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Full Length Research Paper

On conformally Osserman Lorentzian manifolds satisfying a certain condition on the Ricci tensor

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Let (M_1^n, g) , $n \geq 4$, be an *n*-dimensional homogeneous Lorentzian manifold of which the Jacobi operator associated to the Weyl conformal curvature tensor has constant eigenvalues on the bundle of unit timelike (spacelike) tangent vectors (known as conformally Osserman Lorentzian manifolds). Then M_1^n is a conformally Osserman Lorentzian manifold if and only if M_1^n is a conformally flat manifold, (Blazic, 2005). In this paper, by utilizing this equivalence and the similar arguments in Erdogan and Ikawa (1995) and Sekigawa and Takagi (1971), we classify locally conformally flat homogeneous Lorentzian manifolds and, equivalently, as well as conformally Osserman Lorentzian manifolds which satisfy a condition on the Ricci tensor.

Key words: Conformally manifold, Weyl conformal tensor, Conformally Osserman manifold, Conformal Jacobi operator.

INTRODUCTION

Let (M^n, g) be an *n*-dimensional semi-Riemannain manifold and $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ its curvature operator and $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ its curvature tensor. Let $\{e_i\}$ be a local orthonormal frame for the tangent bundle, $g(e_i, e_i) = \varepsilon_i$, $\varepsilon_i = \pm 1$. The Ricci curvature and the scalar curvature are defined by $Ric(X,Y) = \sum_i \varepsilon_i R(X, e_i, e_i, Y)$ and $\tau = \sum_j \varepsilon_j Ric(e_j, e_j)$.

The Weyl conformal curvature tensor is defined by

$$\begin{aligned} W(X,Y) &= R(X,Y) \\ &+ \frac{\tau}{(n-1)(n-2)}\{g(Y,.)X - g(X,.)Y\} \\ &+ \frac{1}{n-2}\left\{\begin{matrix} g(QY,.)X - g(QX,.)Y \\ + g(Y,.)QX - g(X,.)QY \end{matrix}\right\}, \quad n > 2, \end{aligned} \tag{1}$$

where *Q* is the symmetric endomorphism which corresponds to the Ricci tensor *Ric* that is $Ric(X,Y) = g(QX,Y)$.

The Jacobi operator and the Weyl (conformal) Jacobi operator are defined by $J_{R_g}(X) := Y \to R_g(Y,X)X$ and $J_{W_g}(X) := Y \to W_g(Y,X)X$. The study of the Jacobi operator is a central topic in Semi-Riemannian geometry. The geodesic deviation is measured by this part of the curvature tensor and because of its fundamental role in the Jacobi equation, many geometric properties can be derived from the knowledge of the Jacobi operators

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(Blazic et al., 2003; Blazic, 2005; Chi, 1988; Erdogan and Ikawa, 1995; Garcia-Rio et al., 1997; Garcia et al., 2002). Since for each vector *X*, the Jacobi operator is a self-adjoint operator, the study of its eigenvalues is of great interest. İn the Lorentzian case especially, they play a fundamental role in the construction of mathematical models in general relativity. On the other hand, the eigenvalues of the Jacobi operator depend both on a point $p\in M^n$ and a direction $X\in T_pM^n$. The conformally Jacobi operator is conformally invariant, namely, if $h=e^{\alpha}g$ is a conformally equivalent to the metric *g*, then $J_{W_h}=J_{W_g}$. The Weyl curvature tensor *W* as a conformal invariant is important in the understanding of Conformal Semi-Riemannian Geometry. We say (M^n,g) is *conformally Osserman* if the eigenvalues of the symmetric Weyl Jacobi operator $J_{W_g}(X)$ are independent on the choice of the unit timelike (spacelike) direction *X*.

It is well known that an *n*-dimensional semi-Riemannian manifold, $n\geq 4$, is conformally flat if and only if its Weyl operator *W* vanishes. It was proved that if M^n is conformally Osserman Lorentzian manifold then it is conformally flat (Blazic, 2005). In this case, $W\equiv 0$ implies that the curvature operator R_g of M^n satisfies

$$R(X,Y)=\frac{1}{n-2}\left\{\begin{array}{l}g(QX,.)Y-g(QY,.)X\\+g(X,.)QY-g(Y,.)QX\end{array}\right\}-\frac{\tau}{(n-1)(n-2)}\{g(Y,.)X-g(X,.)Y\} \tag{2}$$

for any tangent vector fields *X* and *Y.* If $X\wedge Y$ denotes the endomorphism defined by $(X\wedge Y)Z=g(Y,Z)X-g(X,Z)Y$, then we have

$$R(X,Y)=\frac{1}{n-2}(QX\wedge Y+X\wedge QY)-\frac{TrQ}{(n-1)(n-2)}X\wedge Y. \tag{3}$$

In a recent paper, Honda and Tsukada (2007) classified three-dimensional conformally flat homogeneous Lorentzian manifolds depending on the form of the Ricci operators. In their proof, they use the same method as in Hahn (1984) which is relating the principal curvatures of an isoparametric hypersurface with the eigenvalues of the shape operator *A.* In the present work, assuming that *Q* is diagonalizable, we aim to prove a local classification theorem for *n*-dimensional conformally flat Lorentzian manifolds and equivalently also for the conformally Osserman Lorentzian manifolds which satisfy the condition;

$$R(X,Y).Q=0 \tag{4}$$

for any tangent vectors *X* and *Y*.

THEOREM

We prove the following:

Let M_1^n be an n-dimensional ($n\geq 4$) connected, complete conformally Osserman Lorentzian manifold satisfying the condition (Equation 4). Then M_1^n is one of the following:

1) An Osserman Lorentzian manifold of constant curvature.
2) A locally product manifold of an Osserman Lorentzian manifold of constant curvature k and a Riemannian Osserman manifold of constant curvature $-k$, or a Riemannian Osserman manifold of constant curvature k and a Lorentzian Osserman manifold of constant curvature $-k$; that is, $M_1^m(k)\times M^{n-m}(-k)$ or $M^m(k)\times M_1^{n-m}(-k)$, where $k\neq 0$.
3) A locally product space of a Lorentzian Osserman manifold of constant curvature k and a 1-dimensional Riemannian Osserman manifold, or a Riemannian Osserman manifold of constant curvature k and a 1-dimensional Lorentzian Osserman manifold; that is, $M_1^{n-1}(k)\times M^1$ *or* $M^{n-1}(k)\times M_1^1$*, where* $k\neq 0$*.*

Now (Equation 3) and the condition (Equation 4) imply

$$\begin{aligned}&\frac{1}{n-2}(g(Q^2Y,Z)X-g(Q^2X,Z)Y\\&\quad+g(Y,QZ)QX-g(X,QZ)QY)\\&\quad-\frac{TrQ}{(n-1)(n-2)}(g(Y,QZ)X-g(X,QZ)Y)\\&\quad-\frac{1}{n-2}(g(QY,Z)QX-g(QX,Z)QY\\&\quad+g(Y,Z)Q^2X-g(X,Z)Q^2Y)\\&\quad+\frac{TrQ}{(n-1)(n-2)}(g(Y,Z)QX-g(X,Z)QY)\\&\quad=0\end{aligned} \tag{5}$$

Since *g* is the Lorentzian metric and *Q* is a symmetric endomorphism of the tangent space $T_pM_1^n$ at point *p*, with respect to suitably chosen frames for $T_pM_1^n$, Q_p has one of the following four forms (O'Neill, 1983):

(i) $$Q_p=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix},$$

(ii) $$Q_p = \begin{pmatrix} \lambda & \mu & 0 & \dots & 0 \\ -\mu & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix}, (\mu \neq 0),$$

(iii) $$Q_p = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 \\ 1 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix},$$

(iv) $$Q_p = \begin{pmatrix} \lambda & 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ -1 & 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_4 & \dots & 0 \\ . & . & . & . & \ddots & . \\ 0 & 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

In cases (i) and(ii), Q_p is represented with respect to an orthonormal frame $\{e_1, e_2, ..., e_n\}$; that is, they satisfy $g(e_1, e_1) = -1$, $g(e_i, e_j) = \delta_{ij}$, $g(e_1, e_j) = 0$, $(2 \leq i, j \leq n)$. In cases (iii) and(iv), Q_p is represented with respect to a semi-orthonormal frame $\{u_1, u_2, ..., u_n\}$; that is, they satisfy $g(u_1, u_1) = g(u_2, u_2) = g(u_1, u_i) = g(u_2, u_i) = 0$, $g(u_1, u_2) = 1$, $g(u_i, u_j) = \delta_{ij}$, $(3 \leq i, j \leq n)$.

PRELIMINARIES

Let (M^n, g) be a Semi-Riemannian manifold of dimension $n \geq 3$ and let g be a non-degenerate symmetric metric on M^n of signature (p, q). For $\varepsilon = \pm$ and $p \in M^n$, let $S_p^\varepsilon := \{X \in T_p M^n : g(X, X) = \varepsilon 1\}$ be the set of all unit spacelike ($\varepsilon = +$) and timelike ($\varepsilon = -$) tangent vectors at $p \in M^n$. If $X \in S_p^\varepsilon$, $T_X(S_p^\varepsilon) = \{Y \in T_p M : g(X, Y) = 0\}$. If X is not a null vector, and $X \in T_p M^n$, $p \in M^n$, then one can split a tangent space $T_p M^n = X \oplus X^\perp$. The endomorphism Q_X of $T_X(S_p^\varepsilon)$ is diagonalizable if $X^\perp$ has the definite induced metric, in other case may be also undiagonalizable. Therefore, we only deal with the Lorentzian case, namely Q is always diagonalizable. In this case, Q has at most two real eigenvalues (Honda, 2003). It is known that any local Lorentzian Osserman manifold has constant sectional curvature (Blazic, 1997). One says that M^n is a conformal space form if $W = \lambda_0 R_0$. We easily see that this implies $\lambda_0 = 0$ so M^n is conformally flat, where R_0 is defined by $R_0(X, Y)Z := g(Y, Z)X - g(X, Z)Y$. Then, since $J_{W_g}(X)X = 0$ and $J_{W_g}(X)$ is self-adjoint, J_{W_g} preserves the subspace $X^\perp$. So we may define the reduced conformal Jacobi operator by letting It is then immediate that J_{W_g} has constant eigenvalues on S_p^ε if and only if J'_{W_g} has constant eigenvalues on S_p^ε.

Now we express a theorem and its proof will follow from the purely algebraic lemma which has been given in Blazic, Bokan and Gilkey (1997):

Theorem 2

Let V be a Lorentzian vector space of the signature $(1, n-1)$ and let R be an algebraic curvature tensor and W the corresponding Weyl operator. The following conditions are equivalent:

1) R is a conformally Osserman curvature tensor,
2) $W = \lambda R_0$, $\lambda = const.$,
3) $W = 0$.

If a connected manifold M_1^n is a locally spacelike Osserman or a locally timelike Osserman manifold, then M_1^n has constant curvature c for some c independent of the point of the manifold. So if $n \geq 4$, then Theorem 2 implies that the following conditions are equivalent:

(1) M_1^n is a conformally Osserman manifold,
(2) M_1^n is a conformally flat manifold.

Corollary 2

A Lorentzian manifold M_1^n is timelike conformally Osserman if and only if it is spacelike conformally Osserman.

PROOF OF THEOREM

First we assume that $\dim M_1^n > 3$. The proof of the theorem will be devided into four parts, according to the four possible forms of Q.

Case (i)

Suppose that Q_p is of the Form (i). We know that Q has at most two real eigenvalues (Honda, 2003). Now by putting $X = e_1, Y = Z = e_j$ in Equation (5), at each point, it follows that

$$(\lambda_j - \lambda_1)\left(\lambda_j + \lambda_1 - \frac{TrQ}{n-1}\right) = 0 \qquad (6)$$

for any j, $2 \le j \le n$. If for all $2 \le j \le n$, $\lambda_j = \lambda_1$, then Q is reduced to $Q = \lambda I$, where I is the identity transformation. Hence M_1^n is Einstein, and from Equation (3), it follows that M_1^n is a space of constant curvature. If $\lambda_1 = \lambda_2 = ... = \lambda_{n-m} =: \lambda$ and $\lambda_{n-m+1} = ... = \lambda_n =: \mu$, then Equation (6) implies

$$\lambda + \mu - \frac{TrQ}{n-1} = 0$$

which gives $(n-m-1)\mu = (1-m)\lambda$. (7)

Now if $\lambda = 0$ then Equation (7) implies that $\mu = 0$ or Q_p has the following form:

$$Q_p = \begin{pmatrix} 0 & & & & \\ & \mu & & & \\ & & \mu & & \\ & & & \ddots & \\ & & & & \mu \end{pmatrix}. \qquad (8)$$

If $\lambda \ne 0$ and $m = 1$ then Equation (7) implies that $\mu = 0$ and so Q is reduced to

$$Q_p = \begin{pmatrix} \lambda & & & & & \\ & \lambda & & & & \\ & & \lambda & & & \\ & & & \ddots & & \\ & & & & \lambda & \\ & & & & & 0 \end{pmatrix}. \qquad (9)$$

Otherwise, if $m \ne 1$ and $\lambda\mu < 0$ then Equation (7) implies that

$$Q_p = \begin{pmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \mu & & \\ & & & & \ddots & \\ & & & & & \mu \end{pmatrix}, \qquad (10)$$

where λ and μ have the algebraic multiplicities $n-m$ and m, respectively.

First let us consider the case (Equation 10). We assume that $\lambda > 0$ and set $U = \{x \in M_1^n : Q_x$ has the form (Equation 10), then, by the continuity argument for the characteristic polynomial of Q, U is an open subset of M_1^n. By U_0, we denote a connected component of U. Then m is constant on U_0 and $\lambda(x)$ and $\mu(x)$ are differentiable functions on U_0. Let us define two distributions on U_0 by $T_1(x) = \{X \in S_x^\varepsilon : QX = \lambda(x)X\}$ and $T_2(x) = \{X \in S_x^\varepsilon : QX = \mu(x)X\}$. Then $T_1(x)$ and $T_2(x)$ are differentiable and the restrictions of the metric on $T_x M_1^n$ to $T_1(x)$ and $T_2(x)$ are nondegenerate. Since M_1^n is conformally flat and $X, Y \in T_1(x)$ satisfies $QX = \lambda(x)X$ and $QY = \lambda(x)Y$, it follows that $R(X,Y) = kX \wedge Y,\ X, Y \in T_1(x)$, $k = \dfrac{\lambda - \mu}{n-2}$. Similarly, for $X, Y \in T_2(x)$, we have $R(X,Y) = -kX \wedge Y$. By the second Bianchi identity, we can see that k is a constant. Therefore, M_1^n is locally a product space of an *m*-dimensional Osserman Lorentzian space of constant curvature k and a $(n-m)$-dimensional Osserman Riemannian space of constant curvature $-k$, or an *m*-dimensional Osserman Riemannian space of constant curvature k and an $(n-m)$-dimensional Osserman Lorentzian space of constant curvature $-k$; that is, $M_1^m(k) \times M^{n-m}(-k)$ or $M^m(k) \times M_1^{n-m}(-k)$, where $1 < m < n-1$ and $k = \dfrac{\lambda - \mu}{n-2}$.

Next assume that the rank of Q is $n-1$ at some point p. Let $U = \{x \in M_1^n :\ Q_x \text{ has the rank } n-1\}$. Then U is open and non-zero eigenvalue of Q, say λ, is a differentiable function on U. Two distributions T_0 and T_1 on U are defined by $T_0(x) = \{X \in S_x^\varepsilon : QX = 0\},\ T_1(x) = \{X \in S_x^\varepsilon : QX = \lambda(x)X\}$. Then, it follows that they are differentiable, T_1 is involutive and geodesics whose tangents belong to T_0 and are infinitely extendible. Moreover, T_0 and T_1 are parallel distributions, (Sekigawa and Takagi, 1971). The restriction of the metric on $T_x M_1^n$ to $T_0(x)$ and $T_1(x)$ are non-degenerate. Hence T_1 (resp. T_0) has maximal integral submanifold M^{n-1} (resp. M^1) of M_1^n. Since M_1^n is conformally flat and $X \in T_1(x)$ satisfies $QX = \lambda(x)X$, M^{n-1} has constant curvature $k = \dfrac{\lambda}{n-2}$ by virtue of the second Bianchi identity. Therefore, M_1^n is locally a product space of (n-1)-dimensional Osserman Lorentzian space M_1^{n-1} of constant curvature k and a 1-dimensional

Osserman Riemannian space M^1, or (n-1)-dimensional Osserman Riemannian space M^{n-1} of constant curvature and a 1-dimensional Osserman Lorentzian space M_1^1.

Case (ii)

Suppose that Q_x is of the Form (ii). Then it follows that

$Qe_1 = \lambda e_1 - \mu e_2,\ Qe_2 = \mu e_1 + \lambda e_2,\ Qe_j = \lambda_j e_j\ \ (j = 3,...,n).$

Putting $X = e_1,\ Y = Z = e_j$ in Equation (5), we have

$$\frac{(\lambda_j{}^2 - \lambda_2 + \mu^2)e_1 + 2\lambda\mu e_2}{n-2} - \frac{TrQ}{(n-1)(n-2)}((\lambda - \lambda_j)e_1 - \mu e_2) = 0. \tag{11}$$

Equation (11) implies that $\frac{\mu(2\lambda - TrQ)}{n-1} = 0$ or since $\mu \neq 0$, $\lambda = \frac{TrQ}{2(n-1)}$. Next, putting $X = e_1,\ Y = e_j,\ Z = e_2$ in Equation (5) and taking an inner product with e_j, we get $2\lambda\mu + \frac{(\lambda_j - \mu)TrQ}{n-1} = 0$. Substituting $\lambda = \frac{TrQ}{2(n-1)}$ into this equation, we have $\lambda\lambda_j = 0$. Similarly, putting $X = e_1$, $Y = e_j$, $Z = e_1$ in Equation (5), we obtain

$$(\lambda^2 - \lambda_j{}^2 - \mu^2) - \frac{2\lambda(\lambda - \lambda_j)}{n-2} = 0. \tag{12}$$

If $\lambda = 0$ then Equation(12) reduces to $\lambda^2 + \mu^2 = 0$. This contradicts the assumption that $\mu \neq 0$. Therefore, $\lambda \neq 0$ and $\lambda_j = 0$. So from Equation (12), it follows that $n = 2$ which contradicts the assumption on the dimension. Thus this case can not occur.

Case (iii)

Suppose that Q_x is of the Form (iii). Then it follows that $Qu_1 = \lambda u_1 + u_2,\ Qu_2 = \lambda u_2,\ Qu_j = \lambda_j u_j\ \ (j = 3,...,n).$ Putting $X = u_1,\ Y = u_2,\ Z = u_1$ in Equation (1.5), we obtain $2\lambda u_2 - \frac{(TrQ)u_2}{n-1} = 0,$ so that

$$\lambda = \frac{2TrQ}{n-1}. \tag{13}$$

Putting $X = u_2,\ Y = u_j,\ Z = u_j$ in Equation (5), we have $\frac{(\lambda_j{}^2 - \lambda^2)u_2}{n-2} + \frac{(TrQ)(\lambda - \lambda_j)u_2}{(n-1)(n-2)} = 0.$ By taking the inner product with u_2, it follows that $\lambda = \lambda_j$ or $\lambda = -2\lambda_j$ by virtue of Equation (13). Hence Equation (13) reduces to $\lambda = \frac{2n\lambda}{n-1}$ which implies $\lambda = 0$ because $\dim M_1^n \geq 4$. Therefore, it follows that $Q^2 = 0$ and that $R(X,Y) = 0$ for any tangent vectors *X* and *Y* in S_x^ε, by virtue of Equation (3). Therefore, M_1^n is flat, namely it has constant curvature zero.

Case (iv)

By using the semi-orthogonal basis to evaluate Q_x and following similar argument in Case (iii) we have $R(X,Y) = 0$ for any tangent Vectors *X* and *Y* in S_x^ε, by virtue of Equation (2). Thus, M_1^n has constant curvature zero.

In the case $\dim M = 3$, if Q is of the Form (i), then the result is accomplished by similar arguments in Sekigawa and Takagi (1971) and also for the other cases (Honda, 2003).

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