) = \begin{cases} t_1 & \text{if} \quad x \in A^1, \\ t_j & \text{if} \quad x \in A^j \;, \quad A^{j+1}, \; j \in \{2, 3, \cdots, n\}, \end{cases}$$

for all x ∈ E. We shall show that ν is the fuzzy ideal in E generated by μ.

Suppose that x, y ∈ E, x ≤ y and k is the smallest integer such that y ∈ A^k. Since A^k is an ideal in E, we immediately have x ∈ A^k and so ν(x) ≥ t_k = ν(y). This shows that ν satisfies (F1). Moreover, suppose that x, y ∈ E, y ⊥ x and r, k are the smallest integers such that x ∈ A^r and y ∈ A^k. Without loss of generality, we may assume that r ≥ k. Then $A^k \subseteq A^r$, and so y + x ∈ A^r. It follows that $\nu(y+x) \geq t_r = \min\{t_r, t_k\} = \min\{\nu(x), \nu(y)\}$. Thus we conclude that ν is a fuzzy ideal in E.

It remains to show that ν is generated by μ. To see this, let x ∈ E and μ(x) = t_j for some $j \in \{1, 2, \cdots, n\}$. Then it is clear that $x \in G^j \subseteq A^j$, and so we have $\nu(x) \geq t_j = \mu(x)$. This shows that ν contains μ. Moreover, let ρ be a fuzzy ideal in E containing μ and $j \in \{1, 2, \cdots, n\}$. Then it is easy to see that $\bigcup_{i=1}^{j} G^i = U(\mu; t_j) \subseteq U(\rho; t_j)$ Consequently, $A^j = \langle \bigcup_{i=1}^{j} G^i \rangle \subseteq U(\rho; t_j)$. Let x ∈ E and ν(x) = t_j. It follows that $x \in A^j \subseteq U(\rho; t_j)$ and so $\rho(x) \geq t_j = \nu(x)$. Hence ρ contains ν and we conclude that $\nu = \langle \mu \rangle_f$. Finally, note that ν is finitely valued by its definition, completing the proof.

Definition 6

A po-PEA E is said to be Noetherian if it satisfies the ascending chain condition on ideals in E, that is, if for every chain $A_1 \subseteq A_2 \subseteq \cdots$ of ideals in E, there is an integer n such that $A_i = A_n$ for all i ≥ n.

Theorem 7

If E is a Noetherian po-PEA, then every fuzzy ideal in E is finitely valued.

Proof: Assume that μ is a fuzzy ideal in E which is not finitely valued. Then there is a descending chain $\mu(0) = t_1 > t_2 > \cdots$ of distinct numbers, where $t_i = \mu(x_i)$ for some $x_i \in E$. It is easy to see that this sequence induces the following ascending chain $U(\mu; t_1) \subset U(\mu; t_2) \subset \cdots$ of distinct nonempty level sets. But by Theorem 2, these level sets are indeed ideals in E. Therefore we obtain an ascending chain of distinct ideals in E, which leads to a contradiction.

The following result is an immediate consequence of

Theorem 6 and 7.

Theorem 8

If E is a Noetherian po-PEA, then every fuzzy ideal in E is finitely generated.

CONCLUSIONS

We introduced a new type of quantum structures, namely partially ordered pseudoeffect algebras (po-PEAs), which can be seen as an order-theoretic extension of pseudoeffect algebras. We then focused on studying the ideal theory of po-PEAs in a fuzzy setting, so as to establish a theory of great generality. Specifically, we defined and investigated (finitely generated and finitely valued) fuzzy ideals in po-PEAs, obtaining many interesting results in relation to the characterizations, connections with crisp ideals and several useful constructions. To extend this study, one could examine other types of ideals (or even dual concepts such as filters) in various quantum structures.

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