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Full Length Research Paper

Some iterative algorithms for solving regularized mixed quasi variational inequalities

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In this paper, we use the auxiliary principle technique coupled with the principle of iterative regularization to suggest and analyze some new iterative algorithms for solving mixed quasi variational inequalities. We also study the convergence criteria of these algorithms under some suitable and mild conditions. Several special cases are also considered. Results obtained in this paper continue to hold for these special cases.

Key words: Iterative regularization, mixed variational inequality, convergence, auxiliary principle, convex functions.

INTRODUCTION

Variational inequalities, which were introduced and studied by Stampacchia (1964), are being used to study a wide class of diverse unrelated problems arising in various branches of pure and applied sciences in a unified framework. Variational inequalities have been extended and generalized in different directions using some new and novel methods (Giannessi and Maugeri, 1995, Giannessi et al., 2001; Glowinski et al., 1981; Khan and Rouhani, 2007; Kinderlehrer and Stampacchia, 1980; Noor, 1975, 1988, 1997, 2002, 2004, 2004a, 2009; Noor et al., 1993, 2011, a, b, c). It is well known that a minimum of a sum of differentiable convex function and non differentiable convex function on the convex set can be characterized by a class of variational inequalities, which are called the mixed quasi variational inequalities involving the bifunction. There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle, and resolvent equations methods for solving variational inequalities. However, it is known that projection, Wiener-Hopf equations, and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving mixed quasi variational inequalities due to the presence of the function $\varphi(.,.)$. This fact motivated us to use the auxiliary principle technique of Glowinski et al. (1981). In this paper, we use this technique of the auxiliary principle in conjunction with the Bregman function coupled with the principle of iterative regularization. It was introduced by Bakushinskii (1979) in connection with variational inequalities. An important extension of this approach is presented by Alber and Ryazantseva (2006). In this approach, the regularized parameter is changed at each iteration which is in contrast with the common practice for parameter identification of using a fixed regularization parameter throughout the minimization process.

In this paper, we suggest and analyze an iterative algorithm based on auxiliary principle technique and principle of iterative regularization to solve a class of mixed quasi variational inequalities. For the convergence analysis of the explicit iterative algorithm, we use partially relaxed strongly monotone operator which is weaker condition than strongly monotonicity. In this respect, our results represent an improvement of the results of Khan and Rouhani (2007). We also suggest implicit type version of this algorithm. The convergence of the implicit iterative methods only requires the monotonicity and the skew symmetry of the bifunction. Our results present an improvement of the previously known results. We hope that the technique and the idea of this paper may be

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extended for mixed quasi variational-like inequalities and equilibrium problems. The interested readers are invited to explore new applications of the auxiliary principle technique for solving the dynamic type variational inequalities and its variant forms.

PRELIMINARIES

Let *H* be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed set in *H*. First of all, we recall the following well-known results and concepts.

For a given nonlinear operator $T : H \to H$ and continuous bifunction $\varphi(.,.)$, we consider the problem of finding $u \in H$ such that

$$\langle Tu, v-u \rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \qquad \forall v \in H. \qquad (1)$$

An inequality of Type (1) is called the mixed quasi variational inequality. The existence of a solution of (1) and other aspects of the mixed quasi variational inequalities is given in Noor (2004, 2004a), Kinderlehrer and Stampacchia (1980) and Giannessi and Maugeri 1995, Giannessi et al., (2001).

If $\varphi(.,.) \equiv \varphi(.)$, then the problem (1) is reduced to the problem of finding $u \in H$ such that

$$\langle Tu, v-u \rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in H. \qquad (2)$$

which is called mixed variational inequality or variational inequality of the second type. If the function $\varphi(.)$ is a proper, convex and lower-semicontinous, then problem (2) is equivalent to finding $u \in H$ such that

$$0 \in Tu + \partial\varphi(u),$$

which is known as the variational inclusion. This problem is also known as finding the zero of the sum of two operators.

If $\phi(.)$ is an indicator function on the closed convex set K in the real Hilbert space H, that is,

$$\phi(u) = \begin{cases} 0, & if\ u \in K \\ \infty, & if\ u \notin K, \end{cases}$$

then problem (2) is equivalent to finding $u \in K$ satisfying

$$\langle Tu, v-u \rangle \geq 0, \quad \forall v \in K, \qquad (3)$$

which is known as the classical variational inequality introduced and studied by Stampacchia (1964).

For suitable and appropriate choice of the operators and spaces, one can obtain a number of known and new classes of variational inequalities and related optimization problems. Applications, formulation, numerical methods, sensitivity analysis, dynamical system and other aspects of the variational inequalities and related problems are given in Giannessi and Maugeri 1995, Giannessi et al. (2001), Glowinski et al. (1981), Khan and Rouhani (2007), Kinderlehrer and Stampacchia (1980), Noor (1975, 1997, 2002, 2004, 2004a, 2009) and Noor et al. (1993, 2011, 20111a, 2011b).

We now recall some basic concepts and results, which are well known.

Definition 1

A function f is said to be a strongly convex function on K with modulus μ, if

$$f(u+t(v-u)) \leq (1-t)f(u) + t\,f(v) - t(1-t)\mu\|v-u\|^2, \ \forall v,u \in K,\ t \in [0,1].$$

Clearly, a differentiable strongly convex function f is equivalent to

$$f(v) - f(u) \geq \langle f'(u), v-u \rangle + \mu\|v-u\|^2, \ \forall v,u \in K.$$

Definition 2

An operator $T : H \to H$ is said to be:

1. Monotone, if and only if, $\langle Tu - Tv, u-v \rangle \geq 0, \qquad \forall u,v \in H.$.

2. Partially relaxed strongly monotone, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, z-v \rangle \geq \alpha\|z-u\|^2, \quad \forall u,v,z \in H.$$

We would like to point out that, for $z = u$, partially relaxed strong monotonicity reduces to monotonicity of the operator T.

Definition 3

The bifunction $\varphi(.,.) : H \times H \to H$ is said to be skew-symmetric, if $\varphi(u,u)-\varphi(u,v)-\varphi(v,u)+\varphi(v,v)\geq 0, \quad \forall u,v\in H.$

If the bifunction $\varphi(.,.)$ is linear in both arguments, then,

$\varphi(u,u)-\varphi(u,v)-\varphi(v,u)+\varphi(v,v)=\varphi(u-v,\,u-v)\geq 0, \quad \forall u,v\in H.$

This shows that the bifunction $\varphi(.,.)$ is nonnegative (Noor, 2004).

The following well known lemma plays an important role in convergence analysis

Lemma 1

$$2\langle u,v\rangle \leq \frac{\varepsilon}{2}\|u\|^2+\frac{1}{2\varepsilon}\|v\|^2, \qquad \forall u,v\in H.$$

ITERATIVE REGULARIZATION METHODS

Here, we suggest and analyze some iterative regularization methods for solving the mixed quasi variational inequality (1) using the auxiliary principle technique coupled with the principle of iterative regularization. This is the main motivation of this paper.

For a given $u\in H$, consider the problem of finding $w\in H$ such that

$$\langle \rho Tw+E'(w)-E'(u),v-w\rangle+\rho\varphi(v,w)-\rho\varphi(w,w)\geq 0, \quad \forall v\in H, \tag{4}$$

which is called the auxiliary mixed quasi variational inequality. We note that, if $w=u$, then $w=u$ is a solution of (1). Using (4), we also consider the regularized auxiliary principle problem associated with the mixed quasi variational inequality (1).

For a given $u\in H$, consider the problem of finding a solution $w\in H$ satisfying the auxiliary variational inequality:

$$\rho_n\langle Tw+\varepsilon_n w+E'(w)-E'(u),v-w\rangle+\rho_n\varphi(v,w)-\rho_n\varphi(w,w)\geq 0, \quad \forall v\in H. \tag{5}$$

where $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of positive real, and $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real such that $\varepsilon_n\to 0$ as $n\to\infty$.

Note that, if $w=u$ and $\varepsilon_n\to 0$ as $n\to\infty$, then w is a solution of (1). We now consider the regularized version of (1) as follows: For a fixed but arbitrary $n\in N$ and for $\varepsilon_n>0$, find $u_{\varepsilon_n}\in H$ such that

$$\langle Tu_{\varepsilon_n}+\varepsilon_n u_{\varepsilon_n},v-u_{\varepsilon_n}\rangle+\varphi(v,u_{\varepsilon_n})-\varphi(u_{\varepsilon_n},u_{\varepsilon_n})\geq 0, \quad \forall v\in H. \tag{6}$$

Note that if $w=u$ and $\varepsilon_n\to 0$ as $n\to\infty$, then w is a solution of mixed quasi variational inequality (1). This simple observation enables us to suggest implicit iterative method for solving (1) and this is one of the motivations of this paper.

Algorithm 1

For a given $u_0\in H$, compute $u_{n+1}\in H$ from the iterative scheme

$$\langle \rho_n(T_n u_{n+1}+\varepsilon_n u_{n+1})+E'(u_{n+1})-E'(u_n),v-u_{n+1}\rangle \geq -\rho_n\varphi(v,u_{n+1})+\rho_n\varphi(u_{n+1},u_{n+1})\geq 0, \quad \forall v\in H, \tag{7}$$

which is known as proximal point (or implicit) algorithm for solving regularized mixed quasi variational inequality.

If $\varphi(u,v)=\varphi(u)$ is the indicate = or function of the convex set K then Algorithm 1 reduces to the following Algorithm for solving the variational inequality (3) and appears to be a new one.

Algorithm 2

For a given $u_0\in H$, find the approximate solution $u_{n+1}\in H$ by the following iterative scheme

$$\langle \rho_n(T_n u_{n+1}+\varepsilon_n u_{n+1})+E'(u_{n+1})-E'(u_n),v-u_{n+1}\rangle\geq 0, \quad \forall v\in K, \tag{3}$$

which is known as proximal-point (or implicit) algorithm for solving regularized variational inequality

In a similar way, for suitable and appropriate choice of the operator, bifunction and the space, one can obtain a number of implicit iterative methods for solving the variational inequalities and related optimization problems.

We now study the convergence analysis of Algorithm 1 using the technique of Khan and Rouhani (2007) and this is the main motivation of our next result.

Theorem 4

Let *T* be a monotone operator. For the approximation T_n of T, assume that there exists $\{\delta_n\}$ such that $\delta_n > 0$ and a constant c such that

$$\|T_n(u) - T(v)\| \le c\delta_n(1 + \|u\|), \quad \forall u \in H. \qquad (8)$$

If, for the sequences $\{\varepsilon_n\}$, $\{\delta_n\}$ and $\{\rho_n\}$, we have

$$\sum_{n=0}^{\infty}\delta_n^2 < \infty, \quad \sum_{n=0}^{\infty}(\rho_n^2 + \delta_n^2) < \infty, \quad \sum_{n=0}^{\infty}\varepsilon_n\rho_n < \infty, \quad \sum_{n=0}^{\infty}\alpha_n\rho_n < \infty. \qquad (9)$$

approximate solution $u_{n+1} \in H$ obtained from Algorithm 1 converges to an exact solution $u \in H$ satisfying (1).

Proof

Let $u_{\varepsilon_n} \in H$ satisfying the regularized mixed variational inequality (6), then replacing v by u_{n+1}, we have

$$\left\langle \rho_n(Tu_{\varepsilon_n} + \varepsilon_n u_{\varepsilon_n}),\, u_{n+1} - u_{\varepsilon_n} \right\rangle + \rho_n\varphi(u_{n+1}, u_{\varepsilon_n}) - \rho_n\varphi(u_{\varepsilon_n}, u_{\varepsilon_n}) \ge 0. \qquad (10)$$

Let $u_{n+1} \in H$ be the approximate solution obtained from (7). Replacing v by u_{ε_n}, we have

$$\left\langle \rho_n(T_n u_{n+1} + \varepsilon_n u_{n+1}) + E'(u_{n+1}) - E'(u_n), u_{\varepsilon_n} - u_{n+1} \right\rangle + \rho_n\varphi(u_{\varepsilon_n}, u_{n+1}) - \rho_n\varphi(u_{n+1}, u_{n+1}) \ge 0. \qquad (11)$$

For the sake of simplicity, we take $T + \varepsilon_n = F_n$ and $T_n + \varepsilon_n = \tilde{F}_n$ in (3.7) and (11) respectively. Adding the resultant inequalities and using Definition 2.4, we have

$$\left\langle \rho_n \tilde{F}_n u_{n+1} - \rho_n F_n u_{\varepsilon_n} + E'(u_{n+1}) - E'(u_n), u_{\varepsilon_n} - u_{n+1} \right\rangle \ge 0,$$

from which, we have

$$\left\langle E'(u_{n+1}) - E'(u_n), u_{\varepsilon_n} - u_{n+1} \right\rangle \ge \left\langle \rho_n F_n u_{\varepsilon_n} - \rho_n \tilde{F}_n u_{n+1}, u_{\varepsilon_n} - u_{n+1} \right\rangle \qquad (12)$$

We consider the Bregman function

$$B(u,w) = E(u) - E(w) - \left\langle E'(w), u - w \right\rangle \ge \mu \| u - w \|,$$

from which one can find

$$B(u_{\varepsilon_{n-1}}, u_n) = E(u_{\varepsilon_{n-1}}) - E(u_n) - \left\langle E'(u_n), u_{\varepsilon_{n-1}} - u_n \right\rangle \ge \mu \| u_{\varepsilon_{n-1}} - u_n \|,$$

and

$$B(u_{\varepsilon_n}, u_{n+1}) = E(u_{\varepsilon_n}) - E(u_{n+1}) - \left\langle E'(u_{n+1}), u_{\varepsilon_n} - u_{n+1} \right\rangle \ge \mu \| u_{\varepsilon_n} - u_{n+1} \|.$$

Considering the difference

$$\begin{aligned}
&B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) \\
&= E(u_{\varepsilon_{n-1}}) - E(u_n) - \left\langle E'(u_n), u_{\varepsilon_{n-1}} - u_n \right\rangle \\
&\quad - E(u_{\varepsilon_n}) + E(u_{n+1}) + \left\langle E'(u_{n+1}), u_{\varepsilon_n} - u_{n+1} \right\rangle \\
&= E(u_{\varepsilon_{n-1}}) - E(u_{\varepsilon_n}) + \left\langle E'(u_{n+1}), u_{\varepsilon_n} - u_{n+1} \right\rangle \\
&\quad + E(u_{n+1}) - E(u_n) - \left\langle E'(u_n), u_{\varepsilon_{n-1}} - u_{n+1} \right\rangle - \left\langle E'(u_n), u_{n+1} - u_n \right\rangle \\
&\ge E(u_{\varepsilon_{n-1}}) - E(u_{\varepsilon_n}) + \left\langle E'(u_{n+1}), u_{\varepsilon_n} - u_{n+1} \right\rangle \\
&\quad - \left\langle E'(u_n), u_{\varepsilon_{n-1}} - u_{n+1} \right\rangle + \mu \|u_{n+1} - u_n\|^2, \\
&\ge E(u_{\varepsilon_{n-1}}) - E(u_{\varepsilon_n}) + \left\langle E'(u_{n+1}) - E'(u_n), u_{\varepsilon_n} - u_{n+1} \right\rangle + \\
&\qquad \left\langle E'(u_n), u_{\varepsilon_n} - u_{\varepsilon_{n-1}} \right\rangle + \mu \|u_{n+1} - u_n\|^2 \\
&\ge E(u_{\varepsilon_{n-1}}) - E(u_{\varepsilon_n}) + \rho_n \left\langle F_n u_{\varepsilon_n} - \tilde{F}_n u_{n+1}, u_{\varepsilon_n} - u_{n+1} \right\rangle + \\
&\qquad \left\langle E'(u_n), u_{\varepsilon_n} - u_{\varepsilon_{n-1}} \right\rangle + \mu \|u_{n+1} - u_n\|^2. \\
&\ge \mu \|u_{n+1} - u_n\|^2 + \mu \|u_{\varepsilon_{n-1}} - u_{\varepsilon_n}\|^2 \\
&\qquad + \left\langle E'(u_n) - E'(u_{\varepsilon_n}), u_{\varepsilon_n} - u_{\varepsilon_{n-1}} \right\rangle \\
&\qquad + \rho_n \left\langle F_n u_{\varepsilon_n} - \tilde{F}_n u_{n+1}, u_{\varepsilon_n} - u_{n+1} \right\rangle
\end{aligned}$$

Since T is a monotone operator, $F_n = T + \varepsilon_n$ is strongly monotone with constant $(\alpha + \varepsilon_n) = \alpha_n$. If we have

$$B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) \geq \mu \|u_{n+1} - u_n\|^2 + \mu \|u_{\varepsilon_{n-1}} - u_{\varepsilon_n}\|^2 + \langle E'(u_n) - E'(u_{\varepsilon_n}), u_{\varepsilon_n} - u_{\varepsilon_{n-1}} \rangle + \rho_n \alpha_n \|u_n - u_{\varepsilon_n}\|^2 - \rho_n \langle F_n u_n - F_n u_{n+1}, u_{\varepsilon_n} - u_{n+1} \rangle,$$

from which, we have

$$B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) \geq \mu \|u_{\varepsilon_{n-1}} - u_{\varepsilon_n}\|^2 + \mu \|u_{n+1} - u_n\|^2 + \rho_n \alpha_n \|u_{\varepsilon_n} - u_{n+1}\|^2 + \tau_1 + \tau_2, \tag{13}$$

where

$$\tau_1 = \langle E'(u_n) - E'(u_{\varepsilon_n}), u_{\varepsilon_n} - u_{\varepsilon_{n-1}} \rangle \geq -\frac{\beta^2 \varepsilon^2}{2} \|u_n - u_{\varepsilon_n}\|^2 - \frac{1}{2\varepsilon^2} \|u_{\varepsilon_{n-1}} - u_{\varepsilon_n}\|^2.$$

Using Lemma 1 and Lipschitz continuity of operator E', we have

$$\tau_1 \geq -\frac{\varepsilon_n \rho_n}{2} \|u_n - u_{\varepsilon_n}\|^2 - \frac{\beta^2}{2\varepsilon_n \rho_n} \|u_{\varepsilon_{n-1}} - u_{\varepsilon_n}\|^2. \tag{14}$$

Solving for τ_2 ,we have

$$\tau_2 \geq -\rho_n \langle F_n u_{n+1} - F_n u_{n+1}, u_{\varepsilon_n} - u_{n+1} \rangle \geq -\frac{\varepsilon^2 \rho_n}{2} \|F_n u_{n+1} - F_n u_n\|^2 - \frac{\varepsilon^2 \rho_n}{2} \|F_n u_n - F_n u_{n+1}\|^2 - \frac{\rho_n}{2\varepsilon^2} \|u_{\varepsilon_n} - u_{n+1}\|^2.$$

Using (8), we obtain

$$\tau_2 \geq \frac{-c^2 \delta_n^2 \varepsilon^2 \rho_n}{2} [1 + \|u_n - u_{n+1}\|]^2 - \frac{\varepsilon^2 \rho_n \gamma_n^2}{2} \|u_n - u_{n+1}\|^2 - \frac{\rho_n}{2\varepsilon^2} \|u_{\varepsilon_n} - u_{n+1}\|^2.$$

Here we have used the Lipschitz continuity of $F_n (= T + \varepsilon_n)$ with constant $\gamma_n (= \gamma + \varepsilon_n)$.

Now, we have

$$\tau_2 \geq \frac{-c^2 \delta_n^2 \varepsilon^2 \rho_n}{2} [1 + \|u_n - u_{n+1}\|]^2 - \frac{\varepsilon^2 \rho_n \gamma_n^2}{2} \|u_n - u_{n+1}\|^2 - \frac{\rho_n}{2\varepsilon^2} \|u_{\varepsilon_n} - u_{n+1}\|^2$$
$$\geq \frac{-c^2 \delta_n^2 \varepsilon^2 \rho_n}{2} [t + \|u_{\varepsilon_n} - u_{n+1}\|]^2 - \frac{\varepsilon^2 \rho_n \gamma_n^2}{2} \|u_n - u_{n+1}\|^2 - \frac{\rho_n}{2\varepsilon^2} \|u_{\varepsilon_n} - u_{n+1}\|^2.$$

Thus for any $t \geq 1 + \|u_n - u_{\varepsilon_n}\|$,

$$\tau_2 \geq -c^2 \delta_n^2 \varepsilon^2 \rho_n t^2 - c^2 \delta_n^2 \varepsilon^2 \rho_n \|u_{\varepsilon_n} - u_{n+1}\|^2 - \frac{\varepsilon^2 \rho_n \gamma_n^2}{2} \|u_n - u_{n+1}\|^2 - \frac{\rho_n}{2\varepsilon^2} \|u_{\varepsilon_n} - u_{n+1}\|^2. \tag{15}$$

From (13) to (15), we have

$$\begin{aligned} B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) &\geq \mu \|u_{n+1} - u_n\|^2 + \mu \|u_{\varepsilon_{n-1}} - u_{\varepsilon_n}\|^2 + \rho_n \alpha_n \|u_{\varepsilon_n} - u_{n+1}\|^2 \\ &- \frac{\varepsilon_n \rho_n}{2} \|u_{n+1} - u_{\varepsilon_n}\|^2 - \frac{\beta^2}{2\varepsilon_n \rho_n} \|u_{\varepsilon_{n-1}} - u_{\varepsilon_n}\|^2 - c^2 \delta_n^2 \varepsilon^2 \rho_n t^2 \\ &- c^2 \delta_n^2 \varepsilon^2 \rho_n \|u_{\varepsilon_n} - u_{n+1}\|^2 - \frac{\varepsilon^2 \rho_n \gamma_n^2}{2} \|u_n - u_{n+1}\|^2 - \frac{\rho_n}{2\varepsilon^2} \|u_{\varepsilon_n} - u_{n+1}\|^2 \\ &\geq (\mu - \frac{\gamma_n}{2}) \|u_{n+1} - u_n\|^2 + C_1 \varepsilon_n \rho_n \|u_{\varepsilon_{n-1}} - u_{\varepsilon_n}\|^2 + \rho_n \alpha_n \|u_n - u_{\varepsilon_n}\|^2 \\ &- \frac{\varepsilon_n \rho_n}{2} \|u_n - u_{\varepsilon_n}\|^2 - \frac{c^2 \delta_n^2 t^2}{\gamma_n} - C_2 (\delta_n^2 + \rho_n^2) \|u_{\varepsilon_n} - u_{n+1}\|^2. \end{aligned}$$

Using conditions (8), we have

$$B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) \geq (\mu - \frac{\gamma_n}{2}) \|u_{n+1} - u_n\|^2.$$

If $u_{n+1} = u_n$, it is easily shown that u_n is a solution of the mixed quasi variational inequality (1). Otherwise, the assumption $\gamma_n > 2\mu$ implies that $B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1})$ is non-negative and we must have

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0. \tag{16}$$

From (16), it follows that the sequence $\{u_n\}$ is bounded. Let $\hat{u} \in H$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_{n_i}\}$ of this sequence converges to $\hat{u} \in H$. Now essentially using the technique of Zhu and Marcotte (1996), it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\hat{u} \in H$ satisfying the mixed quasi variational inequality (1).

To implement the proximal method, one has to calculate the solution implicitly, which is itself a difficult problem. We again use the auxiliary principle technique to suggest another iterative method; the convergence of which requires only the partially relaxed strongly monotonicity of the operator.

For a given $u \in H$, consider the problem of finding $w \in H$ such that

$$\langle Tu + E'(w) - E'(u), v - w \rangle + \varphi(v, w) - \varphi(w, w) \geq 0, \quad \forall v \in H. \tag{17}$$

Note that, t if $w = u$, then (17) reduces to (1). Using (17), we develop an iterative scheme for solving (1).

For a given $u \in H$, consider the problem of finding a solution $w \in H$ satisfying the auxiliary variational inequality

$$\rho_n \langle Tu + \varepsilon_n u + E'(w) - E'(u), v - w \rangle + \rho_n \varphi(v, w) - \rho_n \varphi(w, w) \geq 0, \quad \forall v \in H., \tag{18}$$

where $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of positive real, and $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real such that $\varepsilon_n \to 0$ as $n \to \infty$.

Note that, if $w = u$ and $\varepsilon_n \to 0$ as $n \to \infty$, then w is a solution of (1).

Algorithm 3

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\langle \rho_n (T_n u_n + \varepsilon_n u_n) + E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle + \rho_n \varphi(v, u_{n+1}) - \rho_n \varphi(u_{n+1}, u_{n+1}) \geq 0, \quad \forall v \in H, \tag{19}$$

where $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of positive real and $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real such that $\varepsilon_n \to 0$ as $n \to \infty$.

Using the technique of Theorem 4, one can prove the convergence of Algorithm 3, we include its proof for the sake of completeness.

Theorem 4

Let T be a partially relaxed strongly monotone operator with constant $\alpha > 0$. For the approximation T_n of T, assume that (8) holds. If, for the sequences $\{\varepsilon_n\}$, $\{\delta_n\}$ and $\{\rho_n\}$, (9) is satisfied and $\varphi(.,.)$ is skew symmetric, then the approximate solution $u_{n+1} \in H$ obtained from Algorithm 4 converges to an exact solution $u \in H$ satisfying (1).

Proof

Let $u_{\varepsilon_n} \in H$ satisfying the regularized mixed variational inequality (17), then replacing v by u_{n+1}, we have

$$\langle \rho_n (Tu_{\varepsilon_n} + \varepsilon_n u_{\varepsilon_n}),\ u_{n+1} - u_{\varepsilon_n} \rangle + \rho_n \varphi(u_{n+1}, u_{\varepsilon_n}) - \rho_n \varphi(u_{\varepsilon_n}, u_{\varepsilon_n}) \geq 0. \tag{20}$$

Let $u_{n+1} \in H$ be the approximate solution obtained from (19). Replacing v by u_{ε_n}, we have

$$\langle \rho_n (T_n u_n + \varepsilon_n u_n) + E'(u_{n+1}) - E'(u_n), u_{\varepsilon_n} - u_{n+1} \rangle + \rho_n \varphi(u_{\varepsilon_n}, u_{n+1}) - \rho_n \varphi(u_{n+1}, u_{n+1}) \geq 0. \tag{21}$$

For the sake of simplicity, we take $T + \varepsilon_n = F_n$ and $T_n + \varepsilon_n = \tilde{F}_n$ in (20) and (21) respectively. Adding the resultant inequalities, we have

$$\langle \rho_n \tilde{F}_n u_n - \rho_n F_n u_n + E'(u_{n+1}) - E'(u_n), u_{\varepsilon_n} - u_{n+1} \rangle \geq 0,$$

from which, we have

$$\langle E'(u_{n+1}) - E'(u_n), u_{\varepsilon_n} - u_{n+1} \rangle \geq \langle \rho_n F_n u_{\varepsilon_n} - \rho_n \tilde{F}_n u_n, u_{\varepsilon_n} - u_{n+1} \rangle . \tag{21}$$

Now, we investigate the difference

$$\begin{aligned} B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) &= E(u_{\varepsilon_{n-1}}) - E(u_n) - \langle E'(u_n), u_{\varepsilon_{n-1}} - u_n \rangle \\ &\quad - E(u_{\varepsilon_n}) + E(u_{n+1}) + \langle E'(u_{n+1}), u_{\varepsilon_n} - u_{n+1} \rangle \\ &\geq E(u_{\varepsilon_{n-1}}) - E(u_{\varepsilon_n}) + \langle E'(u_{n+1}), u_{\varepsilon_n} - u_{n+1} \rangle \\ &\quad - \langle E'(u_n), u_{\varepsilon_{n-1}} - u_{n+1} \rangle + \mu \|u_{n+1} - u_n\|^2 . \end{aligned}$$

Here we used the strongly convexity of E .

Since T is partially relaxed strongly monotone with constant $\alpha > 0$, $F_n = T + \varepsilon_n$ is partially relaxed strongly monotone with constant $(\alpha + \frac{\varepsilon_n}{4}) = \alpha_n$, then we have

$$\begin{aligned} B(u_{\varepsilon_{n-1}}, u_n) - B(u_{\varepsilon_n}, u_{n+1}) &\geq \mu \|u_{n+1} - u_n\|^2 + \mu \|u_{\varepsilon_{n-1}} - u_{\varepsilon_n}\|^2 + \langle E'(u_n) - E'(u_{\varepsilon_n}), u_{\varepsilon_n} - u_{\varepsilon_{n-1}} \rangle \\ &\quad + \rho_n \alpha_n \|u_n - u_{\varepsilon_n}\|^2 - \rho_n \langle \tilde{F}_n u_n - F_n u_n, u_{\varepsilon_n} - u_{n+1} \rangle . \end{aligned}$$

From which, we have

$$B(u_{\varepsilon_{n-1}},u_n)-B(u_{\varepsilon_n},u_{n+1})\geq \mu\left\|u_{\varepsilon_{n-1}}-u_{\varepsilon_n}\right\|^2+\mu\|u_{n+1}-u_n\|^2 + \rho_n\alpha_n\left\|u_n-u_{\varepsilon_n}\right\|^2+\tau_1+\tau_2,$$

where

$$\begin{aligned}\tau_1 &= \left\langle E'(u_n)-E'(u_{\varepsilon_n}), u_{\varepsilon_n}-u_{\varepsilon_{n-1}}\right\rangle \\ &= -\left\langle E'(u_n)-E'(u_{\varepsilon_n}), u_{\varepsilon_{n-1}}-u_{\varepsilon_n}\right\rangle \\ &\geq -\frac{\varepsilon^2}{2}\left\|E'(u_n)-E'(u_{\varepsilon_n})\right\|^2-\frac{1}{2\varepsilon^2}\left\|u_{\varepsilon_{n-1}}-u_{\varepsilon_n}\right\|^2 \\ &\geq -\frac{\beta^2\varepsilon^2}{2}\left\|u_n-u_{\varepsilon_n}\right\|^2-\frac{1}{2\varepsilon^2}\left\|u_{\varepsilon_{n-1}}-u_{\varepsilon_n}\right\|^2.\end{aligned}$$

Here we used the Lipchitz continuity of operator E'. Taking $\varepsilon=\sqrt{\varepsilon_n\rho_n/\beta^2}$, we have

$$\tau_1 \geq -\frac{\varepsilon_n\rho_n}{2}\left\|u_n-u_{\varepsilon_n}\right\|^2-\frac{\beta^2}{2\varepsilon_n\rho_n}\left\|u_{\varepsilon_{n-1}}-u_{\varepsilon_n}\right\|^2.$$

Solving for τ_2, where

$$\begin{aligned}\tau_2 &\geq -\rho_n\left\langle \tilde{F}_nu_n-F_nu_n, u_{\varepsilon_n}-u_{n+1}\right\rangle \\ &\geq -\frac{\varepsilon^2\rho_n}{2}\left\|\tilde{F}_nu_n-F_nu_n\right\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2 \\ &\geq -\frac{\varepsilon^2\rho_n}{2}\left\|\tilde{F}_nu_n-F_nu_{n+1}\right\|^2-\frac{\varepsilon^2\rho_n}{2}\left\|F_nu_{n+1}-F_nu_n\right\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2.\end{aligned}$$

Using (8), we obtain

$$\tau_2 \geq \frac{-c^2\delta_n^2\varepsilon^2\rho_n}{2}[1+\|u_n-u_{n+1}\|]^2-\frac{\varepsilon^2\rho_n\gamma_n^2}{2}\|u_n-u_{n+1}\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2.$$

Here we used the Lipschitz continuity of $F_n(=T+\varepsilon_n)$ with constant $\gamma_n(=\gamma+\varepsilon_n)$. Now, we have

$$\begin{aligned}\tau_2 &\geq \frac{-c^2\delta_n^2\varepsilon^2\rho_n}{2}[1+\|u_n-u_{n+1}\|]^2-\frac{\varepsilon^2\rho_n\gamma_n^2}{2}\|u_n-u_{n+1}\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2 \\ &\geq \frac{-c^2\delta_n^2\varepsilon^2\rho_n}{2}[t+\|u_{\varepsilon_n}-u_{n+1}\|]^2-\frac{\varepsilon^2\rho_n\gamma_n^2}{2}\|u_n-u_{n+1}\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2 \\ &= \frac{-c^2\delta_n^2\varepsilon^2\rho_nt^2}{2}-\frac{c^2\delta_n^2\varepsilon^2\rho_n}{2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2+\frac{c^2\delta_n^2\varepsilon^2\rho_n}{2}[-\frac{1}{2}\|t\|^2-\frac{1}{2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2] \\ &\quad -\frac{\varepsilon^2\rho_n\gamma_n^2}{2}\|u_n-u_{n+1}\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2 \\ &= -c^2\delta_n^2\varepsilon^2\rho_nt^2-c^2\delta_n^2\varepsilon^2\rho_n\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2 \\ &\quad -\frac{\varepsilon^2\rho_n\gamma_n^2}{2}\|u_n-u_{n+1}\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2.\end{aligned}$$

Combining all the results aforementioned, we have

$$\begin{aligned}B(u_{\varepsilon_{n-1}},u_n)-B(u_{\varepsilon_n},u_{n+1}) &\geq \mu\|u_{n+1}-u_n\|^2+\mu\|u_{n+1}-u_n\|^2+\rho_n\alpha_n\left\|u_n-u_{\varepsilon_n}\right\|^2 \\ &\quad -\frac{\varepsilon_n\rho_n}{2}\left\|u_n-u_{\varepsilon_n}\right\|^2-\frac{\beta^2}{2\varepsilon_n\rho_n}\left\|u_{\varepsilon_{n-1}}-u_{\varepsilon_n}\right\|^2-c^2\delta_n^2\varepsilon^2\rho_nt^2 \\ &\quad -c^2\delta_n^2\varepsilon^2\rho_n\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2-\frac{\varepsilon^2\rho_n\gamma_n^2}{2}\|u_n-u_{n+1}\|^2-\frac{\rho_n}{2\varepsilon^2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2.\end{aligned}$$

With $\varepsilon^2=\dfrac{1}{\gamma_n\rho_n}$, we have

$$\begin{aligned}B(u_{\varepsilon_{n-1}},u_n)-B(u_{\varepsilon_n},u_{n+1}) &\geq \mu\|u_{n+1}-u_n\|^2+\mu\left\|u_{\varepsilon_{n-1}}-u_{\varepsilon_n}\right\|^2+\rho_n\alpha_n\left\|u_n-u_{\varepsilon_n}\right\|^2 \\ &\quad -\frac{\varepsilon_n\rho_n}{2}\left\|u_n-u_{\varepsilon_n}\right\|^2-\frac{\beta^2}{2\varepsilon_n\rho_n}\left\|u_{\varepsilon_{n-1}}-u_{\varepsilon_n}\right\|^2-\frac{c^2\delta_n^2t^2}{\gamma_n} \\ &\quad -\frac{c^2\delta_n^2}{\gamma_n}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2-\frac{\gamma_n}{2}\|u_n-u_{n+1}\|^2-\frac{{\rho_n}^2\gamma_n}{2}\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2 \\ &\geq (\mu-\frac{\gamma_n}{2})\|u_{n+1}-u_n\|^2+C_1\varepsilon_n\rho_n\left\|u_{\varepsilon_{n-1}}-u_{\varepsilon_n}\right\|^2+\rho_n\alpha_n\left\|u_n-u_{\varepsilon_n}\right\|^2 \\ &\quad -\frac{\varepsilon_n\rho_n}{2}\left\|u_n-u_{\varepsilon_n}\right\|^2-\frac{c^2\delta_n^2t^2}{\gamma_n}-C_2(\delta_n^2+{\rho_n}^2)\left\|u_{\varepsilon_n}-u_{n+1}\right\|^2.\end{aligned}$$

Using conditions (9), we have

$$B(u_{\varepsilon_{n-1}},u_n)-B(u_{\varepsilon_n},u_{n+1})\geq(\mu-\frac{\gamma_n}{2})\|u_{n+1}-u_n\|^2.$$

If $u_{n+1}=u_n$, it is easily shown that $u_n\in H$ is a solution of the mixed quasi variational inequality (1).
Otherwise, the assumption $\gamma_n>2\mu$ implies that $B(u_{\varepsilon_{n-1}},u_n)-B(u_{\varepsilon_n},u_{n+1})$ is non-negative and we must have $\lim\limits_{n\to\infty}\|u_{n+1}-u_n\|=0$. Using the similar technique used in the proof of Theorem 3.1, we can show that solution converges strongly.

Conclusion

In this paper, we have used the auxiliary principle technique coupled with regularization principle to suggest and analyze some explicit and proximal point algorithms for solving the mixed quasi variational inequalities. We have also discussed the convergence criteria of the proposed new iterative methods under some suitable weaker conditions. In this sense, our results can be viewed as refinement and improvement of the previously known results. Note that the auxiliary principle technique does not involve the projection and the resolvent of the involved operators. Results proved in this paper may inspire further research in this area.

FUTURE DIRECTIONS

We would like to mention that the problem considered in this paper can be studied from different point of views such as sensitivity analysis, dynamical and stability analysis. It is an interesting problem from both application point of view and numerical analysis to verify the implementation and efficiency of the proposed iterative methods for solving the mixed equilibrium variational inequalities. This is another direction of future research in the dynamic and fast growing field of mathematical and engineering sciences. The ideas and technique of the auxiliary principle technique may be extended for solving the mixed quasi variational-like inequalities and the equilibrium problems. We hope that this direction of research will yield some new and novel applications of these techniques.

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