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Full Length Research Paper

Modified-accelerated Krawczyk's algorithm

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An algorithm that is twice as fast as the original Krawczyk method for finding zeros of nonlinear systems of equations is obtained via the procedures of Wolfe's modification of Krawckzyk method using the ideas derived in Uwamusi (2004). The method was implemented using Moore's interval arithmetic. It is shown that whenever the interval arithmetic evaluation exists the Hausdorff distance R(f,[X]) and f(m([X])) go linearly to zero with the width w[X] as the desired solution is approached.

Key words: nonlinear system of equation, Newton's method, Krawczyk's algorithm.

Subject classification: AMS (2000), 65G20, 65G30, 65G40.

INTRODUCTION

We consider the following system of equation:

$$f(x) = 0 \qquad (1.1)$$

Later in the paper F (as opposed to f) denotes an interval extension of f with $F:ID \to IR^n, ID \subset IR^n, ID$ bounded, $[\underline{x}_i, \overline{x}_i]$ is an interval, $x \in [X]$ and $[X] = [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \times ... \times [\underline{x}_n, \overline{x}_n]$ are a box in the sense of Moore (1977) and Neumaier (1990). [X] can be found using enclosure theorems for the set of zeros. We assume that F possesses a Frechet derivative F' which is continuous on ID. We denote the real solution of (1.1) as $x^* \in IR^n$.

In this paper, we assume the existence of a comparable initial interval vector $X^{(0)}$ = [a,b], where $F(a) \le x \le F(b)$, and construct a sequence of bounded monotone and hence convergence iterate in the sense of Schelin (1973) .

Newton's method is one of the important methods for the enclosure of nonlinear systems (1.1). For continuously differentiable F for which the determinant of the Jacobian $(F'(x)) \neq 0$ and for which that Jacobian is well conditioned, interval computation yields self verifying results given that there exists unique solution within small boxes with x^* reasonably close to the centre –the mid-point of [X].

However, if $F'(x)$ is ill conditioned or singular, such computations of solution to systems (1.1) will undoubtedly break down due to empty intersection of the successive iterates. In this circumstance an extended interval division and the concept of topological degree index can become useful tools in proving existence and non existence of zeros of F.

The treatment of topological degrees is well known in the Literature (Ortega and Rheinboldt, 2000; Kearfott and Dian, 2000; Mourrain et al., 2000; Kearfott et al., 2000).
In Kearfott et al. (2000), a generalization of theory and techniques to calculate arbitrary topological index using an effective heuristic approach were the main focus of studies. The result of these studies was the proof that a non zero topological index signified existence of a solution.

Some of the concepts of interval arithmetic used in this paper are given below.
We define a subset of the set of real numbers R of the form:

$$a = [a_1, a_2] = \left(a \middle| a_1 \le x \le a_2, a_1, a_2 \in R\right)$$

as a closed real interval.

The set of all closed real intervals is denoted by IR. When $a_1 = a_2$, then the interval $a=[a_1, a_2]$ is called a

point interval. In this way one recovers at once the real num-bers R and the corresponding real arithmetic when IR is restricted to the set of degenerate real intervals equipped with the basics of interval arithmetic defined for any interval operation.

For example given that $[a]=[a_1.a_2]$ and $[b]=[b_1,b_2]$ we define that,

$$[a]\,o[b]=\{a\,ob \mid a\in[a], b\in[b]\},\ 0\notin[b]$$

Let us note that the set (IR,+, o) is neither a field nor a ring. This is more so since a non degenerate interval [a] has no inverse with respect to addition and multiplication. The distributive law does not hold in general for interval arithmetic, instead it holds for subdistributivity. For example:

$$[a]([b]+[c])\subseteq[a][b]+[a][c]$$

We define a real interval matrix as the matrix the elements of which are real intervals. The set of $m\times n$ matrices over the real intervals will be denoted by $A_{m\times n}(IR)$. When m=n we speak of an interval matrix of order n. An interval matrix which all components are point intervals is called a point matrix.

For an interval matrix A $=(a_{i,j})$, the following point matrices are associative:

the width matrix w(A) $=(w(a_{i,j}))$, the mid point matrix

m(A) $=(m(a_{i,j}))$.

Further details on the discussion of basic tenets in interval methods can be found in Alefeld and Herzberger (1983), Alefeld and Mayer (2000) and Neumaier (2000). The above preliminary exposition on interval matrices will be sufficient for understanding this paper.

The rest of the paper is organized as follows.

In section 2, the well known Krawczyk's algorithm is discussed which gives the existence theorem for non linear systems of equations by using Moore (1977). In section 3, we revisit the Wolfe's modification of Krawczyk's algorithm using the idea presented in Uwamusi (2004). Thus we were able to construct an algorithm that is twice as fast as the original Krawckzyk's method. In section 4, we discussed the order of convergence of our proposed method. It is proved that the order of convergence of the new method is four. Finally, numerical examples have been given to demonstrate this.

The Krawczyk's algorithm

The Krawczyk's original algorithm according to Moore and Jones (1977) is given by:

$$K(X)=y-H(y)f(y)+\{I-Hf'(X)\}(X-y) \quad (2.1)$$

where y is a point interval vector chosen from X, H is an arbitrary non singular real matrix, and I is an identity matrix.

The functions f(y) and $f'(X)$ have interval extensions $F(y)$ and $F'(X)$ in the sense of Moore (1977). It can easily be verified that if $K(X^{(k)}\subseteq X^{(0)}$ (k=0,1,..,.), then

$$\left\|I-H^{(k)}F'(X^{(k)})\right\|<1\ K(X^{(k)})\subseteq X^{(0)} \text{ and}$$

$$\lim_{k\to\infty}\{w(X^{(k)})\}=0, where\ w(X^{(k+1)})\le\left\|R^{(k)}\right\|w(X^{(k)})$$

and w is the interval width of the vector X , $X^{(k)}$ is the sequence of Krawczyk's iterates for k=0,1,…, and $R^{(k)}=\left\|I-H^{(k)}F'(X^{(k)})\right\|$.

Therefore, the intersection of the form

$$X^{(k)}=X^{(k)}\cap X^{(k-1)}\quad (k=0,1,\ldots.) \qquad (2.2)$$

provides tight inclusion bounds separating the sought zeros of the nonlinear system (1.1).

Krawczyk's algorithm does not involve the inversion of interval matrices. The existence theorems for Krawczyk operator can be found in Moore and Jones (1977), Michelli and Miranker (1975) and Neumaier (1990).

While the Krawczyk iteration formula discussed above can be used to tightly enclose a solution, what is most significant is the ability of this Krawczyk method to provide a test if a solution exists within a given interval and also to check if this solution is unique.

For this, we will review the following theorem:

Theorem 2.1, Neumaier (1990)

Let A be a strongly regular Lipschitz matrix on $x\in ID_0$ for $F: ID_0\subseteq IR^n\to IR^n$. Let H be such that the spectral radius $\rho(|HA-I|)=\beta^*<1$.

Then Krawczyk's iteration (2.1) is strongly convergent and moreover, as long as $X^{(k+1)}\neq\phi$ we have:

$$\beta^{\bullet}\left\|\frac{1}{2}w(X^{(k+1)})\right\|\le\beta^*\left\|\frac{1}{2}(X^{(k)})\right\| \text{ with } \beta^*=\|HA-I\|.$$

Further more the radius converges linearly to zero with asymptotic convergence factor $\le\beta^{\bullet}$. It is assumed that $H=H^{(k)}$ varies with k such that it is the inverse of some matrix $F'([X]^k)$. Let us note in passing that whenever

$I - HF'([X]) = 1$ implies that $F'([X])$ is singular.

Main results- the accelerated Krawczyk's algorithm

The accelerated Krawczyk's algorithm mentioned at the beginning is developed in this section.

Our approach is based on that of Wolfe (1980)'s modification of Krawczyk's original algorithm which is given by the formula:

$$K(X_i^{(k)}) = y_{i-1}^{(k)} - H^{(k)} f(y_{i-1}^{(k)}) + \left(I - H^{(k)} F'(X^{(k)})\right)\left(X_{i-1}^{(k)} - y_{i-1}^{(k)}\right)$$

(i=1,2,…). (3.1)

Here k the iterative step, may be fixed and the iteration in method (3.1) may also be allowed to run as many times as required as inner loop before a complete cycle is obtained. His method is governed by the fixed value of i in advance before iteration is begun.

With the above exposition coming from Wolfe (1980) we are in the spirit of presenting our algorithm that will be easy to use and interpret as follows.

Let $X^{(0)} \in IR^{(n)}$ and a positive integer q be given. Then the following algorithm can be developed using Uwamusi [2004] as well as the approach of Wolfe [1980].

Algorithm 3.1

Input

Define m as the order of accuracy,

$f(x) = 0$ with $x \in R^n$, $f = (f_1, f_2, ..., f_n)^T$,

k,v,q are integers and v=0,1,…,q-1.

(1) Inflate x by $[X]^{(0)} = x_i^{(0)} + [-\varepsilon, \varepsilon]$

(2) Compute $y^{(0)} = m([X]^{(0)})$

(3) Compute $H^{(0)} = m(F'([X]^{(0)}))^{-1}$

(4) Compute $R^{(0)} = \left\| I - H^{(0)} F'([X]^{(0)}) \right\|$

(5) Set k=0,and v=0,1 with q=2 fixed

(6) Set

$$\hat{K}(X^{(k+\frac{v+1}{q})}) = y^{(k+\frac{v}{q})} - H^{(k)} f(y^{(k+\frac{v}{q})}) + (I - H^{(k)} F'(X^{(k)}))(X^{(k+\frac{v}{q})} - y^{(k+\frac{v}{q})})$$

(7) Compute

$$K([X]^{\left(k+\frac{v+1}{q}\right)}) = y^{(k)} - H^{(k)} f(y^{\left(k+\frac{v+1}{q}\right)}) + (I - H^{(k)} F'([X]^{(k)}))(\hat{X}^{\left(k+\frac{v+1}{q}\right)} - y^{\left(k+\frac{v+1}{q}\right)})$$

(8) if $[X]^{(k+1)} = K([X]^{(k)}) \cap [X^{(k)}] = \phi$

Stop this means that $F(y)$ has no solution for some $y \in [X]^{(k)}$, else

(9) Verify that $K([X]^{(k+1)}) \subseteq [X]^{(k)}$

Verify that $\rho(D^{-1}\left|R^{(k)}\right| D) < 1$, D is the diagonal matrix with $d_{ii} = d[x_{ii}], i = 1,2,...,n.$

(10) Set k=0,1,2,….,

(11) Repeat operations beginning from steps 2 to 7

(12) if $\left|K([X]^{(k+1)}) - [X]^{(k)}\right| \le m$

Stop, else repeat operation starting from steps 2 to 7

Finish,

Finish,

End.

The computation of H in the algorithm defined above is afforded because the product of the inverse midpoint matrix H and the interval matrix $F'([X])$ transforms given expression to almost a diagonal matrix. The optimal value for q is found to be 2.The algorithm proposed in the above section is different from that used in Uwamusi (2004) because Uwamusi (2004) used real floating point arithmetic to implement Newtonian steps with Jacobi and Gauss-Siedel methods. But here in this paper we are proposing a method which incorporates Krawczyk's method with itself using two steps length per iteration. To the best of knowledge from all Literatures available in the internet and elsewhere this method for Krawczyk's method is new.

Existence

Theorem 4.1

Let $F : IR^n \to IR^n$ be a continuously differentiable function, let $y \in IR^n, x \in IR^{n\times n}$, $H \in R^{n\times n}$ and $F'([X]) \subseteq IR^{n\times n}$ be given. If

$$-HF(y^{\left(k+\frac{v+1}{q}\right)}) + \left(I - H^{(k)} F'([X]^{(k)})\right).X \subseteq \text{int}(X)$$

then there exists some $y \in y + X$ with F(y) = 0, where y$\in x$

Proof: (Rump (1999).

The convergence analysis of the proposed method goes as follow:

First we observe that

$$H^{(k)} F(y^{\left(k+\frac{v+1}{q}\right)}) = H^{(k)} F(y^{\left(k+\frac{v+1}{q}\right)}) - (H^{(k)}.H^{(k)^{-1}} - I) H^{(k)} F(y^{\left(k+\frac{v+1}{q}\right)})$$

Table 1. Results from our algorithm for problem 1.

Iterations	Results
$X^{(\frac{1}{2})}$	[1.960714437, 1.960732374]
	[-0.214010613, -0.214002433]
	[0.536134319, 0.536150220]
	[-0.023389723, -0.023387331]
$X^{(1)}$	[1.896531951, 1.896531973]
	[-0.209458296, -0.209458294]
	[0.542592796, 0.542592815]
	[-0.022911082, -0.022911076]
$X^{(\frac{3}{2})}$	[1.896514009, 1.896514009]
	[-0.210266768, -0.210266768]
	[0.542056132, 0.542056132]
	[-0.023886970,-0.023886970]

$$\in H^{(k)}F\left(y^{\left(k+\frac{v+1}{q}\right)}\right)-(H^{(k)}F'\left(X^{(k)}\right)-I)x \qquad (4.1)$$

which is valid for all $H^{(k)} \in F'([X]^{(k)})$, $x \in [X]$ and , $F(y^{(k)}) \in F([X^{(k)}])$.

Because $\left(I-H^{(k)}F'([X]^{(k)})\right)$ is compact, then there exists $\alpha < 1$ ∋ $\left(I-H^{(k)}F'([X]^{(k)})\right).X \subset \alpha X$.

This shows that in the norm in which $[X]$ is a ball the expression

$\left|I-H^{(k)}(dF'([X])\right| \leq \alpha < 1$ (4.2) is valid.

Equality is satisfied when $\hat{X} = X$ and this will imply that

$K(\hat{X}, y, H) \subseteq K(X, y, H) \subseteq X$. (4.3)

To prove the order of convergence of our method we proceed as follow:

We set

$$d[\hat{X}]^{(k+1)} \leq \left|I-H^{(k)}F'([X]^{(k)}\right| d[X]^{(k)}$$

$$\leq$$

$$\left|H^{(k)}\right|\left|H^{(k)-1}-F([X]^{(k)})\right| d[X]^{(k)} \leq \hat{H}\left|F'([X]^{(k)})-F'([X]^{(k)})\right| d[X]^{(k)}$$

$$= \hat{H}\, dF'([X]^{(k)})\, d[X]^{(k)}$$

Also,

$$d[X]^{(k+1)} \leq \left|I-H^{(k)}F'([X]^{(k)})\right| d[\hat{X}]^{(k)} = \left|H^{(k)}\right|\left|H^{(k)-1}-F'([X]^{(k)})\right| d[\hat{X}]^{(k)}$$

$$\leq \hat{H}\left|F'([X]^{(k)})-F'(X]^{(k)})\right| d[\hat{X}]^{(k)}$$

$$= \hat{H}\, dF'([X]^{(k)})\, d[X]^{(k)} \times \hat{H}\, dF'([X]^{(k)})\, d[X]^{(k)}$$

$$= \hat{H}^{2}\, d^{2}F'([X]^{(k)})\, d^{2}[X]^{(k)}$$

Taking norms of both sides we have that

$$\left\|d[X^{(k+1)}\right\|_{\infty} \leq \beta\left\|d[X]^{(k)}\right\|_{\infty}^{4}, where\ \beta \geq 0, \quad (4.4)$$

and

$dF'([X]_{ij} \leq \alpha\left\|d[X]\right\|_{\infty}, \alpha \geq 0, 1 \leq i, j \leq n\ for\ all\ [X] \subseteq [X]^{(0)}$

.

The shows that our proposed method is of fourth order of convergence as opposed to Original Krawczyk (1969)'s and Wolfe (1980)'s algorithms which are quadratically convergent. Wolfe (1980)'s algorithm depends mainly on inner iterations as we have found in our implementation of the method. Our proposed method is different from that of Uwamusi (2004) in that it does not depend in knowing apriori the eigenvalues of Jacobi and Gauss-Siedel iteration matrices and it is also not dependent on the use of Jacobi and Gauss-Siedel iterative methods.

Numerical example

Problem 1 (Mancino, 1967)

Consider

$F(X) =$

$$20x_1 - \cos^2 x_2 + x_3 - \sin x_3 = 37$$

$$\cos 2x_1 + 20x_2 + \log_e(1+x_4^2) = -5$$

$$\sin(x_1+x_2) - x_2 + 19x_3 + \arctan x_3 = 12$$

$$2\tanh x_2 + e^{-2x_3^2+0.5} + 21x_4 = 0$$

$x^{(0)} = (2.0154195, -0.3182241, 0.6364483, -0.08784438)^T$

Using ε inflation of 10^{-2} where $[\varepsilon] = (-\varepsilon, \varepsilon)$ the initial inclusion interval is calculated.

We present results for method (3.1) as shown in Table 1.

Problem 2

$$F(x) =$$

$$6x_1 - 2\cos(x_2x_3) - 1 = 0$$

$$9x_2 + \sqrt{x_1^2 + \sin x_3 + 1.06} + 0.9 = 0$$

$$60x_3 + 3e^{-x_1x_2} + 10\pi - 3 = 0$$

$$X^{(0)} = \begin{pmatrix} 0.1 \\ -0.1 \\ 0.1 \end{pmatrix}, \varepsilon = 10^{-2}.$$

Table 2. Results for Wolfe (1980)'s algorithm.

No of inner iterations	Results
1	[1.897186446, 1.897186414]
	[-0.210851902,-0.210851901]
	[0.535873260, 0.535873260]
	[-0.023748854, -0.023748626]
2	[1.896497540, 1.896497540]
	[-0.210287532, -0.210287532]
	[0.536288177, 0.536288178]
	[-0.023867537, -0.023867537]
3	[1.896494401, 1.896494401]
	[-0.210266697, -0.210266697]
	[0.542477942, 0.542477942]
	[-0.023882900, -0.023882900]
4	[1.896494401, 1.896494401]
	[-0.210266697,-0.210266697]
	[0.542477942, 0.542477942]
	[-0.023882900, -0.023882900]

Table 3. Results for problem 1 using Krawczyk's method (2.1).

Iterations(k)	Results
0	[2.0054195,2.0254195]
	[-0.32822441,-0.3082241]
	[0.6264483,0.6464483]
	[-0.0974438,-0.0774438]
$X^{(1)}$	[1.897205777,1.897236841]
	[-0.209878094,-0.208985383]
	[0.535828321,0.535852697]
	[-0.007828442,-0.004244712]
$X^{(2)}$	[1.896521337,1.896521999]
	[-0.210232839,-0.210231329]
	[0.542440341,0.542440487]
	[-0.024116819,-0.023956322]
$X^{(3)}$	[1.896515381,1.896515388]
	[-0.21019125,-0.210191203]
	[0.542069585,0.542069607]
	[-0.02389349,-0.023893094]

The convergence of the proposed algorithm (3.1) is achieved when the predicted values coincide with the evaluated results. This usually happens after some few steps of iteration have begun. The method is different from that of Wolfe (1980)'s algorithm because Wolfe's method allows fixing the iteration step k while allowing the inner iteration to be run as many times as possible in each iteration cycle. Thus Wolfe (1980)'s algorithm could be seen as a method belonging to Class of stationary Newton's method. This allows the computation of Jacobian matrix viz: Predictor –corrector approach and the Jacobian only

Table 4. Results from our algorithm 3.1 for Problem 2.

Iteration	Results
1	[0.498901717,0.498901710]
	[-0.199872459,-0.199873833]
	[-0.530073299,-0.530083371]
2	[0.498144782,0.498144782]
	[-0.199605179,-0.199605179]
	[-0.528826126,-0.528826126]

Table 5. Results for Wolfe (1980) algorithm.

No of inner iterations	Results
1	[0.500078018, 0.500081981]
	[-0.153367353, -0.153321046]
	[-0.527034416, -0.526850525]
2	[0.498913213, 0.498913214]
	[-0.199694064, -0.199693941]
	[-0.527830790, -0.527830448]
3	[0.498149872, 0.498149872]
	[-0.199583638, -0.199583638]
	-0.528835295, -0.528835295]

only once throughout the course of computation. Our method requires just two main steps matrix varies according to the sizes of $[X^{(k)}]$. It is very easy to use. The numerically obtained results are quite good and accurate.

Conclusion

From Table 1, it can be seen that convergence is attained after $1\frac{1}{2}$ successive iterations as the widths of iteration sequence of vectors $\to 0$ as $k \to \infty$.

From the numerical results presented it can be seen that there is a solution to F(x) =0. Results obtained are in agreement with those given in Mancino (1967).

For q=1 the asymptotic rate of convergence of the q-step Krawckzyk's method is precisely the same as that of Krawczyk's original algorithm applied on a nonlinear system F(x)=0. It is then to be expected that the q-step Krawczyk's process is q times faster as the one –step process, since the corresponding nonlinear problem, one iteration of the q-step process amounts to q-Krawczyk's steps. Numerical experiment suggests that for q= 2 one is expected to get better accurate results than using q>2 since higher values of q may not necessarily enhance convergence of the method. It is evident from the results presented that as iteration step goes to infinity, the mid-point of interval results obtained are quite good as approximation to those obtained by Mancino (1967).

Results in table 1 appear to be better than those in tables 2 and 3.This is again demonstrated in problem 2 as evidenced in Tables 4 and 5. Also in Table 4, it can be seen that our proposed method out-performed the Wolfe (1980)'s algorithm due to the results obtained from the given problem 2

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