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Full Length Research Paper

Some oscillating motions of a Burgers' fluid

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In this work, the exact analytic solutions for some unsteady oscillating flows of an incompressible Burgers' fluid in a duct of rectangular cross-section are considered. Here, two types of flows, namely, (1) flow due to the oscillating pressure gradient, and (2) flow due to the oscillation of duct parallel to its length, were considered. The exact analytical expressions for the velocity field and the adequate shear stress are determined by means of the Laplace and Fourier sine transforms. They are written as a sum of steady and transient solutions and satisfy all imposed initial and boundary conditions. The effects of the indispensable parameters of the flow are graphically analyzed. Moreover, similar solutions for Oldroyd-B, Maxwell, Newtonian fluid, and for the flows induced by a constant pressure gradient and impulsive motion of duct are obtained as the limiting cases of the presented solutions.

Key words: Oscillating motions, duct flows, Burgers' fluid, exact solutions.

INTRODUCTION

In recent years, study of non-Newtonian fluids is of paramount importance due to their increasing applications in various manufacturing and processing industries. Many fluids of industrial importance, notably, most particulate slurries (china clay and coal in water, sewage sludge, etc), multiphase mixtures (oil-water emulsion, gas-liquid dispersions such as froths and foams, butter), pharmaceutical formations, cosmetics and toiletries, paints, synthetic lubricants, biological fluids (blood, synovial, saliva), and foodstuffs (jams, jellies, soup, etc) are non-Newtonian in their flow characteristics and are referred to as rheological fluids. That is, they might exhibit dramatic deviation from Newtonian behavior depending on the flow configuration and/or the rate of deformation. The notable points of the rheological behavior are the ability of the fluid to shear thinning or shear thickening, the presence of non-zero normal stress differences in shear flow, the ability of the fluid to yield stress, the ability of the fluid to exhibit relaxation and the ability of the fluid to creep. Non-Newtonian fluids form a broad class of fluids in which the relation connecting the shear stress and the deformation rate is non-linear. Hence, there is no universal constitutive model available which exhibits the characteristics of all non-Newtonian fluids. Here are some of the studies (Teipel, 1981; Rajagopal, 1984, 1982; Kumari et al., 2010; Erdogan, 1995) made by using various non-Newtonian models. Although, many models are accorded to describe the rheological behavior of non-Newtonian fluids, the rate type fluids have attained an increasing attention, because these fluids take into account the elastic and memory effects. As a subclass of the non-Newtonian rate type fluids, the Burgers' fluid (Burger, 1935), yet the model of choice to characterize the response of a variety of geological materials is considered in this study. The Burgers' model has also been used to characterize diverse viscoelastic materials: food products such as cheese, soil, asphalt, etc. There are numerous examples of the use of Burgers' model to study asphalt and asphalt mixes (Majidzadeh and Schweyer, 1967). This model has also been used in calculating the transient creep properties of the earth's mantle and specifically related to the post-glacial uplift (Peltier et al., 1981). The Burgers' model is also used to model other geological structure, like olivine rock (Chopra, 1977). In general, the Burgers' model has not attained much attention in spite of its diverse applications. Very limited studies (Ravindran et al., 2004; Quintanilla and Rajagopal, 2006; Hayat et al., 2006; Corina et al., 2010) have been reported for flows

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involving Burgers' fluids. The duct flows have the significant role in the physical/industrial world. There is large number of experiments about the duct (channel) flows, but a small amount of literature is available (Nazar et al., 2012; Nadeem and Akram, 2010; Schuchkin et al., 2002). The duct phenomenon occurs in human body like gastrointestinal tract, renal duct, bile duct and piping system, fuel chimneys, water tank, etc. are used in every engineering design.

The aim of this paper is to present the exact analytical solutions for the velocity fields and the tangential stresses corresponding to an incompressible Burgers' fluid lying in a duct of rectangular cross-section. The flow is generated by the oscillating pressure gradient as well as due to cosine and sine oscillations of the duct parallel to its length. In order to obtain these starting solutions, double Fourier sine and Laplace transforms are used. These solutions, presented as sum of steady-state and transient solutions, describe the motion of the fluid for some time after its initiation. In the absence of side walls $(\beta = 0)$, the obtained solutions correspond to the case when flow is between the parallel plates. The required times to reach the steady-state for the cosine and sine oscillations of the boundary and the effect of the various pertinent parameters have been examined by the graphical pictures.

GOVERNING EQUATIONS

The unsteady flow of an incompressible fluid, in the absence of body forces, is governed by the following laws:

$$\operatorname{div} \mathbf{V} = 0, \tag{1}$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \nabla \cdot \mathbf{S}, \tag{2}$$

where ρ is the density of fluid, $\mathbf{V}$ is the velocity vector, ∇ is the gradient operator, d/dt is the material time derivative, p is the pressure and the extra stress tensor $\mathbf{S}$ in a Burgers' fluid is given by:

$$\mathbf{S} + \lambda \frac{\delta \mathbf{S}}{\delta t} + \gamma \frac{\delta^2 \mathbf{S}}{\delta t^2} = \mu \left[\mathbf{A}_1 + \lambda_r \frac{\delta \mathbf{A}_1}{\delta t} \right], \tag{3}$$

in which μ is the dynamic viscosity, $\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor, $\mathbf{L}$ is the velocity gradient, λ and $\lambda_r < \lambda$ are relaxation and retardation times, γ is the material constant of Burgers' fluid, and

$$\frac{\delta \mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} - \mathbf{LS} - \mathbf{SL}^T. \tag{4}$$

For the following problems, we assume the velocity and the stress fields of the form:

$$\mathbf{V}(x, y, t) = u(x, y, t)\hat{k}, \qquad \mathbf{S} = \mathbf{S}(x, y, t), \tag{5}$$

with $\hat{k}$ as the unit vector along z-direction of the Cartesian coordinate system. For such flows, the constraint of incompressibility is automatically satisfied.

Substituting Equation 5 into Equations 2 and 3 and taking into account the initial conditions for stress:

$$\mathbf{S}(x, y, 0) = \frac{\partial \mathbf{S}(x, y, 0)}{\partial t} = \mathbf{0}, \tag{6}$$

leads to the following relevant equations:

$$\rho \frac{\partial u(x, y, t)}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\partial S_{xz}(x, y, t)}{\partial x} + \frac{\partial S_{yz}(x, y, t)}{\partial y}, \tag{7}$$

$$\left(1 + \lambda \frac{\partial}{\partial t} + \gamma \frac{\partial^2}{\partial t^2}\right) S_{xz}(x, y, t) = \mu \left(1 + \lambda_r \frac{\partial}{\partial t}\right) \frac{\partial u(x, y, t)}{\partial x}, \tag{8}$$

$$\left(1 + \lambda \frac{\partial}{\partial t} + \gamma \frac{\partial^2}{\partial t^2}\right) S_{yz}(x, y, t) = \mu \left(1 + \lambda_r \frac{\partial}{\partial t}\right) \frac{\partial u(x, y, t)}{\partial y}, \tag{9}$$

where $S_{xz}(x, y, t)$ and $S_{yz}(x, y, t)$ denote the non-trivial tangential stresses.

Substituting Equations 8 and 9 in Equation 7, a straightforward calculations yields the equation of motion as follows:

$$\left(1 + \lambda \frac{\partial}{\partial t} + \gamma \frac{\partial^2}{\partial t^2}\right) \frac{\partial u(x, y, t)}{\partial t} = -\frac{1}{\rho}\left(1 + \lambda \frac{\partial}{\partial t} + \gamma \frac{\partial^2}{\partial t^2}\right) \frac{\partial p}{\partial z} + \nu \left(1 + \lambda_r \frac{\partial}{\partial t}\right) \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] u(x, y, t), \tag{10}$$

with $\nu = \mu / \rho$ as the kinematic viscosity of the fluid. Equations 8 to 10 may be written in dimensionless form by introducing the following relations:

$$x^* = \frac{x}{d},\ y^* = \frac{y}{h},\ z^* = \frac{z}{d}, t^* = \frac{t}{\left(d^2/\nu\right)},\ \omega^* = \frac{\omega}{\left(\nu/d^2\right)},\ u^* = \frac{u}{U},\ \lambda^* = \frac{\lambda}{\left(d^2/\nu\right)},\ \lambda_r^* = \frac{\lambda_r}{\left(d^2/\nu\right)},$$

$$\gamma^* = \frac{\gamma}{\left(d^2/\nu\right)^2},\ p^* = \frac{p}{\left(\mu U/d\right)},\ Q^* = \frac{Q}{\left(\nu U/d^2\right)},\ \tau_1 = \frac{S_{xz}}{\left(\mu U/d\right)} \text{ and } \tau_2 = \frac{S_{yz}}{\left(\mu U/h\right)}, \tag{11}$$

where U denotes the characteristic velocity.

Using these scales, the dimensionless form of the aforementioned equations becomes:

$$\left(1+\lambda\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)\tau_1(x,y,t) = \left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u(x,y,t)}{\partial x}, \tag{12}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)\tau_2(x,y,t) = \left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u(x,y,t)}{\partial y}, \tag{13}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)\frac{\partial u(x,y,t)}{\partial t} = -\left(1+\lambda\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)\frac{\partial p}{\partial z}+\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left[\frac{\partial^2}{\partial x^2}+\beta^2\frac{\partial^2}{\partial y^2}\right]u(x,y,t), \tag{14}$$

where the dimensionless parameter $\beta = d/h$ representing the aspect ratio and the asterisks have been omitted for simplicity.

POISEUILLE FLOW DUE TO AN OSCILLATING PRESSURE GRADIENT

Consider the unsteady flow of an incompressible Burgers' fluid filling the duct of rectangular cross-section whose sides are $x=0,\ x=d,\ y=0$ and $y=h.$ Initially, we assume that both fluid and duct are at rest. The duct remains stationary for all time and the fluid laid in the duct is disturbed due to an oscillating pressure gradient of the form:

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = QH(t)\cos(\omega t) \text{ or } QH(t)\sin(\omega t), \tag{15}$$

applied at time $t=0^+$. In Equation 13, Q is the constant and $H(\cdot)$ denotes the Heaviside unit step function.

The associated initial and boundary conditions of the problem are:

$$u(x,y,0) = \frac{\partial u(x,y,0)}{\partial t} = \frac{\partial^2 u(x,y,0)}{\partial t^2} = 0 \text{ for} (x,y)\in(0,d)\times(0,h), \tag{16}$$

$$u(0,y,t) = u(d,y,t) = u(x,0,t) = u(x,h,t) = 0 \text{ for all } t. \tag{17}$$

Using the non-dimensional scheme of Equation 11, Equations 15 to 17 reduce to the non-dimensional form:

$$-\frac{\partial p}{\partial z} = QH(t)\cos(\omega t) \text{ or } QH(t)\sin(\omega t), \tag{18}$$

$$u(x,y,0) = \frac{\partial u(x,y,0)}{\partial t} = \frac{\partial^2 u(x,y,0)}{\partial t^2} = 0 \text{ for } (x,y)\in(0,1)\times(0,1), \tag{19}$$

and

$$u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0 \text{ for all } t. \tag{20}$$

In order to solve the problem consisting of Equations 14, 18, 19, and 20 for small and large times, we use double Fourier sine and Laplace transforms (Fetecau et al., 2011; Salah et al., 2011; Christov and Jordan, 2012; Anjum et al., 2012). Consequently, multiply both sides of Equation 14 by $\sin(\lambda_m x)\sin(\mu_n y)$, $(\lambda_m = m\pi \text{ and } \mu_n = n\pi)$, integrate the result with respect to x and y over $[0,1]\times[0,1]$ and bearing in mind the conditions of Equations 19 and 20, we arrive at,

$$\gamma\frac{d^3u_{mn}(t)}{dt^3}+\lambda\frac{d^2u_{mn}(t)}{dt^2}+\left(1+\lambda_r\lambda_{mn}^2\right)\frac{du_{mn}(t)}{dt}+\lambda_{mn}^2u_{mn}(t)=-\frac{\left[1-(-1)^m\right]}{\lambda_m}\frac{\left[1-(-1)^n\right]}{\mu_n}\left(1+\lambda\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)\frac{\partial p}{\partial z}, \tag{21}$$

where $\lambda_{mn}^2=\lambda_m^2+\beta^2\mu_n^2$ and the double Fourier sine transform $u_{mn}(t)$ of $u(x,y,t)$ has to satisfy the initial conditions:

$$u_{mn}(0)=\frac{du_{mn}(0)}{dt}=\frac{d^2u_{mn}(0)}{dt^2}=0 \text{ for } m,n=1,2,3... \tag{22}$$

Applying the Laplace transform to Equation 21 together with Equation 18, we deduce:

$$\bar{u}_{mn}(q)=Q\frac{\left[1-(-1)^m\right]}{\lambda_m}\frac{\left[1-(-1)^n\right]}{\mu_n}\frac{\left(\gamma q^2+\lambda q+1\right)}{\left[\gamma q^3+\lambda q^2+\left(1+\lambda_r\lambda_{mn}^2\right)q+\lambda_{mn}^2\right]}\frac{q}{q^2+\omega^2}, \tag{23}$$

respectively,

$$\bar{u}_{mn}(q)=Q\frac{\left[1-(-1)^m\right]}{\lambda_m}\frac{\left[1-(-1)^n\right]}{\mu_n}\frac{\left(\gamma q^2+\lambda q+1\right)}{\left[\gamma q^3+\lambda q^2+\left(1+\lambda_r\lambda_{mn}^2\right)q+\lambda_{mn}^2\right]}\frac{\omega}{q^2+\omega^2}, \tag{24}$$

where $\bar{u}_{mn}(q)$ is the Laplace transform of $u_{mn}(t)$ and q the transform parameter.

Rewriting Equations 23 and 24 in the equivalent form:

$$\bar{u}_{mn}(q)=\frac{Q}{\gamma}\frac{\left[1-(-1)^m\right]}{\lambda_m}\frac{\left[1-(-1)^n\right]}{\mu_n}\frac{1}{q}\bar{G}_{mn}(q), \tag{25}$$

where

$$\bar{G}_{mn}(q)=\frac{\left(\gamma q^2+\lambda q+1\right)q}{\left(q-q_{1mn}\right)\left(q-q_{2mn}\right)\left(q-q_{3mn}\right)}\frac{q}{q^2+\omega^2}=-\frac{q_{1mn}^2\varphi_{1mn}}{\left(q_{2mn}-q_{1mn}\right)\left(q_{1mn}-q_{3mn}\right)}\frac{1}{q-q_{1mn}}$$
$$-\frac{q_{2mn}^2\varphi_{2mn}}{\left(q_{1mn}-q_{2mn}\right)\left(q_{2mn}-q_{3mn}\right)}\frac{1}{q-q_{2mn}}-\frac{q_{3mn}^2\varphi_{3mn}}{\left(q_{3mn}-q_{1mn}\right)\left(q_{2mn}-q_{3mn}\right)}\frac{1}{q-q_{3mn}}$$
$$+\omega^2\varphi_{4mn}\frac{q}{q^2+\omega^2}+\omega\varphi_{5mn}\frac{\omega}{q^2+\omega^2}, \tag{26}$$

respectively,

$$\bar{G}_{mn}(q)=\frac{\left(\gamma q^2+\lambda q+1\right)q}{\left(q-q_{1mn}\right)\left(q-q_{2mn}\right)\left(q-q_{3mn}\right)}\frac{\omega}{q^2+\omega^2}=-\frac{\omega q_{1mn}\varphi_{1mn}}{\left(q_{2mn}-q_{1mn}\right)\left(q_{1mn}-q_{3mn}\right)}\frac{1}{q-q_{1mn}}$$
$$-\frac{\omega q_{2mn}\varphi_{2mn}}{\left(q_{1mn}-q_{2mn}\right)\left(q_{2mn}-q_{3mn}\right)}\frac{1}{q-q_{2mn}}-\frac{\omega q_{3mn}\varphi_{3mn}}{\left(q_{3mn}-q_{1mn}\right)\left(q_{2mn}-q_{3mn}\right)}\frac{1}{q-q_{3mn}}$$
$$-\omega\varphi_{5mn}\frac{q}{q^2+\omega^2}+\omega^2\varphi_{4mn}\frac{\omega}{q^2+\omega^2}, \tag{27}$$

in which

$$\varphi_{jmn} = \frac{\left(\gamma q_{jmn}^{2} + \lambda q_{jmn} + 1\right)}{q_{jmn}^{2} + \omega^{2}}, \quad q_{jmn} = s_{jmn} - \frac{\lambda}{3\gamma}, \quad \left(j = 1, 2, 3\right)$$

$$\varphi_{4mn} = \left[q_{2mn} q_{3mn} + q_{1mn} q_{2mn} + q_{1mn} q_{3mn} + \lambda q_{1mn} q_{2mn} q_{3mn} - \left\{1 + \left(q_{2mn} q_{3mn} + q_{1mn}\left(q_{2mn} + q_{3mn}\right)\right)\gamma \right.\right.$$

$$\left.\left. + \lambda\left(q_{1mn} + q_{2mn} + q_{3mn}\right)\right\}\omega^{2} + \gamma\omega^{4}\right] / \left(q_{1mn}^{2} + \omega^{2}\right)\left(q_{2mn}^{2} + \omega^{2}\right)\left(q_{3mn}^{2} + \omega^{2}\right),$$

$$\varphi_{5mn} = \left[q_{1mn} q_{2mn} q_{3mn} - \left\{q_{2mn} + q_{3mn} + \lambda q_{2mn} q_{3mn} + q_{1mn}\left[1 + \gamma q_{2mn} q_{3mn} + \lambda\left(q_{2mn} + q_{3mn}\right)\right]\right\}\omega^{2}\right.$$

$$\left. + \left(\lambda + \gamma\left(q_{1mn} + q_{2mn} + q_{3mn}\right)\right)\omega^{4}\right] / \left(q_{1mn}^{2} + \omega^{2}\right)\left(q_{2mn}^{2} + \omega^{2}\right)\left(q_{3mn}^{2} + \omega^{2}\right).$$

and

$$s_{1mn} = \sqrt[3]{-\frac{g_{mn}}{2} + \sqrt{\frac{g_{mn}^{2}}{4} + \frac{f_{mn}^{3}}{27}}} + \sqrt[3]{-\frac{g_{mn}}{2} - \sqrt{\frac{g_{mn}^{2}}{4} + \frac{f_{mn}^{3}}{27}}},$$

$$s_{2mn} = Z\sqrt[3]{-\frac{g_{mn}}{2} + \sqrt{\frac{g_{mn}^{2}}{4} + \frac{f_{mn}^{3}}{27}}} + Z^{2}\sqrt[3]{-\frac{g_{mn}}{2} - \sqrt{\frac{g_{mn}^{2}}{4} + \frac{f_{mn}^{3}}{27}}},$$

$$s_{3mn} = Z^{2}\sqrt[3]{-\frac{g_{mn}}{2} + \sqrt{\frac{g_{mn}^{2}}{4} + \frac{f_{mn}^{3}}{27}}} + Z\sqrt[3]{-\frac{g_{mn}}{2} - \sqrt{\frac{g_{mn}^{2}}{4} + \frac{f_{mn}^{3}}{27}}},$$

are the roots of the equation $s^{3} + f_{mn} s + g_{mn} = 0$ and

$$f_{mn} = -\frac{\lambda^{2}}{3\gamma^{2}} + \frac{1 + \lambda_{r}\lambda_{mn}^{2}}{\lambda_{2}}, \quad g_{mn} = \frac{2\lambda^{3}}{27\gamma^{3}} - \frac{\lambda}{3\gamma}\frac{1 + \lambda_{r}\lambda_{mn}^{2}}{\gamma} + \frac{\lambda_{mn}^{2}}{\gamma} \text{ and } Z = \left(-1 + i\sqrt{3}\right)/2.$$

Inverting Equation 25 by means of the Laplace transform, one obtains:

$$u_{mn}(t) = \frac{Q}{\gamma}\frac{\left[1 - (-1)^{m}\right]}{\lambda_{m}}\frac{\left[1 - (-1)^{n}\right]}{\mu_{n}}\int_{0}^{t} G_{mn}(\tau)\,d\tau, \quad (28)$$

Where

$$G_{mn}(t) = -\frac{q_{1mn}^{2}\varphi_{1mn}}{\left(q_{2mn} - q_{1mn}\right)\left(q_{1mn} - q_{3mn}\right)}e^{q_{1mn}t} - \frac{q_{2mn}^{2}\varphi_{2mn}}{\left(q_{1mn} - q_{2mn}\right)\left(q_{2mn} - q_{3mn}\right)}e^{q_{2mn}t}$$

$$-\frac{q_{3mn}^{2}\varphi_{3mn}}{\left(q_{3mn} - q_{1mn}\right)\left(q_{2mn} - q_{3mn}\right)}e^{q_{3mn}t} + \omega^{2}\varphi_{4mn}\cos(\omega t) + \omega\varphi_{5mn}\sin(\omega t), \quad (29)$$

respectively,

$$G_{mn}(t) = -\frac{\omega q_{1mn}\varphi_{1mn}}{\left(q_{2mn} - q_{1mn}\right)\left(q_{1mn} - q_{3mn}\right)}e^{q_{1mn}t} - \frac{\omega q_{2mn}\varphi_{2mn}}{\left(q_{1mn} - q_{2mn}\right)\left(q_{2mn} - q_{3mn}\right)}e^{q_{2mn}t}$$

$$-\frac{\omega q_{3mn}\varphi_{3mn}}{\left(q_{3mn} - q_{1mn}\right)\left(q_{2mn} - q_{3mn}\right)}e^{q_{3mn}t} - \omega\varphi_{5mn}\cos(\omega t) + \omega^{2}\varphi_{4mn}\sin(\omega t). \quad (30)$$

Finally, substituting Equations 29 and 30 in Equation 28 and inverting the result by means of the double Fourier sine transform, we find for the velocity field, the following simple expressions

$$u(x, y, t) = \frac{16Q}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\omega\sin(\omega t)\varphi_{4MN} - \cos(\omega t)\varphi_{5MN}\right\}\frac{\sin(\lambda_{M}x)}{\lambda_{M}}\frac{\sin(\mu_{N}y)}{\mu_{N}}$$

$$-\frac{16Q}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\frac{q_{1MN}\varphi_{1MN}\,e^{q_{1MN}t}}{\left(q_{2MN} - q_{1MN}\right)\left(q_{1MN} - q_{3MN}\right)} + \frac{q_{2MN}\varphi_{2MN}\,e^{q_{2MN}t}}{\left(q_{1MN} - q_{2MN}\right)\left(q_{2MN} - q_{3MN}\right)}\right.$$

$$\left. + \frac{q_{3MN}\varphi_{3MN}\,e^{q_{3MN}t}}{\left(q_{3MN} - q_{1MN}\right)\left(q_{2MN} - q_{3MN}\right)}\right\}\frac{\sin(\lambda_{M}x)}{\lambda_{M}}\frac{\sin(\mu_{N}y)}{\mu_{N}}, \quad (31)$$

respectively,

$$u(x,y,t) = -\frac{16Q}{\gamma}H(t)\sum_{m,n=1}^{\infty}\{\sin(\omega t)\varphi_{5MN} + \omega\cos(\omega t)\varphi_{4MN}\}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}$$

$$-\frac{16Q\omega}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\frac{\varphi_{1MN}\,e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})} + \frac{\varphi_{2MN}\,e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})}\right.$$

$$\left.+\frac{\varphi_{3MN}\,e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3MN})}\right\}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}, \quad (32)$$

where $M = 2m-1$ and $N = 2n-1$.

Note that the starting solutions $u(x,y,t)$ in Equations 31 and 32 are sum of the steady solution u_s (valid for large times) and the transient solutions u_t. The steady solutions are:

$$u_s(x,y,t) = \frac{16Q}{\gamma}\sum_{m,n=1}^{\infty}[\omega\sin(\omega t)\varphi_{4MN} - \cos(\omega t)\varphi_{5MN}]\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}, \quad (33)$$

respectively,

$$u_s(x,y,t) = -\frac{16Q}{\gamma}\sum_{m,n=1}^{\infty}[\sin(\omega t)\varphi_{5MN} + \omega\cos(\omega t)\varphi_{4MN}]\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}, \quad (34)$$

whereas, the transient solutions are

$$u_t(x,y,t) = -\frac{16Q}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\frac{q_{1MN}\varphi_{1MN}\,e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})} + \frac{q_{2MN}\varphi_{2MN}\,e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})}\right.$$

$$\left.+\frac{q_{3MN}\varphi_{3MN}\,e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3MN})}\right\}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}, \quad (35)$$

respectively,

$$u_t(x,y,t) = -\frac{16Q\omega}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\frac{\varphi_{1MN}\,e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})} + \frac{\varphi_{2MN}\,e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})}\right.$$

$$\left.+\frac{\varphi_{3MN}\,e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3MN})}\right\}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}. \quad (36)$$

The non-trivial tangential stresses can be immediately calculated from Equations 12 and 13. Consequently, taking the Laplace transform of Equations 12 and 13, and using the initial conditions of Equation 6, we attained:

$$\bar{\tau}_1(x,y,q) = \frac{(1+\lambda_r q)}{(1+\lambda q+\gamma q^2)}\frac{\partial \bar{u}(x,y,q)}{\partial x}, \quad (37)$$

$$\bar{\tau}_2(x,y,q) = \frac{(1+\lambda_r q)}{(1+\lambda q+\gamma q^2)}\frac{\partial \bar{u}(x,y,q)}{\partial y}, \quad (38)$$

where $\bar{\tau}_1(x,y,q)$, $\bar{\tau}_2(x,y,q)$, and $\bar{u}(x,y,q)$ are the Laplace transforms of $\tau_1(x,y,t)$, $\tau_2(x,y,t)$ and $u(x,y,t)$, respectively.

Now $\bar{u}(x,y,q)$ can be obtained, for instance, by taking the inverse Fourier sine transform of Equation 23, that is,

$$\bar{u}(x,y,q) = 4Q\sum_{m,n=1}^{\infty}\frac{[1-(-1)^m][1-(-1)^n](\gamma q^2+\lambda q+1)}{[\gamma q^3+\lambda q^2+(1+\lambda_r\lambda_{mn}^2)q+\lambda_{mn}^2]}\frac{q}{q^2+\omega^2}\frac{\sin(\lambda_m x)}{\lambda_m}\frac{\sin(\mu_n y)}{\mu_n}. \quad (39)$$

Inserting Equation 39 into Equations 37 and 38, and applying the inverse Laplace transform, it immediately results in the following expressions for the tangential stresses:

$$\tau_1(x,y,t) = \frac{16Q}{\gamma}H(t)\sum_{m,n=1}^{\infty}\{\omega\sin(\omega t)\chi_{4MN}-\cos(\omega t)\chi_{5MN}\}\cos(\lambda_M x)\frac{\sin(\mu_N y)}{\mu_N}$$

$$-\frac{16Q}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\frac{\chi_{1MN}\,e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})}+\frac{\chi_{2MN}\,e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})}\right.$$

$$\left.+\frac{\chi_{3MN}\,e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3mn})}\right\}\cos(\lambda_M x)\frac{\sin(\mu_N y)}{\mu_N}, \quad (40)$$

respectively,

$$\tau_2(x,y,t) = \frac{16Q}{\gamma}H(t)\sum_{m,n=1}^{\infty}\{\omega\sin(\omega t)\chi_{4MN}-\cos(\omega t)\chi_{5MN}\}\frac{\sin(\lambda_M x)}{\lambda_M}\cos(\mu_N y)$$

$$-\frac{16Q}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\frac{\chi_{1MN}\,e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})}+\frac{\chi_{2MN}\,e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})}\right.$$

$$\left.+\frac{\chi_{3MN}\,e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3mn})}\right\}\frac{\sin(\lambda_M x)}{\lambda_M}\cos(\mu_N y), \quad (41)$$

where

$$\chi_{jmn} = \frac{q_{jmn}(1+\lambda_r q_{jmn})}{q_{jmn}^2+\omega^2}, \quad (j=1,2,3)$$

$$\chi_{4mn} = \left[q_{2mn}q_{3mn} + q_{1mn}q_{2mn} + q_{1mn}q_{3mn} + \lambda_r q_{1mn}q_{2mn}q_{3mn} - \left(1+\lambda_r\left(q_{1mn}+q_{2mn}+q_{3mn}\right)\right)\omega^2\right]$$

$$/\left(q_{1mn}^2+\omega^2\right)\left(q_{2mn}^2+\omega^2\right)\left(q_{3mn}^2+\omega^2\right),$$

$$\chi_{5mn} = \left[q_{1mn}q_{2mn}q_{3mn} - \left(q_{1mn}+q_{2mn}+q_{3mn}+\lambda_r q_{2mn}q_{3mn}+\lambda_r q_{1mn}\left(q_{2mn}+q_{3mn}\right)\right)\omega^2 + \gamma\omega^4\right]$$

$$/\left(q_{1mn}^2+\omega^2\right)\left(q_{2mn}^2+\omega^2\right)\left(q_{3mn}^2+\omega^2\right).$$

In-deed, similar to the velocity, the forgoing expressions for the tangential stresses are sum of the steady and transient solutions. More exactly, we have:

$$\tau_{1s}(x,y,t) = \frac{16Q}{\gamma}\sum_{m,n=1}^{\infty}\left\{\omega\sin(\omega t)\chi_{4MN} - \cos(\omega t)\chi_{5MN}\right\}\cos(\lambda_M x)\frac{\sin(\mu_N y)}{\mu_N}, \quad (42)$$

respectively,

$$\tau_{2s}(x,y,t) = \frac{16Q}{\gamma}\sum_{m,n=1}^{\infty}\left\{\omega\sin(\omega t)\chi_{4MN} - \cos(\omega t)\chi_{5MN}\right\}\frac{\sin(\lambda_M x)}{\lambda_M}\cos(\mu_N y), \quad (43)$$

which represent the steady-state tangential stresses.

FLOW DUE TO AN OSCILLATING DUCT PARALLEL TO ITS LENGTH

Here, we consider unsteady flow of an incompressible Burgers' fluid at rest in a duct of rectangular cross-section. At time $t=0^+$, the duct begins to oscillate with velocity $UH(t)\cos(\omega t)$ or $UH(t)\sin(\omega t)$, where U is the amplitude and ω the angular frequency of the velocity of duct. Due to the shear, the fluid will also start oscillating parallel to the length of the duct. Its velocity is of the form $(5)_1$. The resulting dimensionless governing problem is

$$\left(1+\lambda\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)\frac{\partial u(x,y,t)}{\partial t} = \left(1+\lambda_r\frac{\partial}{\partial t}\right)\left[\frac{\partial^2}{\partial x^2}+\beta^2\frac{\partial^2}{\partial y^2}\right]u(x,y,t), \quad (44)$$

$$u(0,y,t)=u(1,y,t)=u(x,0,t)=u(x,1,t)=V(t) \text{ for all } t, \quad (45)$$

where

$$V(t) = H(t)\cos(\omega t) \text{ or } H(t)\sin(\omega t) \quad (46)$$

and the initial conditions are prescribed in Equation 19. Writing

$$u(x,y,t)=V(t)-v(x,y,t). \quad (47)$$

adopting a similar procedure as in Poiseuille flow due to an oscillating pressure gradient, we finally find the expressions for the velocity field:

$$u(x,y,t) = H(t)\cos(\omega t) - \frac{16\omega}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\sin(\omega t)\varphi_{5MN}+\omega\cos(\omega t)\varphi_{4MN}\right\}$$

$$\times\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N} + \frac{16}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\frac{q_{1MN}^2\varphi_{1MN}e^{q_{1MN}t}}{\left(q_{2MN}-q_{1MN}\right)\left(q_{1MN}-q_{3MN}\right)}\right.$$

$$\left.+\frac{q_{2MN}^2\varphi_{2MN}e^{q_{2MN}t}}{\left(q_{1MN}-q_{2MN}\right)\left(q_{2MN}-q_{3MN}\right)}+\frac{q_{3MN}^2\varphi_{3MN}e^{q_{3MN}t}}{\left(q_{3MN}-q_{1MN}\right)\left(q_{2MN}-q_{3mn}\right)}\right\}$$

$$\times\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}, \quad (48)$$

respectively,

$$u(x,y,t) = H(t)\sin(\omega t) - \frac{16\omega}{\gamma}H(t)\sum_{m,n=1}^{\infty}\{\omega\sin(\omega t)\varphi_{4MN} - \cos(\omega t)\varphi_{5MN}\}$$

$$\times\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N} + \frac{16\omega}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\frac{q_{1MN}\varphi_{1MN}\,e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})}\right.$$

$$\left.+\frac{q_{2MN}\varphi_{2MN}\,e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})} + \frac{q_{3MN}\varphi_{3MN}\,e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3mn})}\right\}$$

$$\times\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}. \tag{49}$$

The corresponding steady state and transient solutions are

$$u_s(x,y,t) = \cos(\omega t) - \frac{16\omega}{\gamma}\sum_{m,n=1}^{\infty}\{\sin(\omega t)\varphi_{5MN} + \omega\cos(\omega t)\varphi_{4MN}\}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}, \tag{50}$$

respectively

$$u_s(x,y,t) = \sin(\omega t) - \frac{16\omega}{\gamma}\sum_{m,n=1}^{\infty}\{\omega\sin(\omega t)\varphi_{4MN} - \cos(\omega t)\varphi_{5MN}\}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}, \tag{51}$$

and

$$u_t(x,y,t) = \frac{16}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\frac{q_{1MN}^2\varphi_{1MN}\,e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})} + \frac{q_{2MN}^2\varphi_{2MN}\,e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})}\right.$$

$$\left.+\frac{q_{3MN}^2\varphi_{3MN}\,e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3mn})}\right\}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}, \tag{52}$$

respectively,

$$u_t(x,y,t) = \frac{16\omega}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left\{\frac{q_{1MN}\varphi_{1MN}\,e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})} + \frac{q_{2MN}\varphi_{2MN}\,e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})}\right.$$

$$\left.+\frac{q_{3MN}\varphi_{3MN}\,e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3mn})}\right\}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}. \tag{53}$$

Employing the same methodology as used in Poiseuille flow due to an oscillating pressure gradient, the expressions for the tangential stresses are given by:

$$\tau_1(x,y,t) = -\frac{16\omega}{\gamma} H(t) \sum_{m,n=1}^{\infty} \left\{ \sin(\omega t)\chi_{5MN} + \omega\cos(\omega t)\chi_{4MN} \right\} \cos(\lambda_M x) \frac{\sin(\mu_N y)}{\mu_N}$$

$$+\frac{16}{\gamma} H(t) \sum_{m,n=1}^{\infty} \left\{ \frac{q_{1MN}\chi_{1MN}\, e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})} + \frac{q_{2MN}\chi_{2MN}\, e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})} \right.$$

$$\left. + \frac{\chi_{3MN}\, e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3MN})} \right\} \cos(\lambda_M x) \frac{\sin(\mu_N y)}{\mu_N}, \tag{54}$$

respectively,

$$\tau_2(x,y,t) = -\frac{16\omega}{\gamma} H(t) \sum_{m,n=1}^{\infty} \left\{ \sin(\omega t)\chi_{5MN} + \omega\cos(\omega t)\chi_{4MN} \right\} \frac{\sin(\lambda_M x)}{\lambda_M} \cos(\mu_N y)$$

$$+\frac{16}{\gamma} H(t) \sum_{m,n=1}^{\infty} \left\{ \frac{q_{1MN}\chi_{1MN}\, e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})} + \frac{q_{2MN}\chi_{2MN}\, e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})} \right.$$

$$\left. + \frac{q_{3MN}\chi_{3MN}\, e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3MN})} \right\} \frac{\sin(\lambda_M x)}{\lambda_M} \cos(\mu_N y), \tag{55}$$

with

$$\tau_{1s}(x,y,t) = -\frac{16\omega}{\gamma} \sum_{m,n=1}^{\infty} \left\{ \sin(\omega t)\chi_{5MN} + \omega\cos(\omega t)\chi_{4MN} \right\} \cos(\lambda_M x) \frac{\sin(\mu_N y)}{\mu_N}, \tag{56}$$

$$\tau_{2s}(x,y,t) = -\frac{16\omega}{\gamma} \sum_{m,n=1}^{\infty} \left\{ \sin(\omega t)\chi_{5MN} + \omega\cos(\omega t)\chi_{4MN} \right\} \frac{\sin(\lambda_M x)}{\lambda_M} \cos(\mu_N y), \tag{57}$$

denoting the steady-state tangential stresses.

LIMITING CASES

Oldroyd-B fluid

Taking γ =0 in Equations 23 and 24, following the same procedure and simplify the result, we obtain the velocity field for an oscillating pressure gradient:

$$u(x,y,t)=16QH(t)\sum_{m,n=1}^{\infty}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_M y)}{\mu_M}\left(\frac{\omega\left(1-\lambda^2{}_{mn}(\lambda-\lambda_r)+\lambda^2\omega^2\right)}{\varphi_{6MN}}\sin(\omega t)\right.$$

$$\left.+\frac{\lambda^2{}_{mn}\left(1+\lambda\lambda_r\omega^2\right)}{\varphi_{6MN}}\cos(\omega t)\right)-\frac{6QH(t)}{\lambda}\sum_{m,n=1}^{\infty}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_M y)}{\mu_M}\left(\frac{(1+\lambda r_1)r_1}{(r_2-r_1)(r_1{}^2+\omega^2)}\right.$$

$$\left.\times e^{r_1 t}+\frac{(1+\lambda r_2)r_2}{(r_1-r_2)(r_2{}^2+\omega^2)}e^{r_2 t}\right), \tag{58}$$

respectively,

$$u(x,y,t)=16QH(t)\sum_{m,n=1}^{\infty}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_M y)}{\mu_M}\left(\frac{\lambda^2{}_{mn}\left(1+\lambda\lambda_r\omega^2\right)}{\varphi_{6MN}}\sin(\omega t)\right.$$

$$\left.-\frac{\omega\left(1-\lambda^2{}_{mn}(\lambda-\lambda_r)+\lambda^2\omega^2\right)}{\varphi_{6MN}}\cos(\omega t)\right)-\frac{6QH(t)}{\lambda}\sum_{m,n=1}^{\infty}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_M y)}{\mu_M}$$

$$\left(\frac{(1+\lambda r_1)}{(r_2-r_1)(r_1{}^2+\omega^2)}e^{r_1 t}+\frac{(1+\lambda r_2)}{(r_1-r_2)(r_2{}^2+\omega^2)}e^{r_2 t}\right), \tag{59}$$

corresponding to the Oldroyd-B fluid. In above relations of Equation 58 and 59:

$$\varphi_{6mn}=\left(1+\lambda^2{}_r\,\omega^2\right)\left(\left(\lambda^2{}_{mn}-\frac{(\lambda-\lambda_r)\omega^2}{\left(1+\lambda^2{}_r\,\omega^2\right)}\right)^2+\frac{\omega^2\left(1+\lambda\lambda_r\,\omega^2\right)^2}{\left(1+\lambda^2{}_r\,\omega^2\right)^2}\right),$$

and

$$r_{1,2}=\frac{-\left(1+\lambda_r\lambda^2{}_{mn}\right)\pm\sqrt{\left(1+\lambda_r\lambda^2{}_{mn}\right)^2-4\lambda\lambda^2{}_{mn}}}{2\lambda}.$$

Maxwell fluid

Making $\lambda_{r=0}$ into Equations 58 and 59, we get the velocity field:

$$u(x,y,t)=16QH(t)\sum_{m,n=1}^{\infty}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_M y)}{\mu_M}\left(c_{MN}\sin(\omega t)+d_{MN}\cos(\omega t)\right.$$

$$\left.-e^{-t/2\lambda}\left(d_{MN}\cosh\left(\frac{b_{MN}}{2\lambda}t\right)+\frac{d_{MN}-2\lambda\omega(1-c_{MN})}{b_{MN}}\sin\left(\frac{b_{MN}}{2\lambda}t\right)\right)\right), \tag{60}$$

respectively,

$$u(x,y,t)=16QH(t)\sum_{m,n=1}^{\infty}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_M y)}{\mu_M}\big(d_{MN}\sin(\omega t)-c_{MN}\cos(\omega t)$$
$$-e^{-t/2\lambda}\left(-\frac{c_{MN}}{b_{MN}}\cosh\left(\frac{b_{MN}}{2\lambda}t\right)+\left(2\lambda d_{MN}-\frac{c_{MN}}{\omega}\right)\sin\left(\frac{b_{MN}}{2\lambda}t\right)\right)\Bigg), \tag{61}$$

where

$$c_{mn}=\frac{\omega\left(1-\lambda\left(\lambda^2{}_{mn}-\lambda\omega^2\right)\right)}{\left(\lambda^2{}_{mn}-\lambda\omega^2\right)^2+\omega^2},\quad d_{mn}=\frac{\lambda^2{}_{mn}}{\left(\lambda^2{}_{mn}-\lambda\omega^2\right)^2+\omega^2}\ \text{and}\ b_{mn}=\sqrt{1-4\lambda\lambda^2{}_{mn}},$$

corresponding to the Maxwell fluid for an oscillating pressure gradient.

Newtonian fluid

Taking the limit λ=→0 in Equations 60 and 61, we find the Velocity field corresponding to Newtonian fluid performing the same motion. Hence, we have: velocity

$$u(x,y,t)=16QH(t)\sum_{m,n=1}^{\infty}\frac{\lambda^2{}_{mn}\cos(\omega t)+\omega\sin(\omega t)}{\lambda^4{}_{mn}+\omega^2}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_M y)}{\mu_M}$$
$$-16QH(t)\sum_{m,n=1}^{\infty}\frac{\lambda^2{}_{mn}}{\lambda^4{}_{mn}+\omega^2}\exp\left(-\lambda^2{}_{mn}t\right)\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_M y)}{\mu_M}, \tag{62}$$

respectively,

$$u(x,y,t)=16QH(t)\sum_{m,n=1}^{\infty}\frac{\lambda^2{}_{mn}\sin(\omega t)-\omega\cos(\omega t)}{\lambda^4{}_{mn}+\omega^2}\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_M y)}{\mu_M}$$
$$+16Q\omega H(t)\sum_{m,n=1}^{\infty}\frac{1}{\lambda^4{}_{mn}+\omega^2}\exp\left(-\lambda^2{}_{mn}t\right)\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_M y)}{\mu_M}. \tag{63}$$

Stokes' first problem (ω→0)

By taking ω=0 in Equations 31, 40, 41, 48, 54, and 55, we can easily obtain the solutions for constant pressure gradient and impulsive motion of the duct performing the same motion. For instance, Equations 31 and 48 take the following form:

$$u(x,y,t)=\frac{16Q}{\gamma}H(t)\sum_{m,n=1}^{\infty}\left[\frac{\gamma}{\lambda_{mn}^2}-\left\{\frac{q_{1MN}\varphi'_{1MN}\,e^{q_{1MN}t}}{(q_{2MN}-q_{1MN})(q_{1MN}-q_{3MN})}+\frac{q_{2MN}\varphi'_{2MN}\,e^{q_{2MN}t}}{(q_{1MN}-q_{2MN})(q_{2MN}-q_{3MN})}\right.\right.$$
$$\left.\left.+\frac{q_{3MN}\varphi'_{3MN}\,e^{q_{3MN}t}}{(q_{3MN}-q_{1MN})(q_{2MN}-q_{3MN})}\right\}\right]\frac{\sin(\lambda_M x)}{\lambda_M}\frac{\sin(\mu_N y)}{\mu_N}, \tag{64}$$

and

Table 1. Values of $u(x, y, t)$ for λ=0.5, λ_r=2.0, γ=0.02, ω=0.5 and various values of t for cosine oscillation.

t	*u*(0.5,0.5,*t*) for β=0.5	*u*(0.5,0.5,*t*) for β=0.0
0.1	0.159227	0.151909
0.2	0.462117	0.444357
0.3	0.685231	0.648599
0.4	0.830653	0.785824

Table 2. Values of $u(x, y, t)$ for t=0.2, λ_r=2.0, γ=0.02, ω=0.5 and various values of λ for cosine oscillation.

λ	*u*(0.5,0.5,*t*) for β=0.5	*u*(0.5,0.5,*t*) for β=0.0
3.0	0.735248	0.693832
4.0	0.581220	0.553647
5.0	0.462117	0.444357
6.0	0.370650	0.358614

$$u(x,y,t) = H(t) + \frac{16}{\gamma} H(t) \sum_{m,n=1}^{\infty} \left\{ \frac{q_{1MN}^2 \varphi'_{1MN} e^{q_{1MN} t}}{(q_{2MN} - q_{1MN})(q_{1MN} - q_{3MN})} + \frac{q_{2MN}^2 \varphi'_{2MN} e^{q_{2MN} t}}{(q_{1MN} - q_{2MN})(q_{2MN} - q_{3MN})} + \frac{q_{3MN}^2 \varphi'_{3MN} e^{q_{3MN} t}}{(q_{3MN} - q_{1MN})(q_{2MN} - q_{3mn})} \right\} \frac{\sin(\lambda_M x)}{\lambda_M} \frac{\sin(\mu_N y)}{\mu_N}, \tag{65}$$

where

$$\varphi'_{jmn} = \frac{(\gamma q_{jmn}^2 + \lambda q_{jmn} + 1)}{q_{jmn}^2}, (j = 1, 2, 3)$$

but q_{jmn} (j=1,2,3) are the same as Stokes' first problem.

GRAPHICAL RESULTS AND DISCUSSION

Here, the significant features of rheological characteristics on velocity field were presented. The graphical results illustrate the velocity profiles only for the flow due to an oscillating duct parallel to its length. To illustrate the difference, we have presented the profiles for both cosine and sine oscillations of the boundary. Further, in order to capture the effects of side walls (when $\beta \neq 0$), a comparison of the velocity profiles with those for the flow between two parallel plates (when $\beta = 0$) is provided. The numerical values are shown in Tables 1 to 3 and plotted in Figures 1 to 7. We interpret these results with respect to the variations of emerging parameters of interest.

Figure 1 displays the influence of time t on the velocity profile for both cosine and sine oscillations of the boundary in the presence as well as absence of side walls. It is clearly seen that increasing the values of t produces an increase in the velocity profiles for both cases. That is, the velocity profile for time t = 0.4 is larger than those of starting time t = 0.1. This fact is also reflected in Table 1. Moreover, this figure also depicts the dependence of velocity on side walls, and one can see that the maximum velocity occurs near the side wall, while the velocity is minimum at the centre of the plate. It is evident from Figure 1 and Tables 1 to 3 that the velocity profiles are greater in magnitude in the presence of side walls (when $\beta = 0.5$) when compared with those in the absence of side walls (when $\beta = 0$).

Figures 2 and 3 present the influence of the parameters λ and λ_r on the velocity field. The effect of increasing λ from $\lambda = 3.0$ to 6.0 is to decrease the velocity for both cosine and sine oscillations of duct. The effect of increasing λ_r is opposite qualitatively to that of the parameter λ. Therefore, λ depicts the shear thickening behavior, while λ_r shows the shear thinning effect on the

Table 3. Values of $u(x, y, t)$ for t=0.2, λ=0.5, λ_r=2.0, ω=0.5 and various values of γ for cosine oscillation.

γ	***u*(0.5,0.5,*t*) for β=0.5**	***u*(0.5,0.5,*t*) for β=0.0**
0.010	0.463913	0.445508
0.040	0.457956	0.441903
0.070	0.449483	0.438116
0.095	0.439785	0.435174

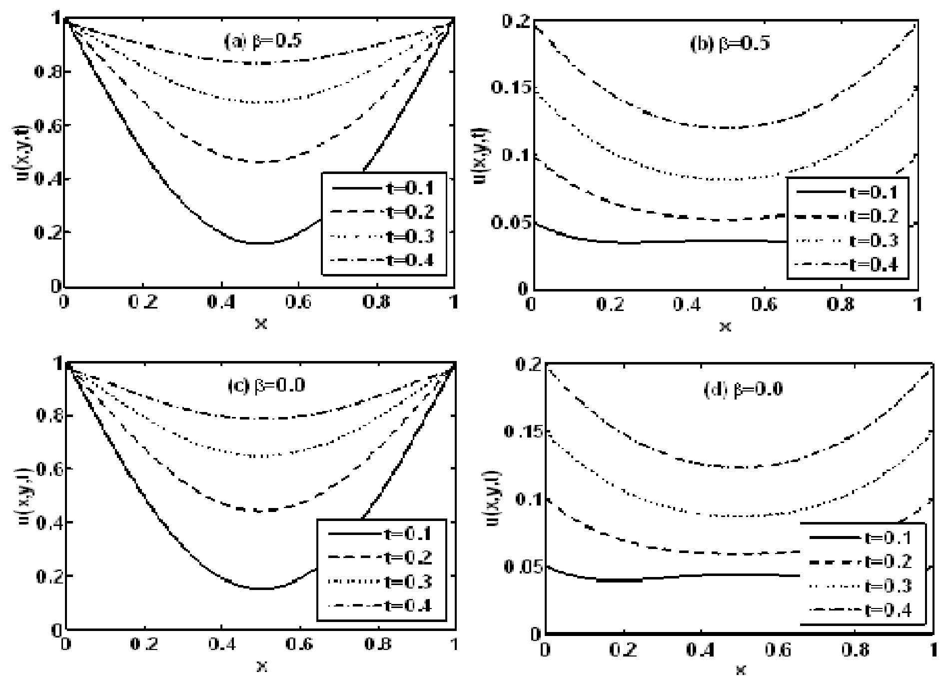

Figure 1. Profiles of the velocity $u(x, y, t)$ given by Equations 48 [left column] and 49 [right column] for $\lambda = 5.0,\quad \lambda_r = 2.0,\quad \gamma = 0.02,\quad y = 0.5$ and different values of time t.

velocity profiles in the presence (when $\beta = 0.5$) as well as in the absence of side walls (when $\beta = 0.0$). Therefore, we can understand that the influence of the relaxation parameter λ as well as the retardation parameter λ_r on the velocity profile is significantly effective and quite opposite qualitatively.

The effects of the rheological parameters γ of Burgers' fluid on the velocity profiles in the presence as well as the absence of side walls are depicted in Figure 4. Here, we can see that the velocity decreases with an increase in γ from $\gamma = 0.010$ to 0.095. However, it is observed that this decrease in the velocity is much in the presence of side walls as compared to that in the absence of side walls. This fact is also reflected in Table 3. Thus, it is concluded that the velocity is severely affected by the rheological parameter γ and the velocity profiles are found to be more sensible to the changes with the rheological parameter γ.

Figures 5 to 7 describe the required time to reach the steady-state or the decay of transient solution for different values of relaxation, retardation and rheological parameters, respectively, for both cosine and sine oscillations in the presence of side walls (when $\beta = 0.5$). Here, we can see that the material parameter λ has the same effect as γ, while λ_r has the opposite effect on the velocity profile. More exactly, it is clearly seen that the required time to reach the steady-state or the decay of transient solution increases if λ and γ increases from 4.0 to 7.0 and 0.01 to 0.30, respectively. However, the required time to reach the steady-state or the decay of

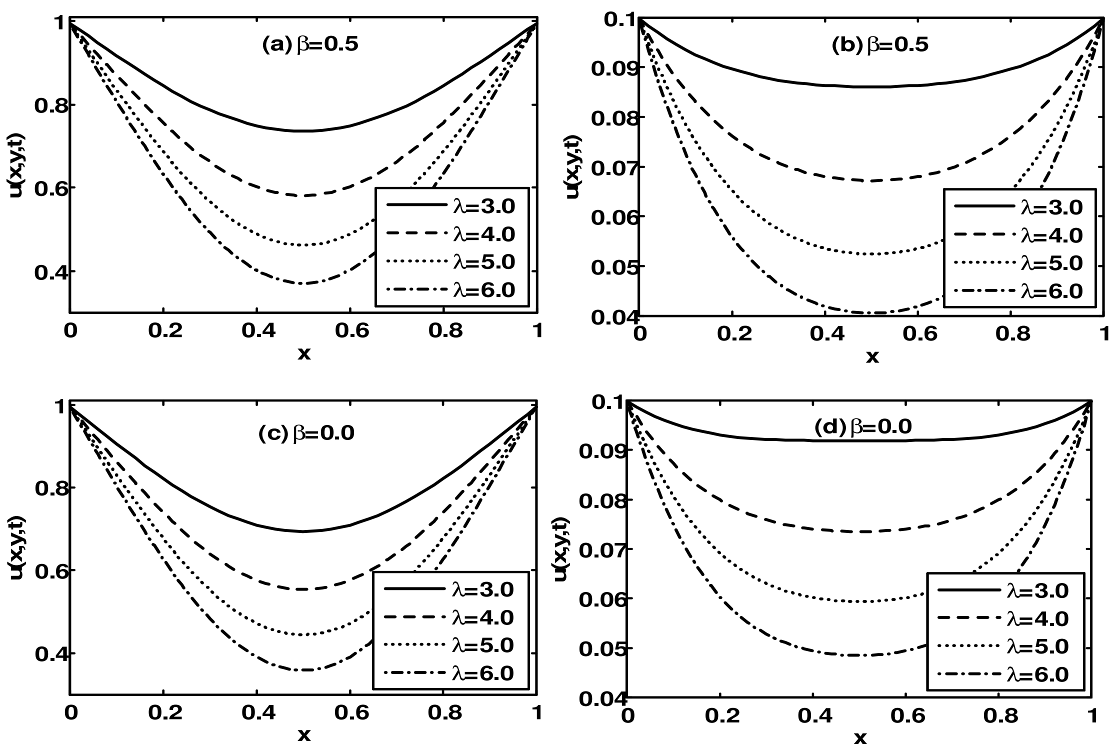

Figure 2. Profiles of the velocity $u(x,y,t)$ given by Equations 48 [left column] and 49 [right column] for $t=0.2,\ \lambda_r=2.0,\ \gamma=0.02,\ y=0.5$ and different values of relaxation time λ.

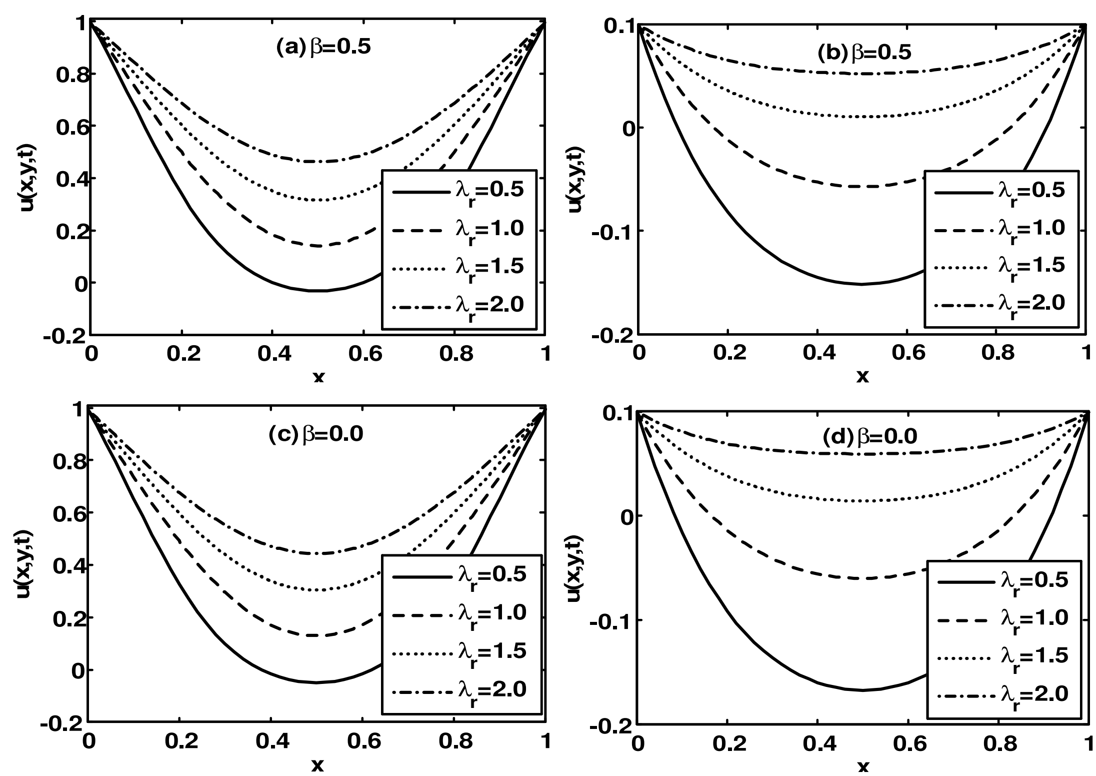

Figure 3. Profiles of the velocity $u(x,y,t)$ given by Equations 48 [left column] and 49 [right column] for $t=0.2,\quad \lambda=5.0,\quad \gamma=0.02,\quad y=0.5$ and different values of retardation time λ_r.

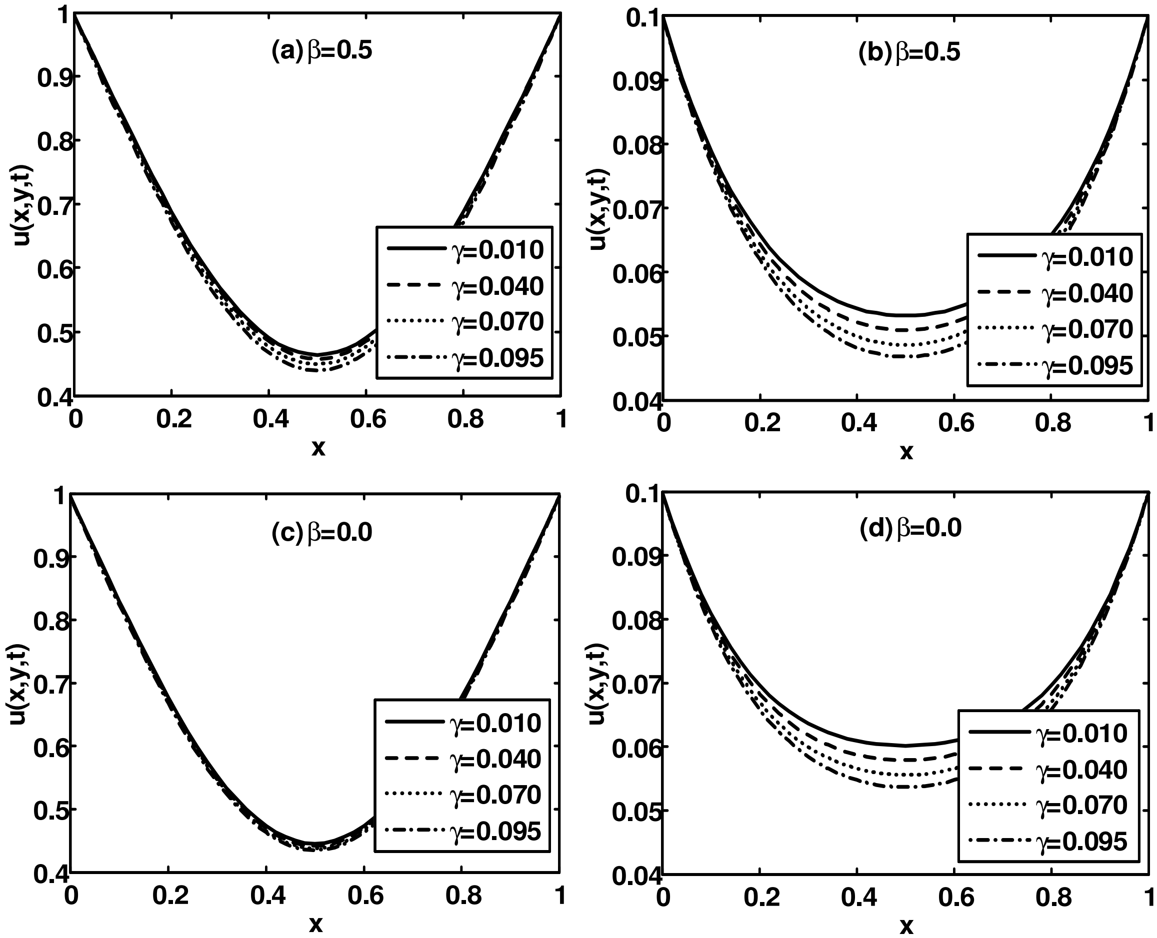

Figure 4. Profiles of the velocity $u(x, y, t)$ given by Equations 48 [left column] and 49 [right column] for $t = 0.2,$ $\lambda = 5.0,$ $\lambda_r = 2.0,$ $y = 0.5$ and different values of rheological parameter γ.

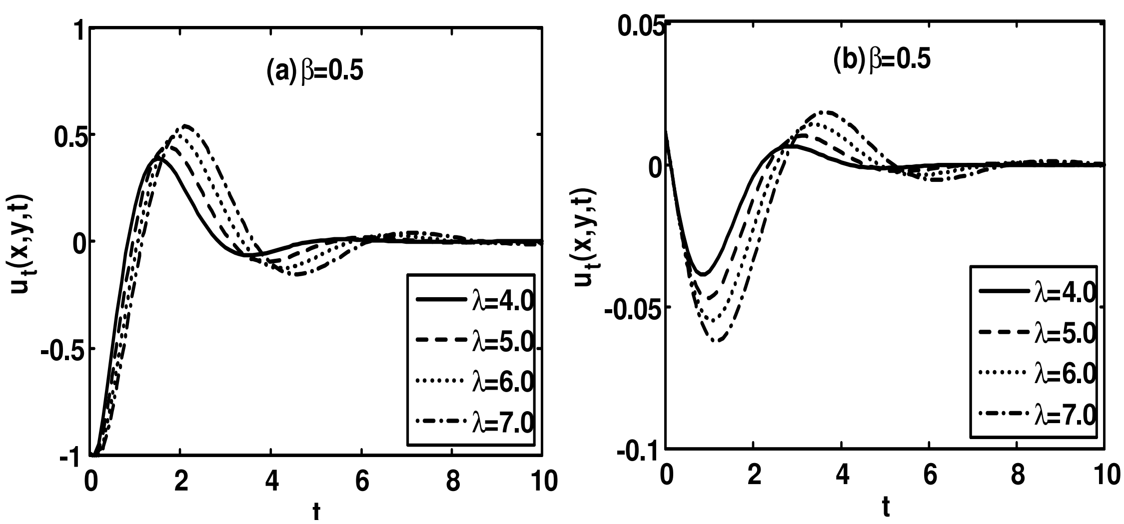

Figure 5. Decay of the transient part of the velocity $u(x, y, t)$ given by Equations 52 [left column] and 53 [right column] for $\lambda_r = 0.5,$ $\gamma = 0.02,$ $\omega = 0.1,$ $y = 0.5$ and different values of relaxation time λ .

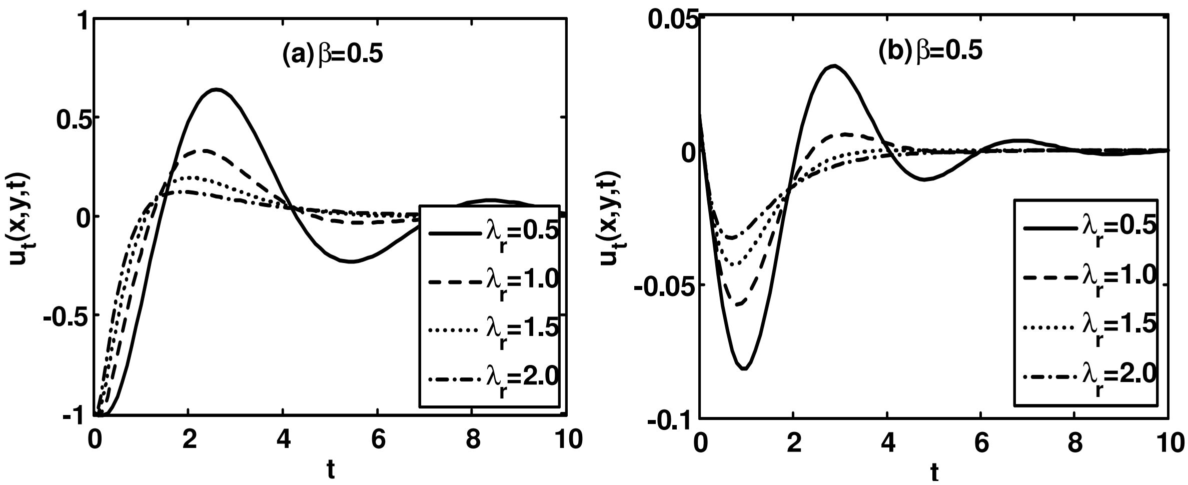

Figure 6. Decay of the transient part of the velocity $u(x, y, t)$ given by Equations 52 [left column] and 53 [right column] for $\lambda = 10,\quad \gamma = 0.02,\quad \omega = 0.1,\quad y = 0.5$ and different values of retardation time λ_r.

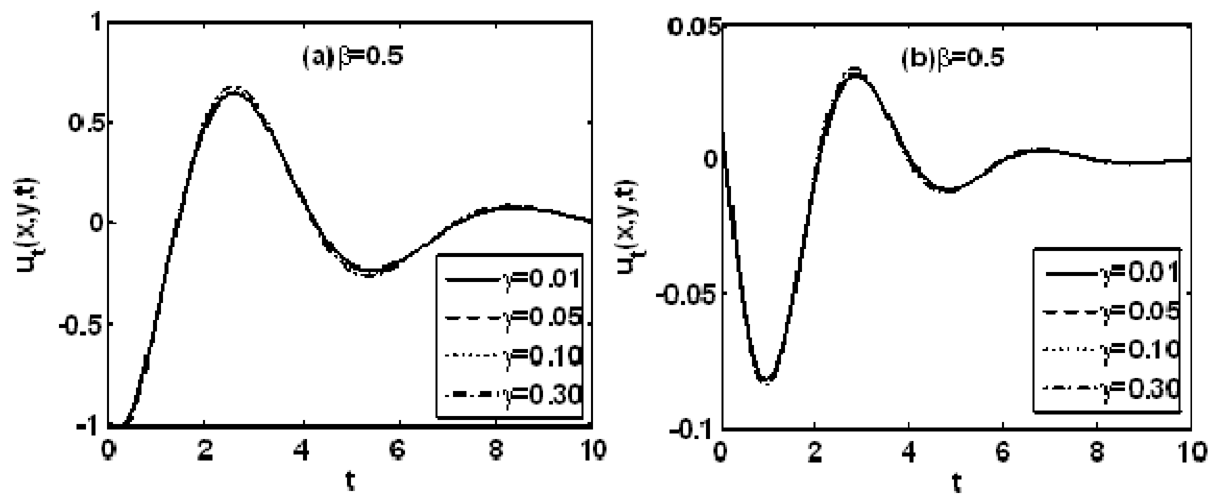

Figure 7. Decay of the transient part of the velocity $u(x, y, t)$ given by Equations 52 [left column] and 53 [right column] for $\lambda = 10,\ \lambda_r = 0.5,\quad \omega = 0.1,\quad y = 0.5$ and different values of rheological parameter γ.

transient solution decreases with an increase in λ_r from 0.5 to 2.0.

Conclusions

In this paper, we have investigated the starting solutions concerning some oscillatory flows of a non-Newtonian fluid. An incompressible Burgers' fluid in a channel of a rectangular cross-section has been considered. The motion was induced by the oscillation of duct parallel to its length as well as the oscillatory pressure gradient. The exact analytical expressions for the velocity field and the corresponding shear stress were established in simple forms by means of integral transforms. These solutions, depending on the initial and boundary conditions, were presented as sum of steady and transient solutions.

Finally, in order to bring light to some relevant physical aspects of the obtained results, the influence of the material parameters on the fluid motion was underlined by graphical illustrations for flow due to the oscillation of duct parallel to its length. On the basis of the earlier discussion, the following important findings were drawn. It is worth pointing out that the velocity profile was found to be more sensible to the changes with rheological parameter γ in the presence of side walls as compared to that in the absence of side walls. Further, it was found that the required time to reach the steady-state or the decay of transient solution increases if λ and γ increases. On the other hand, the required time to reach the steady-state or decay of the transient solution decreases if λ_r increases.

It is hoped that the present investigation would be useful in studying more complex problems and can be utilized as the basis for many scientific and industrial applications.

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REFERENCES

Anjum A, Ayub M, Khan M (2012). Starting solutions for oscillating motions of an Oldroyd-B fluid over a plane wall. Comm. Non-Linear Sci. Numer. Simulat. 1:472-482.
Burgers JM (1935). Mechanical considerations-model systems-phenomenological theories of relaxation of viscosity. In:J.M. Burgers (Ed.), First Report on Viscosity and Plasticity. Nordemann Publishing Company, New York.
Chopra PN (1977). High temperature transient in olivine rocks. Tectonophysics 279:93-111.
Christov IC, Jordan PM (2012). Comments on:“Starting solutions for some unsteady unidirectional flows of a second grade fluid.” [Int. J. Eng. Sci. 43(2005):781]. Int. J. Eng. Sci. 51:326-332.
Corina F, Hayat T, Khan M, Fetecau C (2010). A note on longitudinal oscillations of a generalized Burgers' fluid in cylindrical domains. J. Non-Newtonian Fluid Mech. 165:350-361.
Erdogan ME (1995). Plane surface suddenly set in motion in a non-Newtonian fluid. Acta Mech. 108:179-187.
Fetecau C, Hayat T, Khan M, Fetecau C (2011). Erratum to:"Unsteady flow of an Oldroyd-B fluid induced by the impulsive motion of a plate between two side walls perpendicular to the plate". Acta Mech. 216:359-361.
Hayat T, Fetecau C, Asghar S (2006). Some simple flows of a Burgers' fluid. Int. J. Eng. Sci. 44:1425-1431.
Kumari M, Pop I, Nath G (2010). Transient MHD stagnation flow of a non-Newtonian fluid due to impulsive motion from rest. Int. J. Non-Linear Mech. 45:463-473.
Majidzadeh K, Schweyer HE (1967). Viscoelastic response of asphalts in the vicinity of the glass transition point. Assoc. Asphalt Paving Technol. Proc. 36:80-105.
Nadeem S, Akram S (2010). Peristaltic flow of a Jeffrey fluid in a rectangular duct. Nonlinear Anal.:Real World Appl. 11:4238-4247.
Nazar M, Shahid F, Akram MS, Sultan Q (2012). Flow on oscillating rectangular duct for Maxwell fluid. App. Math. Mech. 33(6):717-730.
Peltier WR, Wu P, Yuen DA (1981). The viscosities of the earth mantle, in:F.D. Stacey, M.S. Paterson, A. Nicholas (Eds.), An elasticity in the Earth. American Geophysical Union, Colorado.
Quintanilla R, Rajagopal KR (2006), On Burgers' fluids. Math. Meth. Appl. Sci. 29:2133-2147.
Rajagopal KR (1982). A note on unsteady unidirectional flows of a non-Newtonian fluid. Int. J. Non-Linear Mech. 17:369-373.
Rajagopal KR (1984). On the creeping flow of the second-order fluid. J. Non-Newtonian Fluid Mech. 15:239-246.
Ravindran P, Krishnan JM, Rajagopal KR (2004). A note on the flow of a Burgers' fluid in an orthogonal rheometer. Int. J. Eng. Sci. 42:1973-1985.
Salah F, Aziz ZA, Ching DLC (2011). Accelerated flows of a magnetohydrodynamic (MHD) second grade fluid over an oscillating plate in a porous medium and rotating frame. Int. J. Phy. Sci. 6(36):8027-8035.
Schuchkin V, Osipov M, Shyy W, Thakur S (2002). Mixing and film cooling in supersonic duct flows. Int. J. Heat Mass Transf. 45:4451-4461.
Teipel I (1981). The impulsive motion of a flat plate in a viscoelastic fluid. Acta Mech. 39:277-279.