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Full Length Research Paper

An analytical approach on a mass grounded by linear and nonlinear springs in series

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In this paper, He's variational approach method is used to solve large amplitude free vibration of a mass grounded by linear and nonlinear springs in series. The conservative oscillation system is formulated as a nonlinear ordinary differential equation having linear and nonlinear stiffness components. By doing some simple mathematical operations on this method, we can obtain their natural frequencies. The main objective of present study is to obtain highly accurate analytical solution, which is valid for whole solution domain, for free vibration of a conservative oscillator with inertia and static type cubic nonlinearities. Other numerical results are finally presented and discussed to validate the present analysis.

Key words: Variational approach method, nonlinear oscillation, analytical method.

INTRODUCTION

Many physical phenomena can be parted into linear or nonlinear according to the type of differential equations of motion. The linearity, nonlinearity or exponential form of a conservative system can be essentially determined by the algebraic relationship between restoring forces and displacement/deflections. Many practical engineering components can be modeled utilizing oscillatory systems such as elastic beams supported by two springs or mass on-moving belt or nonlinear pendulum and vibration of a milling machine (Dimarogonas and Haddad, 1992; Ganji et al., 2009). Due to many usages of "two degree of freedom systems" some of which were discussed above, solving the equations of motion for a mechanical system associated with linear and nonlinear properties was attempted through transformation into a set of differential algebraic equations using intermediate variables; which are introduced here to transform the equations of motion for a TDOF system into the Duffing equation (Telli and Kopmaz, 2006). Many analytical and numerical approaches have been investigated due to the limitation of existing exact solutions and even if an exact solution is obtainable, the required calculations may be too complicated to be practical, or it might be difficult to interpret the outcome. Very recently, some promising approximate analytical solutions are proposed, such as Exp-function method (Mohyud-Din, 2010), Adomian decomposition method (Wazwaz, 2005), variational iteration method (Fouladi et al., 2010), homotopy-perturbation method (Bayat et al., 2010; Shaban et al., 2010), homotopy analysis method (Kimiaeifar et al., 2009a; Kimiaeifar et al., 2009b), Energy balance (Bayat et al., 2011), variational approach method (He, 2007; Liu, 2009; Wang, 2009), Newton–harmonic balancing (Lai et al., 2009), differential transformation (Omidvar et al., 2010), Max-Min approach (He, 2008), Hansan Sengunpta method (Uwamusi, 2009) and rational solution to cosmological puzzles (Yang, 2009).

In this study, He's variational approach method is used to find analytical solutions for nonlinear free vibrations of a mass grounded by linear and nonlinear springs. It is shown that the solutions are quickly convergent and their components can be simply calculated. The results of the VAM are compared with the numerical one, it can be

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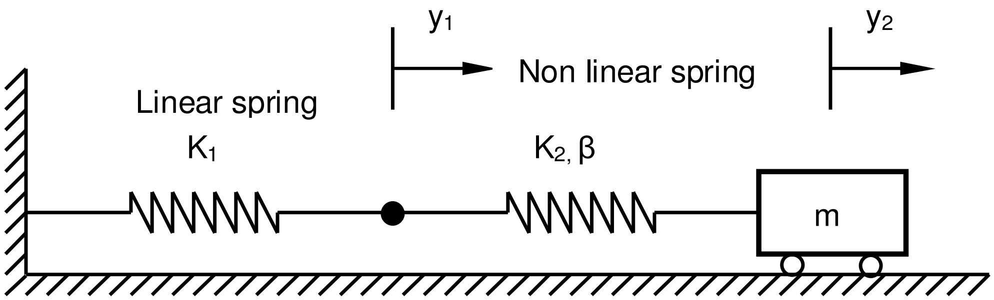

Figure 1. Nonlinear free vibration of a system of mass with serial linear and nonlinear stiffness on a frictionless contact surface.

observed that VAM is accurate and require smaller computational effort. An excellent accuracy of the VAM results indicates that this method can be used for problems in which the strong nonlinearities are taken into account.

GOVERNING EQUATION OF MOTION

System with linear and nonlinear springs in series

In this section, we will consider the system shown in Figure 1. k_2 and β describe the parameters of the second spring since it has a hardening/softening cubic nonlinear characteristic.

The following relationship represents the deflection of this spring and the force acting upon it

$$F_2 = k_2 x + \beta x^3 = k_2 x + \varepsilon k_2 x^3, \tag{1}$$

where k_2 and β are the coefficients associated with the linear and nonlinear portions of spring force, and ε is defined as;

$$\varepsilon = \frac{\beta}{k_2} \tag{2}$$

As it is shown in Figure 1, x is the net deflection of nonlinear spring and defined as

$$x = y_2 - y_1, \tag{3}$$

The case of $\beta > 0$ corresponds to a hardening spring while a negative β indicates a softening one. Here, it is assumed that $\beta > 0, \varepsilon < 1$ and ε will be employed as a perturbation or book keeping parameter. The equations of motion of the system in Figure 1 can easily be obtained as follows:

$$k_1 y_1 - k_2(y_2 - y_1) - \varepsilon k_2(y_2 - y_1)^3 = 0,$$

$$x = y_2 - y_1, \tag{4.a}$$

$$m \ddot{y}_2 + k_2(y_2 - y_1) + \varepsilon k_2(y_2 - y_1)^3 = 0. \tag{4.b}$$

Let the new (intermediate) variables u and υ be defined as follows:

$$y_1 := u, \tag{5.a}$$

Then, we can rewrite the Equation (4) in a different form

$$k_1 u - k_2 \upsilon - \varepsilon k_2 \upsilon^3 = 0, \tag{6.a}$$

$$m(\ddot{u} + \ddot{\upsilon}) + k_2 \upsilon + \varepsilon k_2 \upsilon^3 = 0. \tag{6.b}$$

Solving Equation (6.a) for u yields

$$u = \xi \upsilon + \varepsilon \xi \upsilon^3, \tag{7}$$

where

$$\xi = k_2 / k_1 \tag{8}$$

By differentiating twice with respect to time from Equation (7) and substituted into Equation (4.b) one finds;

$$m(1+\xi+3\varepsilon\xi\upsilon^2)\ddot{\upsilon} + 6m\varepsilon\xi\upsilon\dot{\upsilon}^2 + k_2\upsilon + \varepsilon k_2\upsilon^3 = 0, \tag{9}$$

The problem of solving Equations (4) is reduced to solving Equation (9). It is interesting to observe that a term proportional to velocity squared appears, suggesting that the system contains a dissipative element although this is not the case. Equation (9) can be considered as a kind of the Duffing equation whose mass and linear spring coefficients are time dependent. If one defines Equation (9) as an ordinary differential equation in υ,then analytical solutions to be presented in this section will also be in terms of υ.Therefore, it seems meaningful to give the initial conditions in υIt should be emphasized that any initial condition for υ leads to different initial values of y_1 and y_2.For a general initial condition $\upsilon(0)=\upsilon_0$,one finds;

$$u(0)=y_1(0)=\xi\,\upsilon_0(1+\varepsilon\,\xi\,\upsilon_0{}^2),\qquad (10)$$
$$y_2(0)=u(0)+\upsilon(0)=(1+\xi+\varepsilon\,\xi\,\upsilon_0{}^2)\,\upsilon_0.$$

Equation (10) relates the initial amount of relative variable υ to those of the original ones. In this study, it is assumed that $\dot{y}_1(0)=\dot{y}_2(0)=0 \longrightarrow \dot{\upsilon}(0)=\dot{\upsilon}_0=0$ and $\upsilon(0)=\upsilon_0=A$ for all solutions, either the analytical or the numerical.

Equation (9) can be written as;

$$(1+3\varepsilon\eta\,\upsilon^2)\,\ddot{\upsilon}+6\,\varepsilon\eta\,\upsilon\dot{\upsilon}^2+\omega_e{}^2\upsilon+\varepsilon\,\omega_e{}^2\upsilon^3=0, \qquad (11)$$

where

$$\omega_e{}^2=\frac{k_2}{m(1+\xi)},\ \eta=\frac{\xi}{1+\xi}. \qquad (12)$$

SOLUTION PROCEDURES

Basic concept of VAM

He suggested a variational approach which is different from the known variational methods in open literature (He, 2007). Hereby we give a brief introduction of the method:

$$u''+f(u)=0 \qquad (13)$$

Its variational principle can be easily established utilizing the semi-inverse method (He, 2007):

$$J(u)=\int_0^{T/4}\left(-\frac{1}{2}u'^2+F(u)\right)dt \qquad (14)$$

where T is period of the nonlinear oscillator, $\partial F/\partial u=f$ .Assume that its solution can be expressed as

$$u(t)=A\cos(\omega t) \qquad (15)$$

where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Equation (15) into Equation (14) results in:

$$\begin{aligned}J(A,\omega)&=\int_0^{T/4}\left(-\frac{1}{2}A^2\omega^2\sin^2\omega t+F(A\cos\omega t)\right)dt \qquad (16)\\ &=\frac{1}{\omega}\int_0^{\pi/2}\left(-\frac{1}{2}A^2\omega^2\sin^2\omega t+F(A\cos\omega t)\right)dt\\ &=-\frac{1}{2}A^2\omega\int_0^{\pi/2}\sin^2 t\,dt+\frac{1}{\omega}\int_0^{\pi/2}F(A\cos\omega t)dt\end{aligned}$$

Applying the Ritz method, we require:

$$\frac{\partial J}{\partial A}=0 \qquad (17)$$

$$\frac{\partial J}{\partial \omega}=0 \qquad (18)$$

But with a careful inspection, for most cases we find that

$$\frac{\partial J}{\partial\omega}=-\frac{1}{2}A^2\int_0^{\pi/2}\sin^2 t\,dt-\frac{1}{\omega^2}\int_0^{\pi/2}F(A\cos\omega t)dt<0 \qquad (19)$$

Thus, we modify conditions Equation (17) and Equation (18) into a simpler form:

$$\frac{\partial J}{\partial \omega}=0 \qquad (20)$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

APPLICATION OF SOLUTION PROCEDURES

Applying VAM

In Equation (9), its variational principle can be easily obtained:

$$J(u)=\int_0^t\left(-\frac{1}{2}\dot{u}^2\left(1+\frac{3}{2}\varepsilon\eta u^2\right)+\omega_0{}^2\left(\frac{1}{2}u^2+\frac{1}{4}\varepsilon u^4\right)\right)dt \qquad (21)$$

Choosing the trial function $u(t)=A\cos(\omega t)$ into Equation (21) we obtain:

$$J(A)=\int_0^{T/4}\begin{pmatrix}-\frac{1}{2}A^2\omega^2\sin^2\omega t\left(1+\frac{3}{2}\varepsilon\eta A^2\cos^2\omega t\right)\\ +\omega_0^2\left(\frac{1}{2}A^2\cos^2\omega t+\frac{1}{4}\varepsilon A^4\cos^4\omega t\right)\end{pmatrix}dt \qquad (22)$$

The stationary condition with respect to A leads to:

$$\begin{aligned}\frac{\partial J}{\partial A}&=\int_0^{T/4}\begin{pmatrix}-\frac{1}{2}A^2\omega^2\sin^2\omega t\left(1+\frac{3}{2}\varepsilon\eta A^2\cos^2\omega t\right)\\ +\omega_0^2\left(\frac{1}{2}A^2\cos^2\omega t+\frac{1}{4}\varepsilon A^4\cos^4\omega t\right)\end{pmatrix}dt=0 \qquad (23)\\ &=\int_0^{\pi/2}\begin{pmatrix}-A\omega^2\sin^2 t\left(1+3\varepsilon\eta A^2\cos^2 t\right)\\ +\omega_0^2\left(A\cos^2 t+\varepsilon A^3\cos^4 t\right)\end{pmatrix}dt=0\\ &=-\omega^2\int_0^{\pi/2}(\sin^2 t+3\varepsilon\eta A^2\cos^2 t)\,dt\\ &\quad+\omega_0^2\left(A\int_0^{\pi/2}\cos^2 t\,dt+\varepsilon A^3\int_0^{\pi/2}\cos^4 t\,dt\right)=0\end{aligned}$$

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Table 1. Comparison of frequency corresponding to various parameters of system.

Constant parameter							Relative error (%)
m	A	ε	k_1	k_2	ω_{VAM}	Rung-Kotta (Telli and Kompmaz, 2006)	$\frac{\omega_{VAM} - \omega_{NS}}{\omega_{NS}}$
1	0.5	0.5	50	5	2.220265	2.220231	0.00153
1	0.5	0.5	50	5	3.162277	3.175501	0.41644
1	2	0.5	5	5	1.889822	1.903569	0.72170
1	2	0.5	5	50	2.192645	2.195284	0.12021
3	5	1	8	16	1.612706	1.615107	0.14866
3	5	1	10	5	1.739775	1.749115	0.53398
5	10	2	12	16	1.545360	1.545853	0.03189
5	30	5	15	5	1.731282	1.731382	0.00
10	200	5	5	250	0.707107	0.707107	0.00
10	100	10	5	25	0.707106	0.707106	0.00
1	0.5	- 0.5	50	5	2.038315	2.038209	0.00520
2	2	-0.1	10	10	1.434860	1.446389	0.00520
3	4	-0.02	30	10	1.313064	1.318370	0.40247
4	10	-0.008	6	3	0.703731	0.705412	0.23830

Solving Equation (23), according to ω, we have

$$\omega^2 = \frac{\omega_0^2 (A \int_0^{\pi/2} \cos^2 t \; dt + \varepsilon A^3 \int_0^{\pi/2} \cos^4 t \, dt)}{\int_0^{\pi/2} (\sin^2 t + 3\varepsilon\eta A^2 \cos^2 t) \; dt}) \qquad (24)$$

Then we have

$$\omega_{VAM} = \frac{\omega_0 \sqrt{\left(4+3A^2\varepsilon\eta\right)\left(4+3A^2\varepsilon\right)}}{4+3A^2\varepsilon\eta}, \qquad (25)$$

According to Equations (15) and (25), we can obtain the following approximate solution:

$$v(t) = A\cos\left(\frac{\omega_0 \sqrt{\left(4+3A^2\varepsilon\eta\right)\left(4+3A^2\varepsilon\right)}}{4+3A^2\varepsilon\eta} t\right) \qquad (26)$$

The first-order analytical approximation for $u(t)$ is

$$u(t) = \xi\left(A\cos\left(\frac{\omega_0 \sqrt{\left(4+3A^2\varepsilon\eta\right)\left(4+3A^2\varepsilon\right)}}{4+3A^2\varepsilon\eta} t\right)\right) + \varepsilon\,\xi\left(A\cos\left(\frac{\omega_0 \sqrt{\left(4+3A^2\varepsilon\eta\right)\left(4+3A^2\varepsilon\right)}}{4+3A^2\varepsilon\eta} t\right)^3\right) \qquad (27)$$

Therefore, the first-order analytical approximate displacements $y_1(t)$ and $y_2(t)$ are

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$$y_1 = u(t)$$

$$y_2(t) = u(t) + A\cos\left(\frac{\omega_0 \sqrt{\left(4+3A^2\varepsilon\eta\right)\left(4+3A^2\varepsilon\right)}}{4+3A^2\varepsilon\eta} t\right) \qquad (28)$$

RESULTS AND DISCUSSION

(25)

To demonstrate the accuracy of the VAM, the procedures explained in previous sections are applied to obtain natural frequency and corresponding displacement of a mass grounded by linear and nonlinear springs in series. Comparisons of angular frequencies for different parameters via numerical is presented in Table1. The maximum relative error between the VAM results and numerical results is 0.72170%.To further illustrate and verify the accuracy of the presenting analytical approach, comparison of VAM numerical solution (Telli and Kopmaz, 2006) are presented in Figures 2 to 4 for $v(t)$ .The effect of amplitude A has been studied in Figure (5).A comparison of the time history oscillatory displacement response for the a mass with numerical solutions is presented in Figures 6 to 7 for $y_1(t)$ and Figures 8 to 9 for $y_2(t)$. As shown in Figures 2 to 9, it is apparent that the Variational Approach Method has an excellent agreement with the numerical solution using Rung-Kotta and these expressions are valid for a wide range.

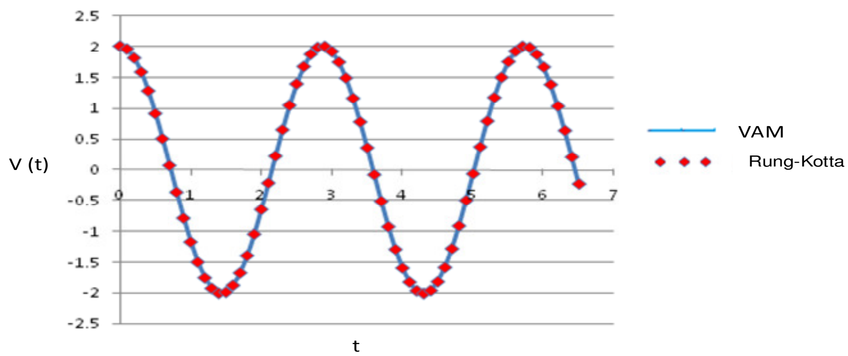

Figure 2. Comparison of analytical solution of $v(t)$ based on time with the numerical solution for $m=1, A=2, \varepsilon=0.5, k_1=5, k_2=50$.

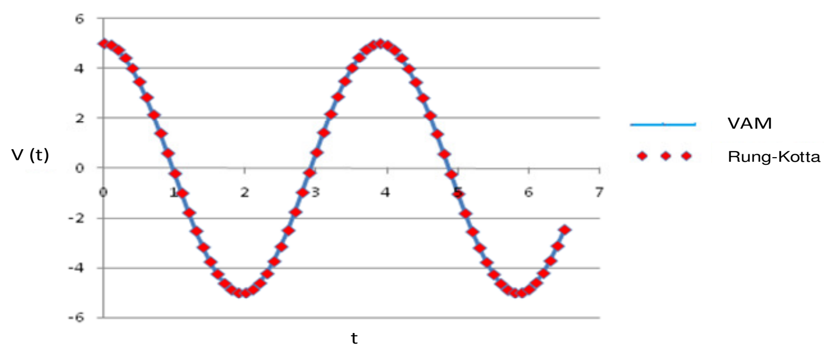

Figure 3. Comparison of analytical solution of $v(t)$ based on time with the numerical solution for $m=3, A=5, \varepsilon=1, k_1=8, k_2=16$.

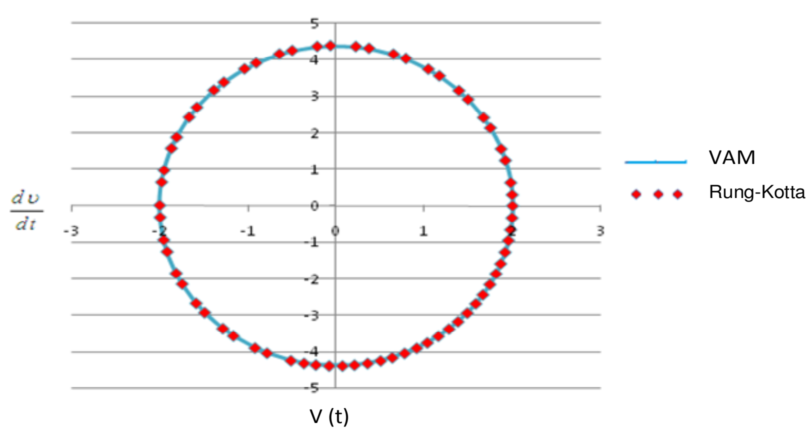

Figure 4. Comparison of analytical solution of $\frac{dv}{dt}$ based on time with the numerical solution for $m=1, A=2, \varepsilon=0.5, k_1=5, k_2=50$.

Conclusion

In this paper, the variational approach method was employed to solve the governing equations of non-linear oscillation of a mass grounded by linear and nonlinear springs in series. Excellent agreement between

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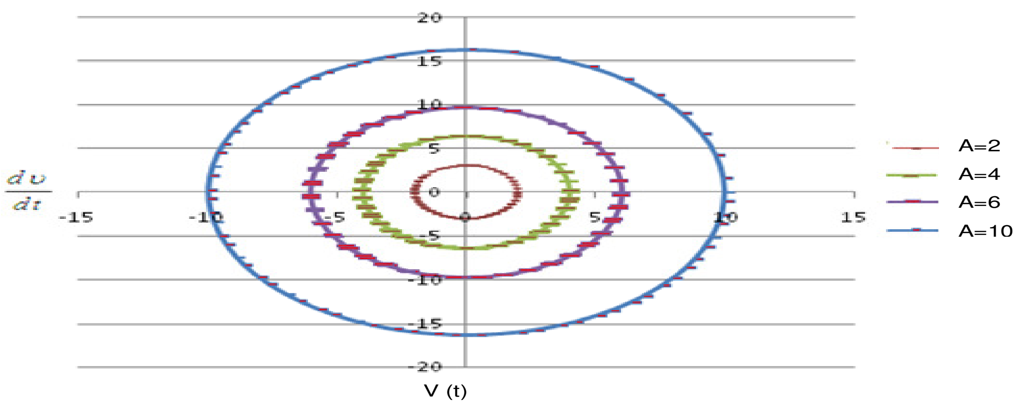

Figure 5. Comparison of analytical solution of $\frac{dv}{dt}$ based on time with the numerical solution for $m=3, \quad A=2, \quad \varepsilon=1, \quad k_1=8, \quad k_2=16$.

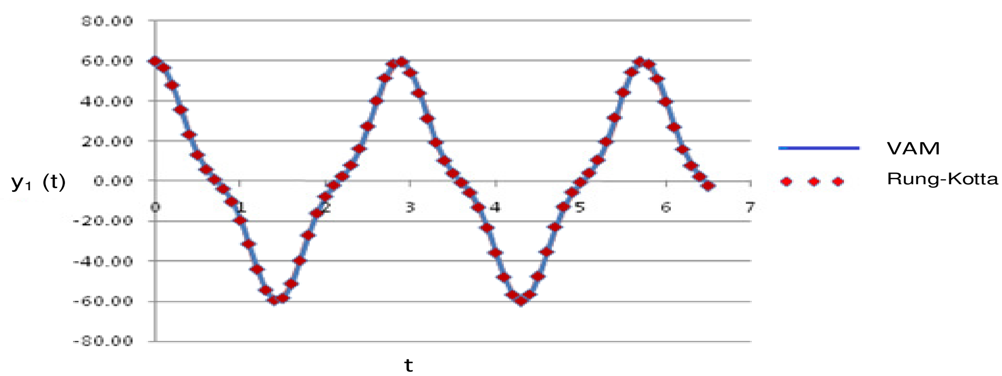

Figure 6. Comparison of analytical solution of $y_1(t)$ based on time with the numerical solution for $m=1, A=2, \varepsilon=0.5, k_1=5, \quad k_2=50$.

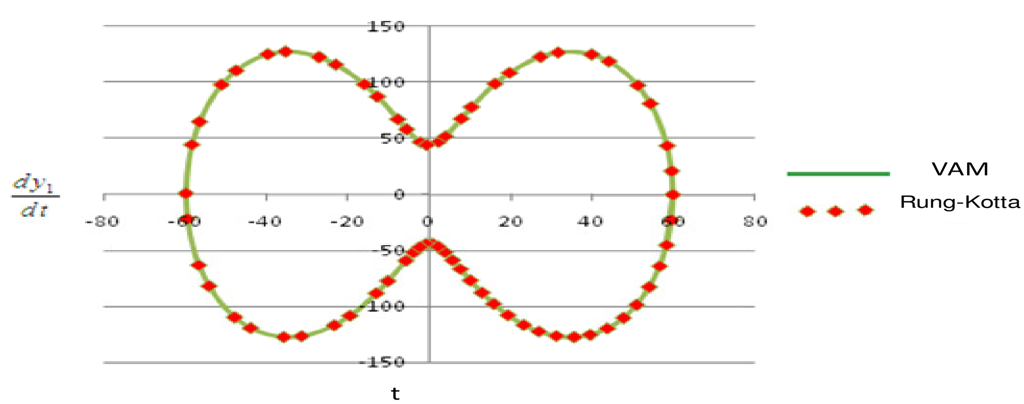

Figure 7. Comparison of analytical solution of $\frac{dy_1}{dt}$ based on time with the numerical solution for $m=1, A=2, \varepsilon=0.5, k_1=5, \quad k_2=50$

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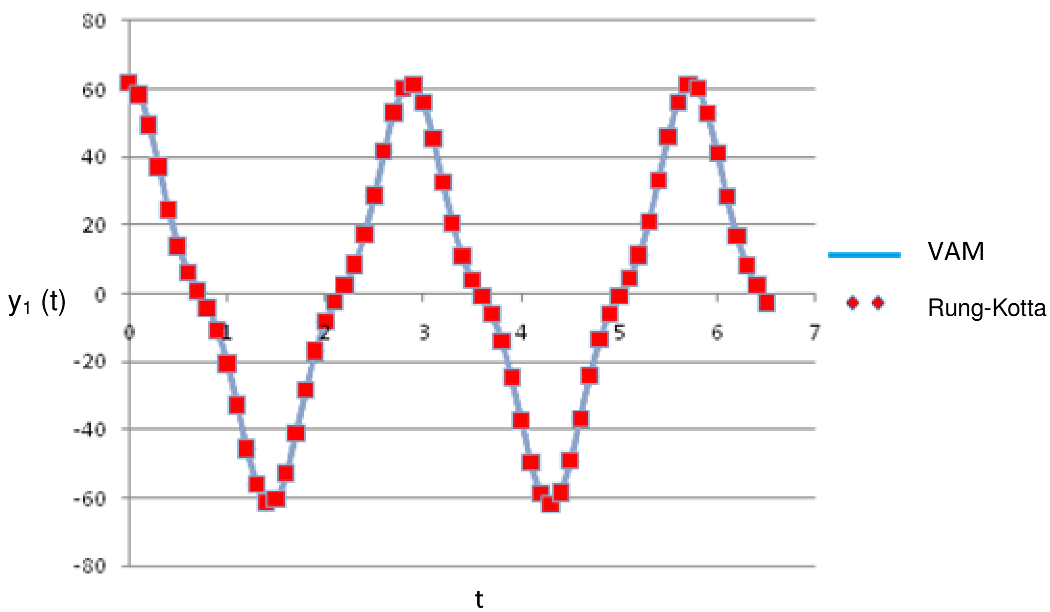

Figure 8. Comparison of analytical solution of $y_2(t)$ based on time with the numerical solution for $m=1, A=2, \varepsilon=0.5, k_1=5, \ k_2=50$.

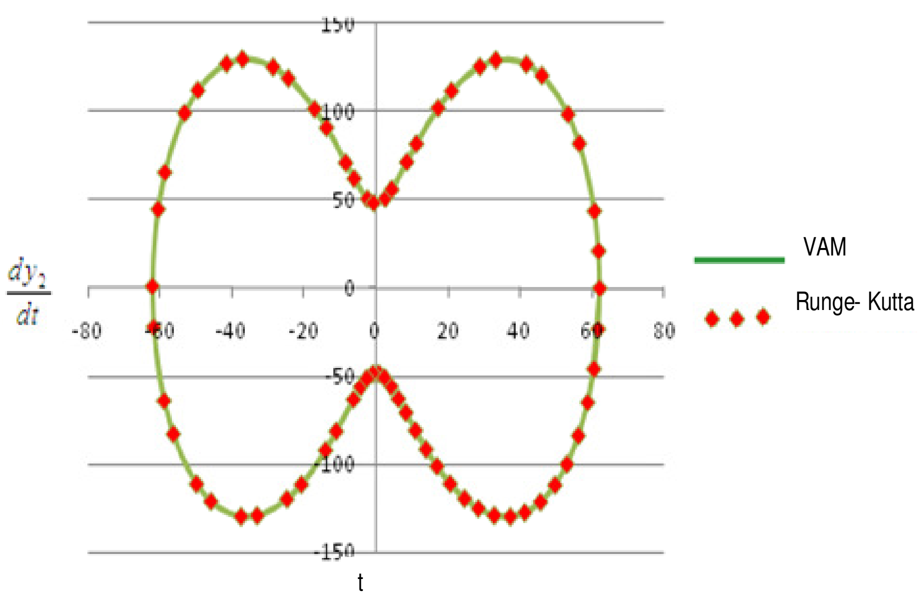

Figure 9. Comparison of analytical solution of $\frac{dy_2}{dt}$ based on time with the numerical solution for $m=1, A=2, \varepsilon=0.5, k_1=5, \ k_2=50$.

approximate frequencies and the exact one are demonstrated and discussed. The method can be a powerful mathematical tool for studying of nonlinear oscillators. According to the results, the precision and convergence rate of the solutions increase using Variational Approach Method. We can suggest VAM as strongly nonlinear method as novel and simple method for oscillation systems which provide easy and direct

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procedures for determining approximations to the periodic solutions.

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