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Full Length Research Paper

Exact solution to longitudinal and torsional oscillations of an electrically conducting second-grade fluid

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In this paper, an attempt has been made to solve the problem related to the motion of a second grade fluid for an oscillating rod. The fluid is electrically conducting in the presence of a uniform transverse magnetic field. The developed differential equation takes into account the effect of the material constants and the applied magnetic field. The modeled problem has been solved analytically and an exact solution has been obtained in terms of Bessel function.

Key words: Second grade fluid, longitudinal and torsional oscillations, exact solution, Bessel functions.

INTRODUCTION

Considerable attention has been paid in recent years to solve problems of non-Newtonian fluids due to their industrial and technological applications. Several models have been developed to describe the behavior of non-Newtonian fluids since the classical Navier-Stokes model fails to describe rheologically complex fluids such as blood, paints, shampoo and polymeric solutions. These fluids exhibit a nonlinear relationship between stresses and the rate of strain. Because of the diversity of non-Newtonian fluids, it is not possible to have single constitutive equation through which all the non-Newtonian fluids can be described; therefore many constitutive equations have been suggested by researchers. Some of them being empirical or semi-empirical.

The equation of motion governing non-Newtonian fluid flow is undoubtedly of higher order than Navier-stokes equation. Among the several models of non-Newtonian fluid model, there is subclass of viscoelastic fluid models namely second grade for which one can reasonably hope to obtain an analytic solution. Many researchers have discussed the flows of second grade fluid model for various situations. Fetecau and Fetecau (2005) has studied starting solutions for some unsteady unidirectional flows of a second grade fluid. Rajagopal (1981) discussed the unsteady unidirectional flows of second grade fluid. The flows are induced either due to the application of pressure gradient or through the motion of boundary. In another paper, Rajagopal (1984) examined the creeping flow of second grade fluid. Bandelli (1995) obtained some unsteady solutions in second grade fluids. In another article, Massoudi and Tran, (2009) has studied unsteady motion of non-linear viscoelastic fluid. In continuation, Massoudi and Tran, (2008) have discussed the motion of the fluid of a second grade fluid due to longitudinal and torsional oscillations of a cylinder.

Magnetohydrodynamics (MHD) fluid flows have vast importance and physical applications in petroleum industry, cooling systems with liquid metals industry, MHD generators and in the extrusion of crude oil etc. Hayat et al. (2001) has studied the motion of a MHD rotating flow of a third grade fluid between two eccentrically placed cylinders. More recently Nicholas et al. (2009) has studied MHD fluid flows spurred by problems encountered in aeronautics and astronautics. In another recent attempt Kipro (2006) has investigated the effects of MHD in the use of constant intensity in metallurgy to direct flow, to stir, and to levitate conducting melts. Whereas, Nouri et al. (2010) has studied the behaviour of MHD fluid flows for damp convection in

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Motivated by these facts in this work an attempt has been made to study the longitudinal and torsional oscillations of a rod under the influence of transverse magnetic field. An exact solution has been obtained for the velocity field by using the theory of Bessels's functions.

GOVERNING EQUATIONS

The basic governing equations for the steady MHD flow of an incompressible second grade fluid in the absence of body forces can be written as:

$$\operatorname{div} V = 0, \tag{1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \operatorname{div} \tau + J \times B, \tag{2}$$

Where V is the velocity vector, ρ the density of the fluid, $\frac{D}{Dt}$ denotes the material derivative, τ the Cauchy stress tensor, p the pressure, J the electric current density and B the total magnetic field. $B = B_\circ + b$, where $B_\circ$ represents the imposed magnetic field and b denotes the induced magnetic field. In the absence of displacement currents, the modified Ohm's law and Maxwell's equations (Hunt and Moreau, 1976; Chang and Yen, 1959; Rossow, 1958) are:

$$J = \sigma\{E + V \times B\}. \tag{3}$$

$$\operatorname{div} B = 0, \qquad \nabla \times B = \mu_m J, \qquad Curl E = -\frac{\partial B}{\partial t} \tag{4}$$

in which σ is the electrical conductivity, E the electric field and μ_m the magnetic permeability. The following assumptions are made in order to lead our discussion:

(i) The density ρ , magnetic permeability μ_m and electric field conductivity σ are constant throughout the flow field region,
(ii) The electrical conductivity σ of the fluid is finite,
(iii) Total magnetic field B is perpendicular to the velocity field V and the induced magnetic field b is negligible compared with the applied magnetic field $B_\circ$ so that the magnetic Reynolds number is small (Hunt and Moreau, 1976; Rossow, 1958).
(iv) There is no energy added or extracted from the fluid by the electric field, which implies that there is no electric field present in the fluid flow region.

Under these assumptions, the magnetohydrodynamic force involved in Equation (2) can be put into the form,:

$$J \times B = -\sigma B_\circ^2 V. \tag{5}$$

The stress tensor τ defining a second grade fluid is given by $\tau = \sum_{i=1}^{2} S_i$, where:

$$S_1 = \mu A_1, S_2 = \alpha_1 A_2 + \alpha_2 A_1^2 \tag{6}$$

and where μ is the coefficient of viscosity and α_1 and α_2, are material constants. The Rivlin-Erickson tensor, A_n , are defined by:

$$A_0 = I, \text{the identity tensor}, \tag{7}$$

and

$$A_n = \frac{DA_{n-1}}{Dt} + A_{n-1}(\nabla V) + (\nabla V)^t A_{n-1}, \qquad n \geq 1. \tag{8}$$

We shall assume a velocity field of the form:

$$V(r,\theta,z,t) = v(r,t)e_\theta + w(r,t)e_z \tag{9}$$

Using Equation 9, Equation 1 is identically satisfied and using Equations 5 to 9, we get:

$$-\frac{\sigma B_0^2 v(r)}{\mu} + \mu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + \alpha_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) - \rho\frac{\partial v}{\partial t} = 0 \tag{10}$$

$$\mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \alpha_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \rho\frac{\partial w}{\partial t} = 0 \tag{11}$$

PROBLEM STATEMENT

Let us consider an incompressible second grade fluid at rest, in a long cylindrical rod of radius r_0. At time $t = 0^+$ the rod starts to oscillate according to Rajagopal and Bhatnagar (1995):

$$(V_1 \cos\omega_1 t)e_\theta + (V_2 \cos\omega_2 t)e_z$$

Where ω_1 and ω_2 are the frequencies of the velocity of the cylindrical rod, e_θ and e_z are the unit vectors corresponding to θ and z directions respectively and V_1 and V_2 are constant amplitudes. The governing equations are given by Equations 10 and 11, while the associated initial and boundary conditions are also given in the study.

Boundary conditions

Since we have assumed that the solid cylindrical rod of radius r_0 is oscillating in a manner that the velocity of the surface of rod is given by $(V_1 \cos\omega_1 t)e_\theta + (V_2 \cos\omega_2 t)e_z$. The conditions that fluid adheres to the surface of the rod then implies that:

$$\mathbf{V} = (V_1 \cos\omega_1 t)e_\theta + (V_2 \cos\omega_2 t)e_z \qquad (12)$$

or

$$v = V_1 \cos\omega_1 t, \qquad w = V_2 \cos\omega_2 t \qquad (13)$$

We shall also require that the fluid is quiescent at infinity,that is:

$$v, w \to 0 \qquad as \qquad r \to \infty \qquad (14)$$

SOLUTIONS OF THE PROBLEM

We shall now proceed to obtain the exact solution to the Equations 10 and 11 subject to the boundary conditions in Equations 13 and 14.

For Equation 10, we assume a separable solution of the form:

$$v(r,t) = V(r)T(t) \qquad (15)$$

Substituting Equation 15 into Equation 10, we obtain:

$$T' - \lambda T = 0 \qquad (16)$$

having the solution:

$$T(t) = C_1 e^{\lambda t} \qquad (17)$$

Where C_1 is constant of integration.

It may be mentioned here that λ is an arbitrary constant, known as separation constant. We have three different cases corresponding to the nature of λ, that is, $\lambda > 0, \lambda < 0$ or $\lambda < 0$ In our case, we had trivial solution for both $\lambda < 0$ or $\lambda < 0$. Therefore we have chosen $\lambda > 0$.

$$V'' + \frac{1}{r}V' - \left(\frac{1}{r^2} + \frac{(\rho\lambda + \mu\sigma B_0^2)}{(\mu + \alpha_1\lambda)}\right)V = 0 \qquad (18)$$

On defining $s = \left(\frac{(\sigma B_0^2 + \mu\rho\lambda)}{\mu(\mu + \lambda\alpha_1)}\right)^{\frac{1}{2}} r$

Equation 18 after some manipulation can be re-written as:

$$s^2 \frac{d^2V}{ds^2} + s\frac{dV}{ds} - (1 + s^2)V = 0 \qquad (19)$$

Equation 19 can be easily identified as a Bessel equation whose general solution is given by:

$$V = k_1 J_\nu(s) + k_2 Y_\nu(s) \qquad (20)$$

substituting the value of s in Equation 20, we get:

$$V = k_1 J_\nu\left(\left(\frac{(\sigma B_0^2 + \mu\rho\lambda)}{\mu(\mu + \lambda\alpha_1)}\right)^{\frac{1}{2}} r\right) + k_2 Y_\nu\left(\left(\frac{(\sigma B_0^2 + \mu\rho\lambda)}{\mu(\mu + \lambda\alpha_1)}\right)^{\frac{1}{2}} r\right) \qquad (21)$$

But we need real part of the solution, therefore we define:

$$I_\nu(s) = i^{-\nu} J_\nu(\imath s) \qquad (22)$$

Putting $\nu = 1$ in Equation 22, we get:

$$I_1(s) = i^{-1} J_1(\imath s) \qquad (23)$$

V now becomes:

$$V = i^{-1} J_1\left(\imath\left(\frac{(\sigma B_0^2 + \mu\rho\lambda)}{\mu(\mu + \lambda\alpha_1)}\right)^{\frac{1}{2}} r\right) \qquad (24)$$

Using results in Equatons 17 and 24 in Equation 15 and applying boundary condition in Equation 13 and 14, we get:

$$V(r,t) = Re\left\{\frac{K_1\left(\left(\frac{\omega_1\imath(\sigma B_0^2 + \imath\mu\rho\lambda\omega_1)}{\mu(\mu + \imath\lambda\alpha_1\omega_1)}\right)^{\frac{1}{2}} r\right)}{K_1\left(\left(\frac{\omega_1\imath(\sigma B_0^2 + \imath\mu\rho\lambda\omega_1)}{\mu(\mu + \imath\lambda\alpha_1\omega_1)}\right)^{\frac{1}{2}} r_0\right)}\right\} C_1 e^{\imath\omega_1 t} \qquad (25)$$

Where Re denotes real part of the solution, $\imath = \sqrt{-1}$, K_1

is a modified Bessel function and ω_1 is the frequency of the velocity of the cylindrical rod.

We next try to solve the equation governing longitudinal oscillations, given by:

$$\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+\alpha_1\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)-\rho\frac{\partial w}{\partial t}=0 \qquad (26)$$

Again by seeking solution $w(r,t)$ of the separable form:

$$w(r,t)=W(r)T(t) \qquad (27)$$

so that Equation 26 gives:

$$T'-\lambda T=0 \qquad (28)$$

having the solution:

$$T(t)=C_2e^{\lambda t} \qquad (29)$$

C_2 being the constant of integration, and

$$W''+\frac{1}{r}W'-\left(\frac{\rho\lambda}{\mu+\alpha_1\lambda}\right)W=0 \qquad (30)$$

Introducing $s=\left(\frac{\rho\lambda}{\mu+\alpha_1\lambda}\right)^{\frac{1}{2}}r$

After some manipulation, Equation 30 can be rewritten as:

$$s^2\frac{d^2W}{ds^2}+s\frac{dW}{ds}-s^2W=0 \qquad (31)$$

which is again a Bessel equation, whose general solution in terms of Bessel functions is given by:

$$W=k_1J_\nu(s)+k_2Y_\nu(s) \qquad (32)$$

Putting the value of s in equation 32, we get:

$$W=k_1J_\nu\left(\left(\frac{\rho\lambda}{\mu+\alpha_1\lambda}\right)^{\frac{1}{2}}r\right)+k_2Y_\nu\left(\left(\frac{\rho\lambda}{\mu+\alpha_1\lambda}\right)^{\frac{1}{2}}r\right)$$

But as we are interested in real value of the solution, therefore we define:

$$I_\nu(s)=i^{-\nu}J_\nu(\imath s) \qquad (33)$$

Putting $\nu=0$ in Equation 33 we have Bessel function of order zero:

$$I_0(s)=i^{-0}J_0(\imath s)$$

or

$$I_0(s)=K_1J_0\left(\left(\frac{\rho\lambda}{\mu+\alpha_1\lambda}\right)^{\frac{1}{2}}r\right) \qquad (34)$$

Using results in Equations 29 and 34 in Equation 33 and applying boundary condition in Equations 13 and 14, we get:

$$W(r,t)=Re\left\{\frac{K_0\left(\left(\frac{\imath\rho\omega_2}{\mu+\imath\alpha_1\omega_2}\right)^{\frac{1}{2}}r\right)}{K_0\left(\left(\frac{\imath\rho V_2}{\mu+\imath\alpha_1\omega_2}\right)^{\frac{1}{2}}r_0\right)}\right\}C_2e^{\imath\omega_2 t} \qquad (35)$$

Where K_0 is a modified Bessel function.

CONCLUSION

In this paper, we have obtained the solution to the problem related to the motion of a second grade fluid for an oscillating rod. The fluid is electrically conducting in the presence of a uniform transverse magnetic field. The developed differential equation takes into account the effect of the material constants and the applied magnetic field. The modeled problem has been solved analytically and an exact solution has been obtained in terms of Bessel function. Remark: In Equation 25, if $\sigma B_0\rightarrow 0$, we recover the solutions obtained by Rajagopal and Bhatnagar (1995) for torsional oscillations.

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