International Journal of Physical Sciences Vol. 7(47), pp. 6135-6143, 16 December, 2012
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS12.560
ISSN 1992 - 1950 ©2012 Academic Journals

Full Length Research Paper

Translation surfaces according to Frenet frame in Minkowski 3-space

Muhammed Çetin[1]*, Hüseyin Kocayiğit[2] and Mehmet Önder[2]

[1]Instutition of Science and Technology, Celal Bayar University, Manisa, Turkey.
[2]Department of Mathematics, Faculty of Arts and Science, Celal Bayar University, Manisa, Turkey.

Accepted 23 November, 2012

In this paper, using non-planar space curves, the translation surfaces were investigated according to Frenet frames in Minkowski 3-space and some properties of these surfaces were given. Furthermore, we calculated first fundamental form, second fundamental form, Gaussian curvature and mean curvature of the translation surface. Also, the Darboux frame of the generator curves of the translation surfaces in Minkowski 3-space was given. Finally, we gave the conditions of being a geodesic, an asymptotic line and a principal line for the generator curves of the translation surface.

Key words: Darboux frame, Minkowski 3-space, translation surface.

INTRODUCTION

The theory of translation surfaces is always one of the interesting topics in mathematics. Translation surfaces have been investigated by some differential geometers. Verstraelen et al. (1994) had studied minimal translation surfaces in n-dimensional Euclidean spaces. Liu (1999) had given the classification of the translation surfaces with constant mean curvature or constant Gaussian curvature in 3-dimensional Euclidean space and 3-dimensional Minkowski space. Yoon (2002) had studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map G satisfies the condition $\Delta G = AG$, $A \in Mat(3,R)$ where Δ denotes the Laplacian of the surface with respect to the induced metric and $Mat(3,R)$ the set of $3x3$ real matrices. Munteanu and Nistor (2011) studied the second fundamental form of translation surfaces in E^3. They had given a non-existence result for polynominal translation surfaces in E^3 with vanishing the second Gaussian curvature K_{II}. They classified those translation surfaces for which K_{II} and H are proportional. Baba-Hamed et al. (2010) studied the translation surfaces in the 3-dimensional Lorentz-Minkowski space under the condition $\Delta r_i = \lambda_i r_i$ where $\lambda_i \in R$ and Δ denotes the Laplace operator with respect to the first fundamental form and they obtained the complete classification theorems for those ones. They also gave explicit forms of these surfaces. Çetin et al. (2011) investigated the translation surfaces in 3-dimensional Euclidean space by using non-planar space curves and they gave the differential geometric properties for both translation surfaces and minimal translation surfaces.

In this paper, by using non-planar curves and their Frenet frames, we studied the translation surface in E_1^3. We gave some differential geometric properties of the translation surfaces. Also, we gave the Darboux frame of the generator curves of the translation surfaces in Minkowski 3-space.

PRELIMINARIES

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by $\langle , \rangle = -x_1{}^2 + x_2{}^2 + x_3{}^2$ where $\{x_1, x_2, x_3\}$ is a rectangular coordinate system of E_1^3. Since $\langle , \rangle$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian causal characters: it can be spacelike, if

*Corresponding author. E-mail: mat.mcetin@hotmail.com.

$\langle v,v\rangle > 0$ or $v=0$, timelike, if $\langle v,v\rangle < 0$ and null (lightlike), if $\langle v,v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha=\alpha(s)$ in E_1^3 can locally be spacelike, timelike, or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike, or null (lightlike) (O'Neill, 1983).

Let $\alpha(s)$ be a regular curve in Minkowski 3-space. Denote by $\{T,N,B\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^3. If α is a timelike curve, then the Frenet formulae were given by:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (1)$$

where, $\langle T,T\rangle = -1$, $\langle N,N\rangle = 1$, $\langle B,B\rangle = 1$, $\langle T,N\rangle = \langle N,B\rangle = \langle T,B\rangle = 0$. For an arbitrary spacelike curve $\alpha(s)$ in the space E_1^3, the following Frenet formulae were given by:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\varepsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (2)$$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = \varepsilon = \pm 1$, $\langle B,B\rangle = -\varepsilon$, $\langle T,N\rangle = \langle N,B\rangle = \langle T,B\rangle = 0$, and k_1 and k_2 are curvature and torsion of the spacelike curve $\alpha(s)$, respectively. Here, ε determines the kind of spacelike curve $\alpha(s)$. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal N and timelike binormal B. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal N and spacelike binormal B (Walrave, 1995).

Definition 1

A timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively.

Definition 2

Hyperbolic angle

Let x and y be future pointing (or past pointing) timelike vectors in R_1^3. Then, there is a unique real number $\theta \geq 0$ such that $\langle x,y\rangle = -|x||y|\cosh\theta$. This number is called the hyperbolic angle between the vectors x and y.

Central angle

Let x and y be spacelike vectors in R_1^3 that span a timelike vector subspace. Then, there is a unique real number $\theta \geq 0$ such that $\langle x,y\rangle = |x||y|\cosh\theta$. This number is called the central angle between the vectors x and y.

Spacelike angle

Let x and y be spacelike vectors in R_1^3 that span a spacelike vector subspace. Then, there is a unique real number $\theta \geq 0$ such that $\langle x,y\rangle = |x||y|\cos\theta$. This number is called the spacelike angle between the vectors x and y.

Lorentzian timelike angle

Let x be a spacelike vector and y be a timelike vector in R_1^3. Then there is a unique real number $\theta \geq 0$ such that $\langle x,y\rangle = |x||y|\sinh\theta$. This number is called the Lorentzian timelike angle between the vectors x and y (O'Neill, 1983).

Definition 3

A surface in the Minkowski 3-space R_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, that is, the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector (Beem and Ehrlich, 1981).

Let $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$ be vectors in Minkowski 3-space E_1^3. Then the scalar product of x and y is defined by:

$$\langle x,y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

Furthermore, the cross product of x and y is defined by:

$$x \times y = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2).$$

Let M be a non-null surface in E_1^3. The mean curvature H and the Gaussian curvature K are given by:

$$H = \frac{Gl + En - 2Fm}{2|EG - F^2|} \quad (3)$$

and

$$K=\langle U,U\rangle\frac{\ln-m^2}{EG-F^2} \quad (4)$$

respectively, where U is the unit normal vector field of the surface (Baba-Hamed et al., 2010). If the surface M is a spacelike surface, then the curve $\alpha(s)$ lying on M is a spacelike curve. If the surface M is a timelike surface, then the curve $\alpha(s)$ lying on M can be a spacelike or a timelike curve.

Since the curve $\alpha(s)$ lies on the surface M there exists another frame along the curve $\alpha(s)$ which is called Darboux frame and denoted by $\{T,g,U\}$. In this frame T is the unit tangent of the curve, U is the unit normal of the surface M along $\alpha(s)$ and g is a unit vector given by $g=U\wedge T$. Since the unit tangent T is common in both Frenet frame and Darboux frame, the vectors N,B,g, and U lie on the same plane.

If the surface M is an oriented timelike surface, then the curve $\alpha(s)$ lying on M is a timelike or a spacelike curve. So, the relations between these frames can be given as follows: If the curve $\alpha(s)$ is timelike, then:

$$\begin{bmatrix}T\\g\\U\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cos\gamma&\sin\gamma\\0&-\sin\gamma&\cos\gamma\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix} \quad (5)$$

and if the curve $\alpha(s)$ is spacelike, then

$$\begin{bmatrix}T\\g\\U\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cosh\gamma&\sinh\gamma\\0&\sinh\gamma&\cosh\gamma\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix}. \quad (6)$$

If the surface M is an oriented spacelike surface, then the curve $\alpha(s)$ lying on M is a spacelike curve. So, the relations between the frames can be given as follows:

$$\begin{bmatrix}T\\g\\U\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cosh\gamma&\sinh\gamma\\0&\sinh\gamma&\cosh\gamma\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix} \quad (7)$$

where γ is the angle between the vectors g and N.

According to Lorentzian causal characters of the surface M and the curve $\alpha(s)$ lying on M, the derivative formulae of the Darboux frame can be changed as follows: If the surface M is a timelike surface, then the curve $\alpha(s)$ lying on M can be a spacelike or a timelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by:

$$\begin{bmatrix}T'\\g'\\U'\end{bmatrix}=\begin{bmatrix}0&\kappa_g&-\varepsilon\kappa_n\\\kappa_g&0&\varepsilon\tau_g\\\kappa_n&\tau_g&0\end{bmatrix}\begin{bmatrix}T\\g\\U\end{bmatrix} \quad (8)$$

where $\langle T,T\rangle=\varepsilon=\pm1$, $\langle g,g\rangle=-\varepsilon$, $\langle U,U\rangle=1$.

If the surface M is a spacelike surface, then the curve $\alpha(s)$ lying on M is a spacelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by:

$$\begin{bmatrix}T'\\g'\\U'\end{bmatrix}=\begin{bmatrix}0&\kappa_g&\kappa_n\\-\kappa_g&0&\tau_g\\\kappa_n&\tau_g&0\end{bmatrix}\begin{bmatrix}T\\g\\U\end{bmatrix} \quad (9)$$

where $\langle T,T\rangle=1$, $\langle g,g\rangle=1$, $\langle U,U\rangle=-1$. In this formulae κ_g,κ_n and τ_g are called the geodesic curvature, the normal curvature, and the geodesic torsion, respectively.

In the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface M, the followings are well-known:

i) $\alpha(s)$ is a geodesic curve $\Leftrightarrow \kappa_g=0$,

ii) $\alpha(s)$ is an asymptotic line $\Leftrightarrow \kappa_n=0$,

iii) $\alpha(s)$ is principal line $\Leftrightarrow \tau_g=0$ (O'Neill, 1966).

TRANSLATION SURFACES WITH SPACE CURVES IN MINKOWSKI 3-SPACE

Here, we investigated the translation surfaces according to Frenet frame in Minkowski 3-space. So, we gave fundamental forms, Gaussian curvature, and mean curvature.

The translation surface M determined by curves $\alpha,\beta:(a,b)\to R$ is the patch

$$M(u,v)=\alpha(u)+\beta(v).$$

It is the surface formed by moving α parallel to itself in such a way that a point of the curve moves along β (Gray, 1998).

Let $\{T_\alpha,N_\alpha,B_\alpha\}$ be the Frenet frame field of $\alpha(u)$ with curvature k_1^α and torsion k_2^α. Also, let $\{T_\beta,N_\beta,B_\beta\}$ be the Frenet frame field of $\beta(v)$ with curvature k_1^β and torsion k_2^β.

A surface that can be generated from two space curves by translating either one of them parallel to itself in such a way that each of its points describes a curve that is a translation of the other curve. For the surface M, there are two cases; first one is that both the surface M and the generator curves $\alpha(u)$ and $\beta(v)$ of M are spacelike. The second case is that the surface M is timelike, so the generator curves $\alpha(u)$ and $\beta(v)$ of M can be timelike or spacelike.

The spacelike translation surfaces

Let $M(u,v)$ be a spacelike translation surface. Then, the generator curves $\alpha(u)$ and $\beta(v)$ of $M(u,v)$ are spacelike curves. So, we can give the following cases:

Case 1

Let $\alpha(u)$ is a spacelike curve with spacelike binormal and $\beta(v)$ is a spacelike curve with spacelike binormal, so there exist the following equalities, and the unit normal of the surface can be defined by:

$$U(u,v)=\frac{1}{\sin\varphi}T_\alpha\wedge T_\beta$$

where $\varphi(u)$ is the angle between tangent vectors of $\alpha(u)$ and $\beta(v)$. The first fundamental form I of the surface is defined by:

$$I=Edu^2+2Fdudv+Gdv^2$$

where $E=1$, $F=\cos\varphi$ and $G=1$ are the coefficients of I. Then,

$$I=du^2+2\cos\varphi dudv+dv^2.$$

The second fundamental form II of the surface is defined by:

$$II=ldu^2+2mdudv+ndv^2$$

where $l=-k_1^\alpha\cosh\theta_\alpha$, $m=0$ and $n=-k_1^\beta\cosh\theta_\beta$ are the coefficients of II. Also θ_α and θ_β are the angle between U and N_α, N_β, respectively. Then,

$$II=-k_1^\alpha\cosh\theta_\alpha du^2-k_1^\beta\cosh\theta_\beta dv^2.$$

On the other hand the Gaussian curvature K and mean curvature H of the surface are:

$$K=\frac{-k_1^\alpha k_1^\beta\cosh\theta_\alpha\cosh\theta_\beta}{\sin^2\varphi}$$

and

$$H=\frac{-k_1^\alpha\cosh\theta_\alpha-k_1^\beta\cosh\theta_\beta}{2\sin^2\varphi}$$

respectively.

By considering the similar calculations and Lorentzian casual characters of the curves, we can give the followings:

Case 2

Let $\alpha(u)$ is a spacelike curve with spacelike binormal and $\beta(v)$ is a spacelike curve with spacelike principal normal, so there exist following equalities:

$$I=du^2+2\cos\varphi dudv+dv^2$$
$$II=-k_1^\alpha\cosh\theta_\alpha du^2+k_1^\beta\sinh\theta_\beta dv^2$$

$$K=\frac{k_1^\alpha k_1^\beta\cosh\theta_\alpha\sinh\theta_\beta}{\sin^2\varphi}\quad H=\frac{-k_1^\alpha\cosh\theta_\alpha+k_1^\beta\sinh\theta_\beta}{2\sin^2\varphi}.$$

Case 3

Let $\alpha(u)$ be a spacelike curve with spacelike principal normal and $\beta(v)$ be a spacelike curve with spacelike binormal, so there exist the following equalities:

$$I=du^2+2\cos\varphi dudv+dv^2$$
$$II=k_1^\alpha\sinh\theta_\alpha du^2-k_1^\beta\cosh\theta_\beta dv^2$$

$$K=\frac{k_1^\alpha k_1^\beta\sinh\theta_\alpha\cosh\theta_\beta}{\sin^2\varphi}\quad H=\frac{k_1^\alpha\sinh\theta_\alpha-k_1^\beta\cosh\theta_\beta}{2\sin^2\varphi}.$$

Case 4

Let $\alpha(u)$ is a spacelike curve with spacelike principal normal and $\beta(v)$ is a spacelike curve with spacelike principal normal, so there exist following equalities:

$$I=du^2+2\cos\varphi dudv+dv^2$$
$$II=k_1^\alpha\sinh\theta_\alpha du^2+k_1^\beta\sinh\theta_\beta dv^2$$
$$K=\frac{-k_1^\alpha k_1^\beta\sinh\theta_\alpha\sinh\theta_\beta}{\sin^2\varphi}\quad H=\frac{k_1^\alpha\sinh\theta_\alpha+k_1^\beta\sinh\theta_\beta}{2\sin^2\varphi}$$

The timelike translation surfaces

Let $M(u,v)$ be a timelike translation surface. Then, the generator curves $\alpha(u)$ and $\beta(v)$ of $M(u,v)$ can be timelike or spacelike. So, we can give the following cases for the timelike translation surface M :

Case 1

Let both $\alpha(u)$ and $\beta(v)$ are timelike curves, so there exist following equalities.

The unit normal of the surface can be defined by:

$$U(u,v)=\frac{1}{\sinh\varphi}T_\alpha\wedge T_\beta$$

where $\varphi(u)$ is the angle between tangent vectors of $\alpha(u)$ and $\beta(v)$. The first fundamental form I of the surface is defined by:

$$I=Edu^2+2Fdudv+Gdv^2$$

where $E=-1$, $F=-\cosh\varphi$ and $G=-1$ are the coefficients of I. Then,

$$I=-du^2-2\cosh\varphi dudv-dv^2 .$$

The second fundamental form II of the surface is defined by:

$$II=ldu^2+2mdudv+ndv^2$$

Where $l=k_1^\alpha\cosh\theta_\alpha$, $m=0$ and $n=k_1^\beta\cosh\theta_\beta$ are the coefficients of II. Also θ_α and θ_β are the angle between U and N_α, N_β, respectively. Then,

$$II=k_1^\alpha\cosh\theta_\alpha du^2+k_1^\beta\cosh\theta_\beta dv^2 .$$

On the other hand the Gaussian curvature K and mean curvature H of the surface are:

$$K=\frac{-k_1^\alpha k_1^\beta\cosh\theta_\alpha\cosh\theta_\beta}{\sinh^2\varphi}$$

and

$$H=\frac{-k_1^\alpha\cosh\theta_\alpha-k_1^\beta\cosh\theta_\beta}{2\sinh^2\varphi}$$

respectively.

By considering the similar calculations and Lorentzian casual characters of the curves we can give the followings:

Case 2

Let $\alpha(u)$ be a spacelike curve with spacelike binormal and $\beta(v)$ be a spacelike curve with spacelike binormal, so there exist the following equalities:

$$U(u,v)=\frac{1}{\sinh\varphi}T_\alpha\wedge T_\beta$$

$$I=du^2+2\cosh\varphi dudv+dv^2$$
$$II=k_1^\alpha\sinh\theta_\alpha du^2+k_1^\beta\sinh\theta_\beta dv^2$$

$$K=\frac{-k_1^\alpha k_1^\beta\sinh\theta_\alpha\sinh\theta_\beta}{\sinh^2\varphi}\quad H=\frac{k_1^\alpha\sinh\theta_\alpha+k_1^\beta\sinh\theta_\beta}{2\sinh^2\varphi} .$$

Case 3

Let $\alpha(u)$ be a spacelike curve with spacelike binormal and $\beta(v)$ be a spacelike curve with spacelike principal normal, so there exist the following equalities:

$$U(u,v)=\frac{1}{\sinh\varphi}T_\alpha\wedge T_\beta$$

$$I=du^2+2\cosh\varphi dudv+dv^2$$
$$II=k_1^\alpha\sinh\theta_\alpha du^2+k_1^\beta\cosh\theta_\beta dv^2$$

$$K=\frac{-k_1^\alpha k_1^\beta\sinh\theta_\alpha\cosh\theta_\beta}{\sinh^2\varphi}\quad H=\frac{k_1^\alpha\sinh\theta_\alpha+k_1^\beta\cosh\theta_\beta}{2\sinh^2\varphi} .$$

Case 4

Let $\alpha(u)$ be a spacelike curve with spacelike principal normal and $\beta(v)$ be a spacelike curve with spacelike binormal, so there exist the following equalities:

$$U(u,v)=\frac{1}{\sinh\varphi}T_\alpha\wedge T_\beta$$

$$I=du^2+2\cosh\varphi dudv+dv^2$$
$$II=k_1^\alpha\cosh\theta_\alpha du^2+k_1^\beta\sinh\theta_\beta dv^2$$

$$K=\frac{-k_1^\alpha k_1^\beta\cosh\theta_\alpha\sinh\theta_\beta}{\sinh^2\varphi}\quad H=\frac{k_1^\alpha\cosh\theta_\alpha+k_1^\beta\sinh\theta_\beta}{2\sinh^2\varphi} .$$

Case 5

Let $\alpha(u)$ be a spacelike curve with spacelike principal normal and $\beta(v)$ be a spacelike curve with spacelike principal normal, so there exist the following equalities:

$$U(u,v)=\frac{1}{\sinh\varphi}T_\alpha\wedge T_\beta$$

$$I=du^2+2\cosh\varphi dudv+dv^2$$
$$II=k_1^\alpha\cosh\theta_\alpha du^2+k_1^\beta\cosh\theta_\beta dv^2$$

$$K=\frac{-k_1^\alpha k_1^\beta\cosh\theta_\alpha\cosh\theta_\beta}{\sinh^2\varphi}\quad H=\frac{k_1^\alpha\cosh\theta_\alpha+k_1^\beta\cosh\theta_\beta}{2\sinh^2\varphi}.$$

Case 6

Let $\alpha(u)$ be a timelike curve and $\beta(v)$ be a spacelike curve with spacelike binormal, so there exist the following equalities:

$$U(u,v)=\frac{1}{\cosh\varphi}T_\alpha\wedge T_\beta$$

$$I=-du^2+2\sinh\varphi dudv+dv^2$$
$$II=k_1^\alpha\cosh\theta_\alpha du^2+k_1^\beta\sinh\theta_\beta dv^2$$

$$K=\frac{-k_1^\alpha k_1^\beta\cosh\theta_\alpha\cosh\theta_\beta}{\cosh^2\varphi}\quad H=\frac{k_1^\alpha\cosh\theta_\alpha-k_1^\beta\cosh\theta_\beta}{2\cosh^2\varphi}.$$

Case 7

Let $\alpha(u)$ be a timelike curve and $\beta(v)$ be a spacelike curve with spacelike principal normal, so there exist the following equalities:

$$U(u,v)=\frac{1}{\cosh\varphi}T_\alpha\wedge T_\beta$$

$$I=-du^2+2\sinh\varphi dudv+dv^2$$
$$II=k_1^\alpha\cosh\theta_\alpha du^2+k_1^\beta\cosh\theta_\beta dv^2$$

$$K=\frac{-k_1^\alpha k_1^\beta\cosh\theta_\alpha\cosh\theta_\beta}{\cosh^2\varphi}\quad H=\frac{k_1^\alpha\cosh\theta_\alpha-k_1^\beta\cosh\theta_\beta}{2\cosh^2\varphi}.$$

Case 8

Let $\alpha(u)$ be a spacelike curve with spacelike binormal and $\beta(v)$ be a timelike curve, so there exist the following equalities:

$$U(u,v)=\frac{1}{\cosh\varphi}T_\alpha\wedge T_\beta$$

$$I=du^2+2\sinh\varphi dudv-dv^2$$
$$II=k_1^\alpha\sinh\theta_\alpha du^2+k_1^\beta\cosh\theta_\beta dv^2$$
$$K=\frac{-k_1^\alpha k_1^\beta\sinh\theta_\alpha\cosh\theta_\beta}{\cosh^2\varphi}\quad H=\frac{-k_1^\alpha\sinh\theta_\alpha+k_1^\beta\cosh\theta_\beta}{2\cosh^2\varphi}.$$

Case 9

Let $\alpha(u)$ be a spacelike curve with spacelike principal normal and $\beta(v)$ be a timelike curve, so there exist the following equalities:

$$U(u,v)=\frac{1}{\cosh\varphi}T_\alpha\wedge T_\beta$$

$$I=du^2+2\sinh\varphi dudv-dv^2$$
$$II=k_1^\alpha\cosh\theta_\alpha du^2+k_1^\beta\cosh\theta_\beta dv^2$$

$$K=\frac{-k_1^\alpha k_1^\beta\cosh\theta_\alpha\cosh\theta_\beta}{\cosh^2\varphi}\quad H=\frac{-k_1^\alpha\cosh\theta_\alpha+k_1^\beta\cosh\theta_\beta}{2\cosh^2\varphi}$$

DARBOUX FRAME OF THE GENERATOR CURVES

Here, we investigate Darboux frame of the generator curves of the translation surface. There exist two cases; first one is that, both the surface M, the generator curves $\alpha(u)$, and $\beta(v)$ of M are spacelike. The second case is that, the surface M is timelike, so the generator curves $\alpha(u)$ and $\beta(v)$ of M can be timelike or spacelike.

Case 1

If the surface M is spacelike, then the curve $\alpha(u)$ and $\beta(v)$ are spacelike.

Case a

Let $\alpha(u)$ be a spacelike curve with spacelike principal normal.

From Equation 7, we can write for the curve $\alpha(u)$ as follows:

$$g_\alpha = \cosh\gamma_\alpha N_\alpha + \sinh\gamma_\alpha B_\alpha \quad (10)$$

$$U = \sinh\gamma_\alpha N_\alpha + \cosh\gamma_\alpha B_\alpha \quad (11)$$

where γ_α is the angle between g_α and N_α. Differentiating Equation 10 with respect to u we have:

$$g_\alpha' = -k_1^\alpha \cosh\gamma_\alpha T_\alpha + (\gamma_\alpha' + k_2^\alpha)\sinh\gamma_\alpha N_\alpha + (\gamma_\alpha' + k_2^\alpha)\cosh\gamma_\alpha B_\alpha.$$

From Equation 9, we can write for the curve $\alpha(u)$ as follows:

$$T_\alpha' = \kappa_g^\alpha g_\alpha + \kappa_n^\alpha U \quad (12)$$

$$g_\alpha' = -\kappa_g^\alpha T_\alpha + \tau_g^\alpha U \quad (13)$$

$$U' = \kappa_n^\alpha T_\alpha + \tau_g^\alpha g_\alpha.$$

Taking the inner product of Equation 12 with g_α, we get:

$$\kappa_g^\alpha = k_1^\alpha \cosh\gamma_\alpha. \quad (14)$$

Taking the inner product of Equation 12 with U, we get:

$$\kappa_n^\alpha = -k_1^\alpha \sinh\gamma_\alpha. \quad (15)$$

Taking the inner product of Equation 13 with U, we get:

$$\tau_g^\alpha = \gamma_\alpha' + k_2^\alpha. \quad (16)$$

Case b

Let $\alpha(u)$ be a spacelike curve with spacelike binormal.
From Equation 7, we can write for the curve $\alpha(u)$ as follows:

$$g_\alpha = \cosh\gamma_\alpha N_\alpha + \sinh\gamma_\alpha B_\alpha \quad (17)$$

$$U = \sinh\gamma_\alpha N_\alpha + \cosh\gamma_\alpha B_\alpha \quad (18)$$

where γ_α is the angle between g_α and N_α. Differentiating Equation 17 with respect to u we have:

$$g_\alpha' = k_1^\alpha \cosh\gamma_\alpha T_\alpha + (\gamma_\alpha' + k_2^\alpha)\sinh\gamma_\alpha N_\alpha + (\gamma_\alpha' + k_2^\alpha)\cosh\gamma_\alpha B_\alpha.$$

From Equation 9, we can write for the curve $\alpha(u)$ as follows:

$$T_\alpha' = \kappa_g^\alpha g_\alpha + \kappa_n^\alpha U \quad (19)$$

$$g_\alpha' = -\kappa_g^\alpha T_\alpha + \tau_g^\alpha U \quad (20)$$

$$U' = \kappa_n^\alpha T_\alpha + \tau_g^\alpha g_\alpha.$$

Taking the inner product of Equation 19 with g_α, we get:

$$\kappa_g^\alpha = -k_1^\alpha \cosh\gamma_\alpha. \quad (21)$$

Taking the inner product of Equation 19 with U, we get:

$$\kappa_n^\alpha = k_1^\alpha \sinh\gamma_\alpha. \quad (22)$$

Taking the inner product of Equation 20 with U, we get:

$$\tau_g^\alpha = -(\gamma_\alpha' + k_2^\alpha). \quad (23)$$

Case 2

If the surface M is a timelike, then the curve $\alpha(u)$ and $\beta(v)$ can be timelike or spacelike.

Case a

Let $\alpha(u)$ be a timelike curve.
From Equation 5, we can write for the curve $\alpha(u)$ as follows:

$$g_\alpha = \cos\gamma_\alpha N_\alpha + \sin\gamma_\alpha B_\alpha \quad (24)$$

$$U = -\sin\gamma_\alpha N_\alpha + \cos\gamma_\alpha B_\alpha \quad (25)$$

where γ_α is the angle between g_α and N_α. Differentiating Equation 24 with respect to u we have:

$$g_\alpha' = k_1^\alpha \cos\gamma_\alpha T_\alpha - (\gamma_\alpha' + k_2^\alpha)\sin\gamma_\alpha N_\alpha + (\gamma_\alpha' + k_2^\alpha)\cos\gamma_\alpha B_\alpha.$$

From Equation 8, we can write for the curve $\alpha(u)$ as follows.

$$T_\alpha' = \kappa_g^\alpha g_\alpha + \kappa_n^\alpha U \quad (26)$$

$$g_\alpha' = \kappa_g^\alpha T_\alpha - \tau_g^\alpha U \tag{27}$$

$$U' = \kappa_n^\alpha T_\alpha + \tau_g^\alpha g_\alpha .$$

Taking the inner product of Equation 26 with g_α, we get:

$$\kappa_g^\alpha = k_1^\alpha \cos\gamma_\alpha . \tag{28}$$

Taking the inner product of Equation 26 with U, we get:

$$\kappa_n^\alpha = -k_1^\alpha \sin\gamma_\alpha . \tag{29}$$

Taking the inner product of Equation 27 with U, we get:

$$\tau_g^\alpha = -(\gamma_\alpha' + k_2^\alpha) . \tag{30}$$

Case b

Let $\alpha(u)$ be a spacelike curve with spacelike principal normal. From Equation 6, we can write for the curve $\alpha(u)$ as follows:

$$g_\alpha = \cosh\gamma_\alpha N_\alpha + \sinh\gamma_\alpha B_\alpha \tag{31}$$

$$U = \sinh\gamma_\alpha N_\alpha + \cosh\gamma_\alpha B_\alpha \tag{32}$$

where γ_α is the angle between g_α and N_α. Differentiating Equation 31 with respect to u we have:

$$\begin{aligned} g_\alpha' = &-k_1^\alpha \cosh\gamma_\alpha T_\alpha + (\gamma_\alpha' + k_2^\alpha)\sinh\gamma_\alpha N_\alpha \\ &+ (\gamma_\alpha' + k_2^\alpha)\cosh\gamma_\alpha B_\alpha \end{aligned} .$$

From Equation 8, we can write for the curve $\alpha(u)$ as follows:

$$T_\alpha' = \kappa_g^\alpha g_\alpha - \kappa_n^\alpha U \tag{33}$$

$$g_\alpha' = \kappa_g^\alpha T_\alpha + \tau_g^\alpha U \tag{34}$$

$$U' = \kappa_n^\alpha T_\alpha + \tau_g^\alpha g_\alpha .$$

Taking the inner product of Equation 33 with g_α, we get:

$$\kappa_g^\alpha = -k_1^\alpha \cosh\gamma_\alpha . \tag{35}$$

Taking the inner product of Equation 33 with U, we get:

$$\kappa_n^\alpha = -k_1^\alpha \sinh\gamma_\alpha . \tag{36}$$

Taking the inner product of Equation 34 with U, we get:

$$\tau_g^\alpha = -(\gamma_\alpha' + k_2^\alpha) . \tag{37}$$

Case c

Let $\alpha(u)$ be a spacelike curve with spacelike binormal. From Equation 6, we can write for the curve $\alpha(u)$ as follows:

$$g_\alpha = \cosh\gamma_\alpha N_\alpha + \sinh\gamma_\alpha B_\alpha \tag{38}$$

$$U = \sinh\gamma_\alpha N_\alpha + \cosh\gamma_\alpha B_\alpha \tag{39}$$

where γ_α is the angle between g_α and N_α. Differentiating Equation 38 with respect to u we have:

$$\begin{aligned} g_\alpha' = &k_1^\alpha \cosh\gamma_\alpha T_\alpha + (\gamma_\alpha' + k_2^\alpha)\sinh\gamma_\alpha N_\alpha \\ &+ (k_2^\alpha + \gamma_\alpha')\cosh\gamma_\alpha B_\alpha \end{aligned} .$$

From Equation 8, we can write for the curve $\alpha(u)$ as follows:

$$T_\alpha' = \kappa_g^\alpha g_\alpha + \kappa_n^\alpha U \tag{40}$$

$$g_\alpha' = \kappa_g^\alpha T_\alpha + \tau_g^\alpha U \tag{41}$$

$$U' = \kappa_n^\alpha T_\alpha + \tau_g^\alpha g_\alpha .$$

Taking the inner product of Equation 40 with g_α, we get:

$$\kappa_g^\alpha = k_1^\alpha \cosh\gamma_\alpha . \tag{42}$$

Taking the inner product of Equation 40 with U, we get:

$$\kappa_n^\alpha = -k_1^\alpha \sinh\gamma_\alpha . \tag{43}$$

Taking the inner product of Equation 41 with U, we get:

$$\tau_g^\alpha = \gamma_\alpha' + k_2^\alpha . \tag{44}$$

Theorem 1

Let $\alpha(u)$ be a space curve with non zero curvature. The curve $\alpha(u)$ is a geodesic curve if and only if $\alpha(u)$ is a timelike curve and, g_α and B_α are linear dependent.

Proof

Let $\alpha(u)$ be a space curve with non zero curvature. If $\alpha(u)$ is a geodesic curve, then $\kappa_g^\alpha = 0$. From Equations 14, 21, 28, 35, and 42 the curve $\alpha(u)$ must be timelike. Since, when $\alpha(u)$ is a spacelike curve, $\kappa_g^\alpha \neq 0$. If $\kappa_g^\alpha = 0$, $\cos\gamma_\alpha = 0$. This means that g_α is perpendicular to N_α. Then we can say that g_α and B_α are linear dependent.

If $\alpha(u)$ is a timelike curve and, g_α and B_α be linear dependent. So, g_α is perpendicular to N_α and $\cos\gamma_\alpha = 0$. Then $\kappa_g^\alpha = 0$. By the definition we can obtain that $\alpha(u)$ is a geodesic curve.

Theorem 2

Let $\alpha(u)$ be a space curve with non zero curvature. The curve $\alpha(u)$ is an asymptotic line if and only if g_α and N_α are linear dependent.

Proof

Let $\alpha(u)$ be a space curve with non zero curvature. If $\alpha(u)$ is an asymptotic line, by the definition $\kappa_n^\alpha = 0$. From Equations 15, 22, 29, 36, and 43 the angle γ_α must be zero. This means that g_α and N_α are linear dependent.

If g_α and N_α be linear dependent. We can say that $\sin\gamma_\alpha = 0$ and $\sinh\gamma_\alpha = 0$. Then $\kappa_n^\alpha = 0$. By the definition we can obtain that $\alpha(u)$ is an asymptotic line.

Theorem 3

Let $\alpha(u)$ be a space curve. The curve $\alpha(u)$ is a principal line if and only if $\gamma_\alpha' + k_2^\alpha = 0$.

Proof

Let $\alpha(u)$ be a space curve. If $\alpha(u)$ is a principal line, by the definition $\tau_g^\alpha = 0$. From Equations 16, 23, 30, 37, and 44 it can be written that $\gamma_\alpha' + k_2^\alpha = 0$.

When $\gamma_\alpha' + k_2^\alpha = 0$ and so $\tau_g^\alpha = 0$. This means that $\alpha(u)$ is a principal line.

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