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Short Communication

Analytical study of electromagnetic wave scattering behaviour using Lippmann-Schwinger equation

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An analytical study of electromagnetic wave scattering using Lippmann-Schwinger equation is presented in this work. Lippomann-Schwinger equation was derived first from hamiltonian that involves non-interacting and interacting terms. The solution for non-interacting and out going scattered waves was obtained using a boundary conditions specified on the green's function $[E_n - H_0]^{-1}$ which specifically handles the singularity that facilitates expressing Lippman-Schwinger equation correctly. The scattering amplitude of the wave was obtained analytically.

Key words: Lippmann-Schwinger equation, eigenstates, hamiltonian, green's function.

INTRODUCTION

Scattering of electromagnetic wave when it encounters an obstacle has been studied by many researchers. When an EM wave encounters an obstacle, obviously, bound charges are set unto oscillation and secondary wave are scattered in all directions (Barron, 1982). Within a medium, the obstacle responsible for light scattering can be as a result of impurities in crystals or medium, water droplet or dust particles in the atmosphere and colloidal matter suspended in liquids.

Light scattering also occurs in transparent material due to the in homogeneities at the molecular level. In this case, scattering occurs without a change in frequency as observed in Rayleigh scattering (Rayleigh, 1900). Scattering in which there is a frequency shift as observed by Raman and Krishnan is known as Raman scattering (Raman et al., 1928).

From the time of Rayleigh and Raman till date, many scholars had delved into the study of scattering using different tools and the application. Green's function has been one of the tools that has been strongly used to study electromagnetic wave scattering (Morse et al., 1953). Other work like formulations based on the principles of scattering superposition have been constructed coupled with scattering dyadic green's function (Le-Wei et al., 1004). The idea of EM wave scattering combined with acoustic excitation for detecting buried objects has been carried out (Stewart, 1960; Scott et al., 1998 and Scott et al., 1999). Others had developed scattering models for specific landmines and had demonstrated the phenomena associated with the acoustic-electromagnetic objects (Daniel et al., 2001; Sarabands et al., 1997).

The purpose of this paper is not however to discuss the applications of EM wave scattering nor to develop any scattering model for specific application, but rather to analytically look at electromagnetic wave scattering using Lippmann-Schwinger equation. In the analysis, consideration was given to the first exponential, the contour closed with a large semicircle in the upper half plane and also for second case when the semicircle is closed in the lower half plane. The singularities that contribute to the poles which enhance the target energy in the out going state and the scattering amplitude were analyzed.

THEORETICAL PROCEDURE

Lipmann-Schwinger equation was derived starting with hamiltonian $H_0 + V$ where H_0 is non-interacting Hamiltonian while V is the interacting term. The energy eigenstates of H_0 are

$$H_0 \left|\Phi_n\right\rangle = E_n \left|\Phi_n\right\rangle \tag{1}$$

and are normalized so that a completeness relation for the identity operator can be written as

$$\int s_n \left|\Phi_n\right\rangle \left\langle\Phi_n\right| = 1 \tag{2}$$

Where n is the quantum number with the integral including sum over any bound state.

Rearranging the Schrödinger equation, the eigenstates of H_0+V are solutions of

$$\left(E_n-H_0\right)\Phi_n\rangle = V\left|\Phi_n\right\rangle \quad (3)$$

The solution for a non-interacting term with the addition of out going scattered waves can be written as

$$\left|\Phi_n\right\rangle = \left|\Phi_n\right\rangle + \frac{1}{E_n-H_0}\Big|_{b.c}V\left|\Phi_n\right\rangle \quad (4)$$

Where $b.c$ is the boundary conditions specified on the green's functions $\left[E_n-H_0\right]^{-1}$. As energy is conserved, non-interacting components of the solutions far from the scattering center will have energy E_n. Therefore specifying boundary conditions means specifying how to handle the singularity when the eigenvalue of H_0 equals E_n. The most common choices are to remove the singularities by writing the Lippmann-Schwinger equations as

$$\left|\varphi^{\pm}{}_n\right\rangle = \left|\Phi_n\right\rangle + \frac{1}{E_n-H_0+\eta}V\left|\varphi^{\pm}{}_n\right\rangle \quad (5)$$

When η is a positively infinitesimal and $\left|\varphi^{\pm}\right\rangle$ is in state while $\left|\Phi^{-}\right\rangle$ is an outgoing state.

Amplitude of the scattered wave

If we insert the completeness state, one writes

$$\left|\varphi^{\pm}\right\rangle = \left|\Phi_n\right\rangle + d\beta\frac{1}{E_n-H_0+i_\eta}\left|\Phi_\beta\right\rangle\left\langle\ \Phi_\beta\right|\ \left|V\varphi^{\pm}{}_{n^{\pm}}\right. \quad (6)$$

Where H_0 operates on $\left|\Phi_\beta\right\rangle$ to give E_β. Using the definition of the T matrix for the matrix element, we obtain

$$\left|\varphi^{\pm}\right\rangle = \left|\Phi_n\right\rangle + \int d\beta\frac{\left|\Phi_\beta\right\rangle T^{\pm}{}_{\beta_n}}{E_n-E_\beta\pm i\eta} \quad (7)$$

When we multiply equation 7 on the left by $\left\langle\Phi_\beta\right|V$ and put dummy state index as γ, then we have

$$\Phi_\beta\left|V\right|\varphi^{\pm}\rangle = \left\langle\Phi_\beta\left|V\right|\Phi_n\right\rangle + \int d\gamma\frac{\left\langle\Phi_\beta\left|V\right|\Phi_\gamma\right.}{E_n-E_\beta+i\eta} \quad (8)$$

And writing V matrix element as $V_{\beta n}$ etc, we have

$$T^{\pm}{}_{\beta n} = V_{\beta n} + \int d\gamma\frac{V_{\beta\gamma}T\gamma^{\pm}n}{E_n-E_\gamma\pm i\eta} \quad (9)$$

Using the Lippmann-Schwinger equation, the wave packets can be written as

$$\left|\varphi_g{}^{\pm}(t)\right\rangle = \int dng(n)e^{-i\frac{1}{\hbar}E_n t}\left[\left|\Phi_n\right\rangle + \int d\beta\frac{\left|\Phi_\beta\right\rangle T^{\pm}{}_{\beta n}}{E_n-E_\beta\pm i\eta}\right. \quad (10)$$

$$= \int d\beta\int dng(n)e^{-i\frac{1}{\hbar}E_n t}\ \frac{\left|\Phi_\beta\right\rangle T^{\pm}{}_\beta}{E_n-E_\beta\pm i\eta} \quad (11)$$

The matrix element with $\left\langle r\right|$ when inserted to a complete set of r and Φ_k state, it gives

$$\left\langle r|\psi_k\right\rangle = \left\langle r\right|\Phi_k + \int d^3r'\int\frac{d^3k'}{(2\pi)^3}\left\langle r\right|\frac{1}{E_k-H_0+i_\eta}\left|\psi_k\right\rangle\left\langle\Phi_k\left|V\right|r^1\right\rangle\left\langle r'|\psi_k\right\rangle \quad (12)$$

Where H_0 operates on $\left|\Phi_{k^1}\right\rangle$ to give $E_{k'}$ and operates on $\left|r'\right\rangle$ to give $V\left(r^1\right)$

Thus in terms of wave function we obtain

$$\Phi_k\left(r\right) = e^{ik.r} + \int d^3r'\int\frac{d^3k'}{(2\pi)^3}\frac{e^{-ik\left(r-r'\right)}}{E_k-E_{k'}+i\eta}V\left(r'\right)\psi_k\left(r'\right) \quad (13)$$

The green's function associated with the formalism is

$$G\left(r\right) = \int\frac{d^3k'}{(2\pi)^3}\frac{e^{-ik.r}}{E_k-E_{k'}+i\eta}$$

$$= \frac{1}{2\pi^2}\int_0^\infty dk'k'^2\frac{\sin\left(k'r\right)}{k'r}\frac{1}{E_k-E_{K'}+i\eta}$$

$$= -\frac{m}{\pi^2\hbar^2 r}\int_0^\infty dk'\ \frac{k'\sin\left(k'r\right)}{k'^2-\left(k+i\eta\right)^2}$$

$$= -\frac{m}{2\pi^2\hbar^2 r}\int_{-\infty}^\infty dk'\ \frac{k'\sin\left(k'r\right)}{k'^2-\left(k+i\eta\right)^2}$$

$$= -\frac{m}{8i\pi^2\hbar^2 r}\int_{-\infty}^\infty dk'\left[e^{ik'.r}-e^{-ik'.r}\right]\left[\frac{1}{k'-\left(k+i\eta\right)}+\frac{1}{k'+\left(k+i\eta\right)}\right] \quad (14)$$

Since r is positive, we can close the contour for the first exponential with a large semicircle in the upper half plane, and only the pole $k+i\eta$ contributes

Similarly, for the second exponential, we close the lower half plane and only the pole at $-k-i\eta$ contributes. Thus the result becomes

$$G(r) = -\frac{m}{2\pi h^2 r} e^{ikr}$$

When η is taken to be zero, putting this into equation 14 results to

$$\psi_k(r) = e^{ik.r} - \frac{m}{2\pi h^2} \int d^3 k' \frac{e^{ik|r-r'|}}{|r-r'|} v(r') \psi_k(r')$$

Where the scattering amplitude is

$$f = -\frac{m}{2\pi h^2} \int d^3 k' e^{-ik.r} V(r') \psi_k(r')$$

$$= -\frac{1}{4\pi} \frac{2m}{h^2} T_{k'k}$$

DISCUSSION OF RESULT

Analytical study of the scattering behaviour of electromagnetic wave using Lippmann-Schwinger equation for a specific hamiltonian starting with a Hamiltonian $H_0 + V$ involving non-interacting and interacting eigenstates.

From the results of equation 11, we can view the η integral as an integral along the real axis of the energy where for any degenerate states, the integrand contains an additional integration over the degenerate states and for energies without states the integrand is multiplied by zero. For $t \to +\infty$, we close the contour with large semicircle in the lower half plane, while for $t \to -\infty$, we close the upper half plane. As a result, the T matrix has with it the imaginary part that gives exponentially damped contributions for large magnitude, t. The only singularity that contribute is therefore the simple poles at $E_n = E_\beta$ which give $2\pi i$ times the residue for the contour close in the upper half plane and $-2\pi i$ for the contour close in the lower half plane. The $\pm i\eta$ factors displace the poles slightly so that they only contribute for positive time for Φ^+ and for negative time for Φ^- with the result given as

$$|\Phi^t{}_g(t)\rangle \to |\varphi_g(t)\rangle,$$

for $t \to \pm\infty$,

$$|\Phi_g^\pm(t) \to \int d\beta \phi_\beta e^{\frac{1}{h}E_g(t)} [g(\beta) \pm 2\pi] \int dn \delta(E_n - E_g) g(n) T_{\beta n}^\pm$$

With these considerations, if the result as obtained in equation 15 is compared with the expressions for wave packets formed from the non-interacting states and the interacting states with same amplitudes, the scattering amplitude is f which is strongly dependent on target energy in the outgoing state from equation 15, k^1 is the wave vector in the r direction that satisfies energy conservation. The plane wave together with the target state from the energy conservation state $|\Phi_B\rangle$, and these doted with potential just give the $T.$ matrix of the scattering amplitude as

$$T_{mk',nk} = -\frac{\rho^1}{2\pi h^2 V^1} Tmk', nkk$$

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