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Full Length Research Paper

A new technique for systems of Abel-Volterra integral equations

Jafar Biazar[1]* and H. Ebrahimi[2]

[1]Department of Mathematics, Rasht branch, Islamic Azad University, P. O. Box 41335-3516, Rasht, Iran.
[2]Faculty of Mathematical Science, University of Guilan, P. O. Box. 41635-19141, P. C. 4193833697, Rasht, Iran.

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Extension of a computational method for solving a special kind of singular system is the novelty of this paper. These systems are called systems of Abel Volterra integral equations. This method is based on the application of Legendre wavelets, as a basis functions for numerical solutions. Some examples are presented to illustrate the efficiency and the simplicity of the method.

Key words: Systems of Abel Volterra integral equations, Legendre wavelets method, operational matrices.

INTRODUCTION

Mathematical modeling of many physical systems leads to functional equations in various fields of physics and engineering. In recent years some methods have been used by many authors to obtain approximate solutions (He, 1999; Biazar et al., 2003, 2009; Faraz et al., 2010; Khan and Faraz, 2011). System of Volterra integral equations arise in mathematical modeling of many phenomena (Delves and Mohamed, 1988; Jerri, 1999; Linz, 1985) and several methods have been proposed in the literature to solve these systems. These systems have been solved by Adomian decomposition method (Biazar et al., 2003), homotopy perturbation method (Biazar et al., 2009), variational iteration method (Biazar and Ebrahimi, 2010) and radial basis function networks (Golbabai et al., 2009).

In the present paper, special kind of singular systems of Volterra integral equations, called systems of Abel integral equations are studied. Historically, Abel is the first person who had studied integral equations, during the 1820 decade (Jerri, 1999; Linz, 1985). He obtained the following equation, when he was generalizing the tautochrone problem.

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\,dt = f(x). \tag{1}$$

where $f(x)$ is a known function and $u(x)$ is an unknown function which could be determined. This equation is a particular case of a linear Volterra integral equation of the first kind. The kernel of Abel integral equations has weak singularity and Abel integral equations are somewhat ill-posed and any small changes in the measurement data may cause unpredictable huge errors in the numerical approximate solutions.

Some methods for solving the system of Abel integral equations are known. The idea of the fractional calculus has been used for a special kind of these systems (Mandal et al., 1996), an operational matrix method based on block-pulse functions for singular integral equations has been introduced (Maleknejad and Salimi, 2008). In Pandey and Mandal (2010), Bernstein polynomials have been used for numerical solutions of systems of generalized Abel integral equations.

The method introduced in this paper consists of reducing a system of Abel integral equations into a system of algebraic equations, by expanding the unknown functions, as a series in terms of Legendre wavelets with unknown coefficients (Maleknejad and Sohrabi, 2007; Mahmoudi, 2005; Yousefi, 2006; Biazar and Ebrahimi, 2010). The general form of these systems is considered as the following.

$$\sum_{j=1}^{m} F_{ij}(x, u_1(x), \ldots, u_n(x)) + \sum_{j=1}^{m} \int_a^x \frac{G_{ij}(u_1(t), \ldots, u_n(t))}{(x-t)^{\alpha_{ij}}}\,dt = f_i(x), \tag{2}$$
$$i = 1, 2, \ldots, n, \quad m = 1, 2, \ldots,$$

*Corresponding author. E-mail: biazar@guilan.ac.ir. Tel: +981313233509. Fax: 981313233509.

where $0 \le x \le 1$, $0 < \alpha_{ij} < 1$, and also $f_i(x),\ i = 1, 2, ..., n,$ are known functions.

This paper is organized as follows: Legendre wavelets method is explained, applications of the method for introduced systems are studied, numerical examples are presented and conclusions are given, finally.

LEGENDRE WAVELETS METHOD

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet (Daubeches, 1992; Christensen and Christensen, 2004). When the dilation parameter a and the translation parameter b, vary continuously the following family of continuous wavelets will appear:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0. \tag{3}$$

Legendre wavelets are defined on the interval $[0,1]$ as follows:

$$\psi_{nm}(t) = \psi(t; k, n, m) = \begin{cases} \sqrt{m + \frac{1}{2}}\, 2^{\frac{k}{2}} P_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & otherwise. \end{cases} \tag{4}$$

where $n = 1, 2, ..., 2^{k-1}$, k is any positive integer, m is the degree of Legendre polynomials, $m = 0, 1, ..., M-1$ and t is the normalized time. $P_m(t)$ is the famous Legendre polynomial of order m. These polynomials are orthogonal with respect to the weight function $w(t) = 1$. The set of Legendre wavelets are an orthonormal set.

A function $f(x) \in L^2[0,1]$ may be expanded as follows:

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x), \tag{5}$$

where $c_{nm} = \left(f(x), \psi_{nm}(x) \right)$, stands for the inner product of $f(x)$ and $\psi_{nm}(x)$. Let's consider truncated series in Equation 5, as the following.

$$f(x) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) = C^T \psi(x). \tag{6}$$

where C and $\psi(x)$ are $2^{k-1}M \times 1$ matrices given by:

$$\begin{aligned} C &= \left[c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, c_{21}, \ldots, c_{2M-1}, \right. \\ &\quad \left. \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}, M-1} \right]^T \\ &= \left[c_1, c_2, \ldots, c_M, c_{M+1}, \ldots, c_{2^{k-1}M} \right]^T, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \psi(x) &= \left[\psi_{10}(x), \ldots, \psi_{1,M-1}(x), \psi_{20}(x), \ldots, \right. \\ &\quad \left. \psi_{2,M-1}(x), \ldots, \psi_{2^{k-1}0}(x), \ldots, \psi_{2^{k-1},M-1}(x) \right]^T \\ &= \left[\psi_1(x), \ldots, \psi_M(x), \psi_{M+1}(x), \ldots, \psi_{2^{k-1}M}(x) \right]^T. \end{aligned} \tag{8}$$

Also, a function $f(x, y) \in L^2[0,1]^2$ can be approximated by:

$$f(x, y) \simeq \psi^T(x) K \psi(y). \tag{9}$$

Here the entries of the matrix $K = [k_{ij}]_{2^{k-1}M \times 2^{k-1}M}$ will be obtain by:

$$k_{i,j} = \left(\psi_i(x), \left(f(x, y), \psi_j(y) \right) \right), \quad i, j = 1, 2, ..., 2^{k-1}M. \tag{10}$$

The integration of the vector $\psi(x)$, defined in Equation 8, can be achieved as the following.

$$\int_0^x \psi(t)\, dt = P \psi(x). \tag{11}$$

where P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix for integration (Razzaghi and Yousefi, 2001).

The following property of the product of two Legendre wavelet vector functions is well known as:

$$\psi(x) \psi^T(x) Y \simeq \tilde{Y} \psi(x). \tag{12}$$

where Y is a given vector and $\tilde{Y}$ is a

$2^{k-1}M \times 2^{k-1}M$ matrix. This matrix is called the operational matrix of product.

SOLUTION OF SYSTEMS OF ABEL VOLTERRA INTEGRAL EQUATIONS

Here, two cases of these systems will be studied.

Case 1

Consider the system (Equation 2) with the limits 0 and x for integral signs. To solve this system by Legendre wavelets method, unknown functions, $u_i(x),\ i=1,2,\dots,n$ are considered as a linear combination of these wavelets as the following.

$$u_i(x) \simeq C_i^T \psi(x), \quad i=1,2,\dots,n. \tag{13}$$

where

$$C_i = \left[c_{i,1}, c_{i,2}, \dots, c_{i,M}, c_{i,M+1}, \dots, c_{i,2^{k-1}M} \right]^T, \quad i=1,2,\dots,n.$$

Other terms also will be considered as the following general expansions:

$$f_i(x) \simeq F_i^T \psi(x),$$

$$F_{ij}(x, u_1(x), \dots, u_n(x)) \simeq X_{ij}^T \psi(x),$$

$$G_{ij}\left(u_1(t), u_2(t), \dots, u_n(t)\right) = \sum_{k=0}^{s} A_{ijk} t^k, \tag{14}$$

$$i=1,2,\dots,n, \quad j=1,2,\dots,m,$$

$$s = 2^{k-1}M, 2^{k-1}M+1, \dots .$$

where F_i are the $2^{k-1}M \times 1$ matrices and X_{ij} are the $2^{k-1}M \times 1$ matrices with the entries which are in terms of the components of the vectors C_i for $i=1,2,\dots,n,\ j=1,2,\dots,m$.

By substituting Equations 13 and 14 into the system, one gets:

$$\begin{aligned}
F_i^T\psi(x) &= \sum_{j=1}^{m} X_{ij}^T\psi(x) + \sum_{j=1}^{m} \int_0^x \frac{\sum_{k=0}^{s} A_{ijk} t^k}{(x-t)^{\alpha_{ij}}} dt \\
&= \sum_{j=1}^{m} X_{ij}^T\psi(x) + \sum_{j=1}^{m}\sum_{k=0}^{s} A_{ijk} \int_0^x \frac{t^k}{(x-t)^{\alpha_{ij}}} dt \\
&= \sum_{j=1}^{m} X_{ij}^T\psi(x) + \sum_{j=1}^{m}\sum_{k=0}^{s} A_{ijk} Z_k^{\alpha_{ij}}(x),
\end{aligned} \tag{15}$$

$$i=1,2,\dots,n, \quad s=2^{k-1}M, 2^{k-1}M+1, \dots .$$

where A_{ijk} and $Z_k^{\alpha_{ij}}(x)$ can be determined by the following formulas:

$$A_{ijk}(c_{10}, \dots, c_{n,2^{k-1}M}) = \frac{d^k G(C_1^T\psi(t), \dots, C_2^T\psi(t))}{k!}\Big|_{t=0}, \tag{16}$$

$$i=1,2,\dots,n, j=1,2,\dots,m, k=0,1,\dots,s.$$

and

$$\begin{aligned}
Z_k^{\alpha_{ij}}(x) &= \int_0^x \frac{t^k}{(x-t)^{\alpha_{ij}}} dt \\
&= \frac{\Gamma(k+1)\Gamma(1-\alpha_{ij})}{\Gamma(k-\alpha_{ij}+2)} x^{k-\alpha_{ij}+1},
\end{aligned} \tag{17}$$

$$k=0,1,\dots,s.$$

Now let's consider:

$$\sum_{k=0}^{s} A_{ijk} Z_k^{\alpha_{ij}}(x) = Y_{ij}^T \psi(x), \quad i=1,2,\dots,n, \quad j=1,2,\dots,m. \tag{18}$$

Substitution of Equation 18 into the system (Equation 15), leads to the following system:

$$F_i^T\psi(x) = \sum_{j=1}^{m}\left(X_{ij}^T + Y_{ij}^T \right)\psi(x), \quad i=1,2,\dots,n, \tag{19}$$

Multiplying $\psi^T(x)$, in both sides of the system (Equation 19) and applying $\int_0^1 (\ .\ )dx$, a linear or non-linear system in terms of the elements of

$C_i,\ i=1,2,\dots,n$, will be obtained.

Case 2

If in Equation 2, limits of integral are different from 0 and x, then the formula (Equation 17) would not be applicable, and another approach should be considered. One can write the kernels as:

$$\frac{1}{(x-t)^{\alpha_{ij}}} \cong \psi^T(x) K_{ij}\, \psi(t), \qquad i=1,2,\dots,n,\ j=1,2,\dots,m. \tag{20}$$

where $K_{i,j}$ are $2^{k-1}M \times 2^{k-1}M$ complex matrices with the entries according to the Equation 10. In applying Legendre wavelets method, obtained solutions will be complex, because of complex entries of matrices K_{ij}, and the real parts of the solutions can be considered as approximate solutions.

Since the truncated Legendre wavelets series are approximate solutions of system (Equation 2), one has an error function $e(u_i(x))$ as follows:

$$e(u_i(x)) = \left| u_i(x) - \sum_{n=1}^{2^{k-1}M} c_{in}\, \psi_n(x) \right|. \tag{21}$$

If one set $x = x_j$, where $x_j \in [0,1]$, the error values can be obtained. Therefore we can check the accuracy of the method by using error functions.

NUMERICAL EXAMPLES

To illustrate the method, some systems are considered and solved by the proposed method.

Example 1

Consider the following system of Abel integral equations on $[0,1]$:

$$\begin{cases} (x^2-1)\displaystyle\int_0^x \frac{1}{\sqrt{x-t}}u(t)dt + x\int_x^1 \frac{1}{\sqrt[3]{x-t}}v(t)dt = f_1(x), \\ x^3\displaystyle\int_x^1 \frac{1}{\sqrt[4]{x-t}}u(t)dt + (1-x)\int_0^x \frac{1}{\sqrt[5]{x-t}}v(t)dt = f_2(x), \end{cases} \tag{22}$$

where

$$f_1(x) = \frac{14x^{\frac{5}{2}}}{15} + \frac{16x^{\frac{9}{2}}}{15} - 2\sqrt{x} - \frac{243(x-1)^{\frac{2}{3}}x^4}{440} - \frac{81(x-1)^{\frac{2}{3}}x^3}{220} - \frac{27(x-1)^{\frac{2}{3}}x^2}{88} - \frac{3(x-1)^{\frac{2}{3}}x}{11},$$

$$f_2(x) = -\frac{56(x-1)^{\frac{3}{4}}x^3}{33} - \frac{128(x-1)^{\frac{3}{4}}x^5}{231} - \frac{32(x-1)^{\frac{3}{4}}x^4}{77} + \frac{625x^{\frac{19}{5}}}{1596} - \frac{625x^{\frac{24}{5}}}{1596}.$$

The exact solutions are $u(x) = x^2+1$, and $v(x) = x^3$. This system is of the case 2. Let's write the system (Equation 22) as the following.

$$\begin{cases} (x^2-1)\displaystyle\int_0^x \frac{1}{\sqrt{x-t}}u(t)dt + x\int_0^1 \frac{1}{\sqrt[3]{x-t}}v(t)dt \\ \qquad -x\displaystyle\int_0^x \frac{1}{\sqrt[3]{x-t}}v(t)dt = f_1(x), \\ x^3\left(\displaystyle\int_0^1 \frac{1}{\sqrt[4]{x-t}}u(t)dt - \int_0^x \frac{1}{\sqrt[4]{x-t}}u(t)dt\right) \\ \qquad +(1-x)\displaystyle\int_0^x \frac{1}{\sqrt[5]{x-t}}v(t)dt = f_2(x), \end{cases} \tag{23}$$

Consider $k=1$, $M=8$, and

$u(x) \cong C_1^T \psi(x)$, $\qquad v(x) \cong C_2^T \psi(x)$,

$f_1(x) \cong F_1^T \psi(x)$, $\qquad f_2(x) \cong F_2^T \psi(x)$,

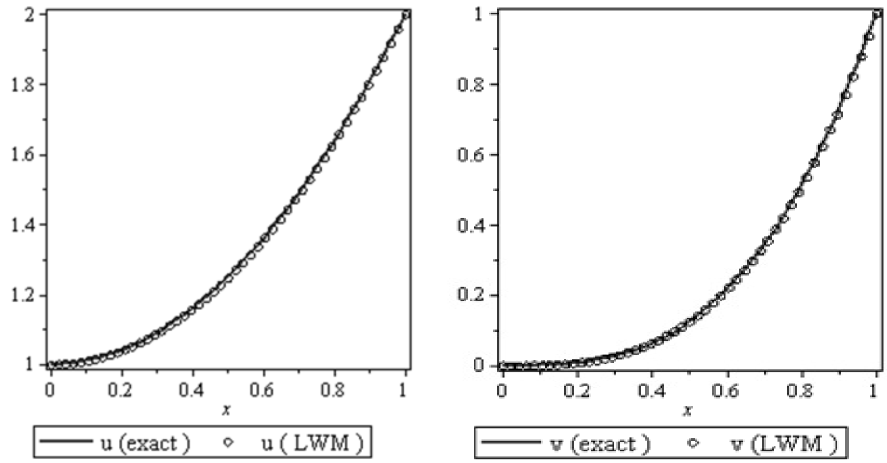
$x^2-1 \cong A_{11}^T \psi(x)$, $\qquad x \cong A_{12}^T \psi(x)$,

$x^3 \cong A_{21}^T \psi(x)$, $\qquad 1-x \cong A_{22}^T \psi(x)$,

$$\frac{1}{\sqrt[3]{x-t}} \cong \psi^T(x) K_1 \psi(t),$$

$$\frac{1}{\sqrt[4]{x-t}} \cong \psi^T(x) K_2 \psi(t),$$

$$\int_0^x \frac{1}{\sqrt{x-t}} u(t)dt \cong Y_1^T \psi(x),$$



(a1) (b1)

Figure 1. The exact and approximate solutions of Example 1 (a1 and b1).

$$\int_0^x \frac{1}{\sqrt[3]{x-t}} v(t)\,dt \simeq Y_2^T \psi(x),$$

$$\int_0^x \frac{1}{\sqrt[4]{x-t}} u(t)\,dt \simeq Y_3^T \psi(x),$$

$$\int_0^x \frac{1}{\sqrt[5]{x-t}} v(t)\,dt \simeq Y_4^T \psi(x).$$

Substituting into the system (Equation 23) and solving the obtained system, the following results will be achieved.

$c_{1,0} = 1.332139718 - 0.0001712234033I,$

$c_{1,1} = 0.2881733408 + 0.0003360200620I,$

$c_{1,2} = 0.07495349421 + 0.0003778882289I,$

$c_{1,3} = 0.0004166935511 - 0.00007348614476I,$

$c_{1,4} = 0.00009264304313 - 0.0002144572776I,$

$c_{1,5} = -0.0001266940664 - 0.00009326947553I,$

$c_{1,6} = -0.0001029232530 + 0.00001070354157I,$

$c_{1,7} = -0.0000079722934141 - 0.00001920837799I,$

$c_{2,0} = 0.2501175456 - 0.001356238161I,$

$c_{2,1} = 0.2595052996 - 0.0009835260024I,$

$c_{2,2} = 0.1112783354 + 0.0002757174589I,$

$c_{2,3} = 0.01885257238 + 0.0006609553538I,$

$c_{2,4} = 0.0003527252537 + 0.0002674685299I,$

$c_{2,5} = 0.0002311584955 - 0.000081849724430I,$

$c_{2,6} = -0.0000632693604 - 0.0001176234874I,$

$c_{2,7} = -0.0001577272089 - 0.00001908555302I.$

Since entries of the matrices K_1 and K_2 are complex, the real parts are considered as approximate solutions.

$$\begin{aligned} u(x) = {} & -0.1059683526x^7 + 0.0279973931x^6 \\ & + 0.4092470195x^5 - 0.5281462460x^4 \\ & + 0.2415872910x^3 + 0.9532702993x^2 \\ & + 0.00157711934x + 0.9998653975, \end{aligned}$$

$$\begin{aligned} v(x) = {} & -2.096522496x^7 + 7.127044981x^6 \\ & -9.334519464x^5 + 5.928096298x^4 \\ & -0.9036584798x^3 + 0.2968255414x^2 \\ & -0.01824363652x + 0.0002621496533. \end{aligned}$$

Plots of the exact and approximate solutions are shown in Figure 1, and plots of error functions are shown in Figure 6.

Example 2

Consider the following linear system of Abel integral equations, with the exact solutions $u(x) = x$, and $v(x) = \sqrt{x}$ on $[0,1]$ (Maleknejad and Salimi, 2008).

$$\begin{cases} 4v(x)-u(x)+\int_0^x \frac{1}{\sqrt{(x-t)^3}}u(t)dt=-x, \\ v(x)-\frac{\pi}{2}u(x)+\int_0^x \frac{1}{\sqrt{x-t}}(v(t)-u(t))dt=\sqrt{x} \\ \qquad -\frac{4}{3}x^{3/2}. \end{cases} \tag{24}$$

In Maleknejad and Salimi (2008), the authors obtained Laplace transforms of this system, and then found inversion of Laplace transform by operational matrices. Let's take $k=1$ and $M=10$, and such that:

$$u(x) \cong C_1^T \psi(x), \qquad v(x) \cong C_2^T \psi(x),$$

$$x \cong F_1^T \psi(x), \qquad \sqrt{x}-\frac{4}{3}x^{3/2} \cong F_2^T \psi(x),$$

$$\int_0^x \frac{1}{\sqrt{(x-t)^3}}u(t)dt \cong \sum_{k=0}^{10} A_1 Z_k^{3/2}(x)=Y_1^T \psi(x),$$

$$\int_0^x \frac{1}{\sqrt{x-t}}u(t)dt \cong \sum_{k=0}^{10} A_2 Z_k^{1/2}(x)=Y_2^T \psi(x),$$

$$\int_0^x \frac{1}{\sqrt{x-t}}v(t)dt \cong \sum_{k=0}^{10} A_3 Z_k^{1/2}(x)=Y_3^T \psi(x).$$

Therefore, the following system will be obtained:

$$\begin{cases} 4C_2^T - C_1^T + Y_1^T = -F_1^T, \\ C_2^T - \frac{\pi}{2}C_1^T + Y_3^T - Y_2^T = F_2^T. \end{cases} \tag{25}$$

Elements of vectors C_1 and C_2 are computed by solving a linear system, with twenty equations and the same number of unknowns, as follows:

$$c_{2,0}=0.66666678324, \qquad c_{2,1}=0.2309422413,$$
$$c_{2,2}=-0.04258874862, \qquad c_{2,3}=0.01680244769,$$
$$c_{2,4}=-0.008652711337, \qquad c_{2,5}=0.005161156387,$$
$$c_{2,6}=-0.003351718010, \qquad c_{2,7}=0.002352618714,$$
$$c_{2,8}=0.001673003306, \qquad c_{2,9}=0.001429765418.$$

Therefore, the following approximate solutions will be resulted:

$$\begin{aligned} u(x) = {} & 5.885144996x^9 - 26.84654734x^8 \\ & +51.26379778x^7 - 53.12871470x^6 \\ & +32.39185155x^5 - 11.76755039x^4 \\ & +2.458395282x^3 - 0.2686523078x^2 \\ & +1.012477244x - 0.0001419708482, \\ v(x) = {} & 303.0097084x^9 - 1452.320553x^8 \\ & +2953.049126x^7 - 3320.767151x^6 \\ & +2258.593806x^5 - 956.3816582x^4 \\ & +251.5888474x^3 - 40.83119833x^2 \\ & +5.012913708x + 0.04957533208. \end{aligned}$$

Plots of the exact and approximate solutions are presented in Figure 2. Error functions are plotted in Figure 7.

Example 3

Consider the following non-linear system of Abel integral equations:

$$\begin{cases} u(x)-2v(x)+\int_0^x \frac{1}{\sqrt[5]{x-t}}(u^2(t)+v^2(t))dt \\ \qquad =x^2-2x^3+\frac{390625}{1573656}x^{34/5}+\frac{3125}{9576}x^{24/5}, \\ v(x)-u(x)-\int_0^x \frac{1}{\sqrt[3]{x-t}}u(t)v(t)dt \\ \qquad =x^3-x^2-\frac{2187}{5236}x^{17/3}, \quad 0 \le x \le 1. \end{cases} \tag{26}$$

With the exact solutions, $u(x)=x^2$ and $v(x)=x^3$.

Applying the Legendre wavelets method for $k=1$ and $M=6$ results in the following.

$$c_{1,0}=0.3333333332, \qquad c_{1,1}=0.2886751346,$$
$$c_{1,2}=0.07453559914, \qquad c_{1,3}=-0.1593029682\times10^{-10},$$
$$c_{1,4}=-0.3468681244\times10^{-12}, \quad c_{1,5}=0.1888710861\times10^{-11},$$
$$c_{2,0}=0.2499999998, \qquad c_{2,1}=0.2598076210,$$
$$c_{2,2}=0.1118033988, \qquad c_{2,3}=0.01889822361,$$
$$c_{2,4}=-0.1116409590\times10^{-10}, \quad c_{2,5}=-0.2114302164\times10^{-11}.$$

Therefore, one gets the following approximate solutions:

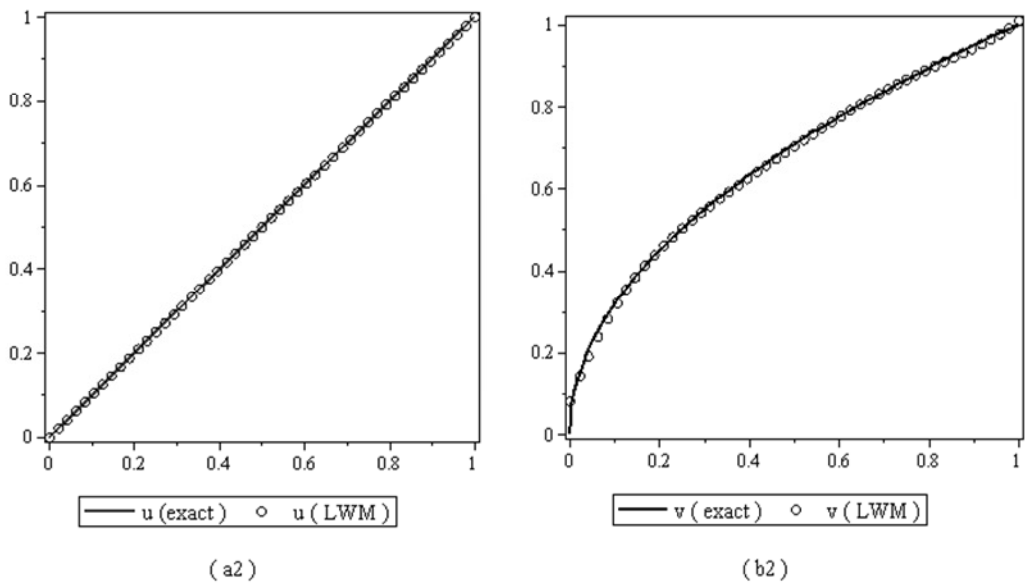


Figure 2. The exact and approximate solutions of Example 2 (a2 and b2).

$$u(x)=0.1578564606\times10^{-8}x^5-0.4019253820\times10^{-8}x^4+0.2810653885\times10^{-8}x^3+0.9999999981x^2+0.1508736445\times10^{-8}x-0.4641165416\times10^{-9},$$

$$v(x)=-0.1767111437\times10^{-8}x^5+0.2073318451\times10^{-8}x^4+0.9999999981x^3+0.458286971\times10^{-9}x^2+0.1405246551\times10^{-9}x-0.18\times10^{-9}.$$

Plots of the exact and approximate solutions are plotted in Figure 3 and error functions are shown in Figure 8.

Example 4

Consider the following non-linear system with the exact solutions $u(x)=x^3-1$ and $v(x)=x$.

$$\begin{cases} u^2(x)+\int_0^x \frac{1}{\sqrt{(x-t)^3}}v^3(t)dt=x^6-2x^3-\frac{32}{5}x^{5/2}+1, \\ v(x)-\int_0^x \frac{1}{\sqrt[4]{x-t}}u(t)v^2(t)dt=x-\frac{32768}{100947}x^{23/4}+\frac{128}{231}x^{11/4}, 0\le x\le 1. \end{cases} \tag{27}$$

Let's consider $k=1$ and $M=5$. Entries of the vectors C_1 and C_2 can be computed as the following.

$c_{1,0}=-0.7500000001,$ $c_{1,1}=0.2598076212,$
$c_{1,2}=0.1118033985,$ $c_{1,3}=0.011889822359,$
$c_{1,4}=0.1532165690\times10^{-9},$ $c_{2,0}=0.4999999999,$
$c_{2,1}=0.2886751347,$ $c_{2,2}=-0.1136139673\times10^{-10},$
$c_{2,3}=-0.1068365696\times10^{-10},$ $c_{2,4}=-0.3978712670\times10^{-11}.$

The approximate solutions are:

$$u(x)=0.3217547949\times10^{-7}x^4+0.9999999324x^3+0.4136847363\times10^{-7}x^2-0.5692994140\times10^{-8}x-1,$$

$$v(x)=-0.8355296607\times10^{-9}x^4+0.1105733333\times10^{-8}x^3-0.3786925712\times10^{-7}x^2+x-0.3717337006\times10^{-9}.$$

Plots of the exact and approximate solutions are shown in Figure 4 and plots of error functions are shown in Figure 9.

Example 5

Consider the following system of Abel Volterra integral of the first kind:

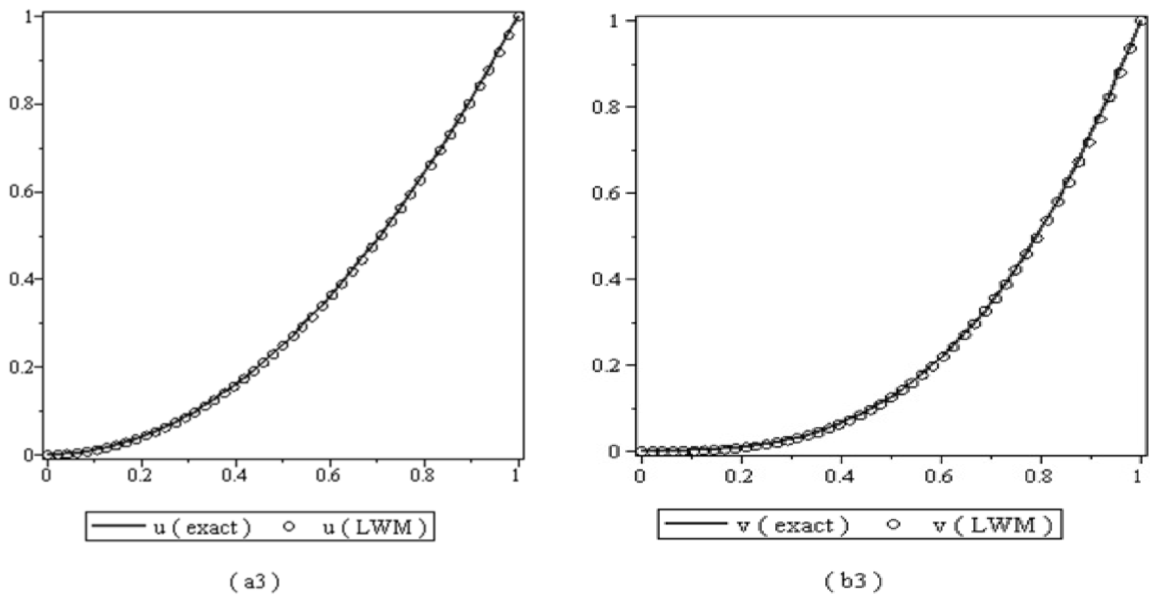


Figure 3. The exact and approximate solutions of Example 3 (a3 and b3).

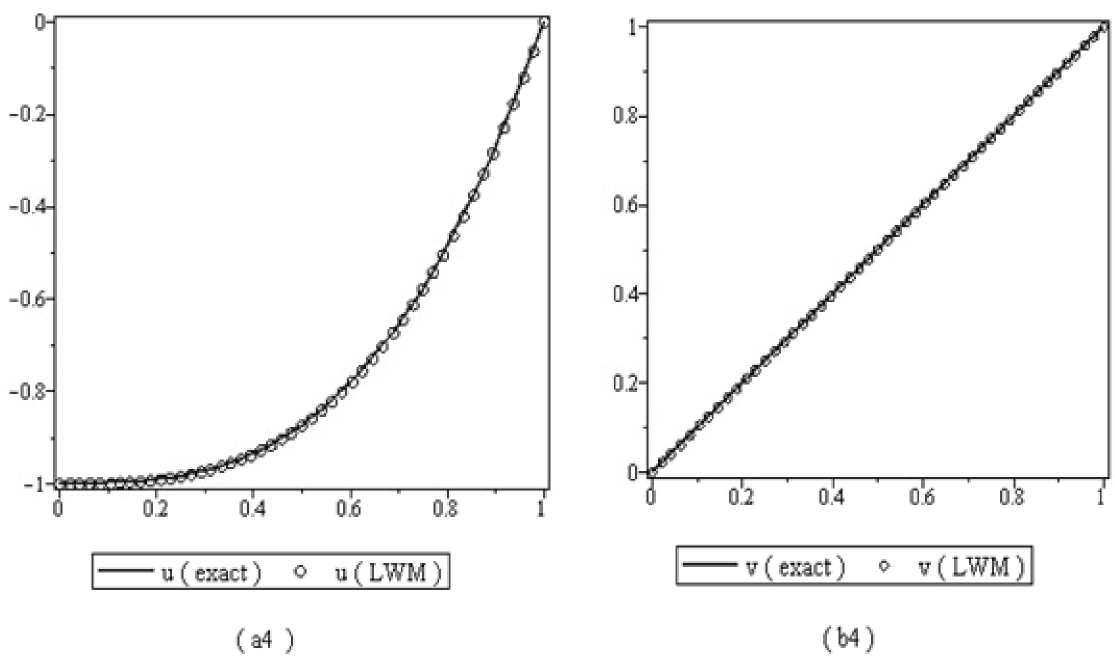


Figure 4. The exact and approximate solutions of Example 4 (a4 and b4).

$$\begin{cases} \int_0^x \frac{1}{\sqrt{x-t}}(u(t)+v(t))dt = e^x + \frac{x}{2}, \\ \int_0^x \frac{1}{\sqrt{x-t}}(u(t)-2v(t))dt = e^x - x - 3, \quad 0 \le x \le 1. \end{cases} \tag{28}$$

With the exact solutions $u(x) = \frac{e^x erf(\sqrt{x})}{\sqrt{\pi}}$ and $v(x) = \frac{1+x}{\pi\sqrt{x}}$

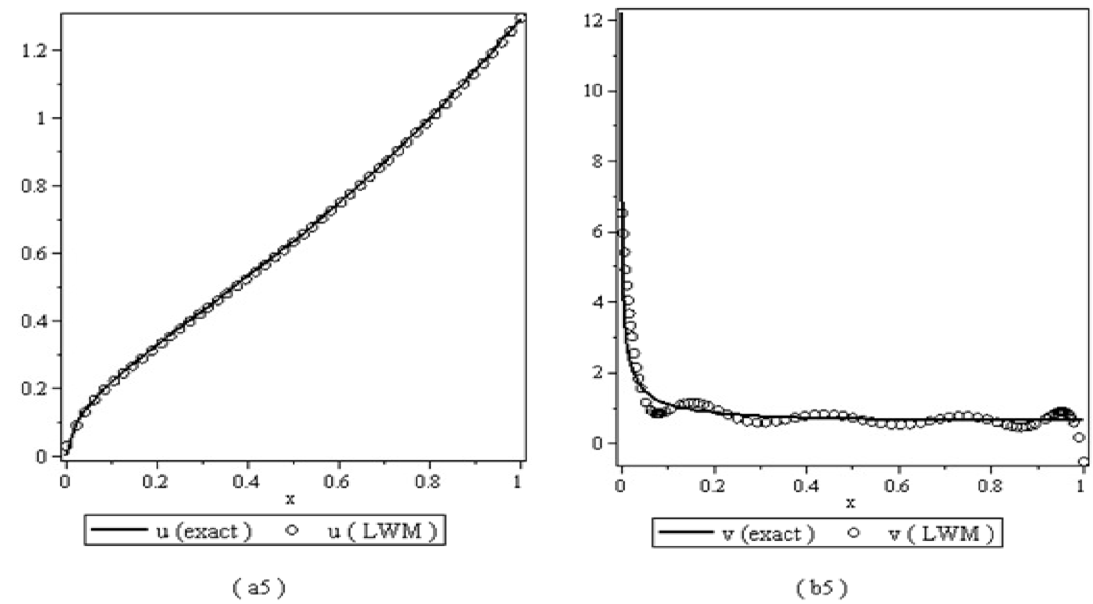


Figure 5. The exact and approximate solutions of Example 5 (a5 and b5).

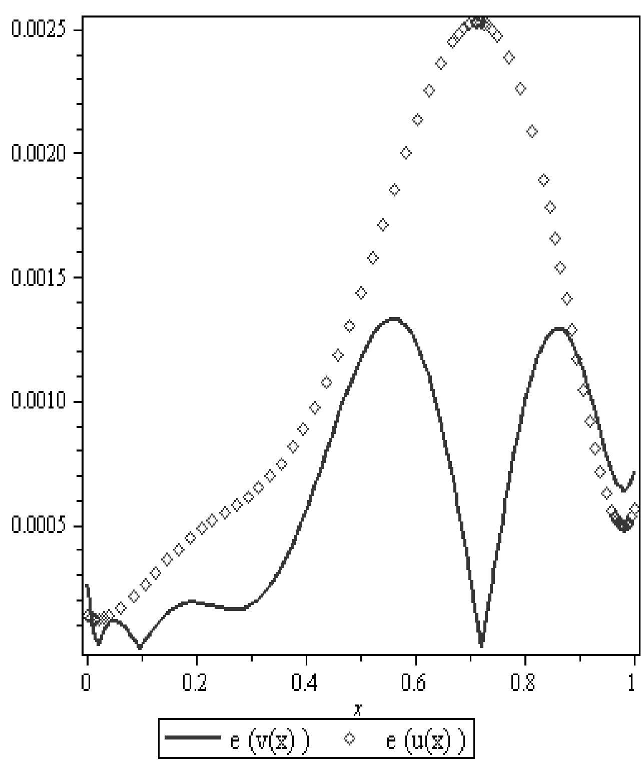


Figure 6. Plots of error functions of Example 1.

By applying the Legendre wavelets approach for $k=1$ and $M=10$, the following solutions would be obtained.

$$\begin{aligned} u(x) = {} & 226.0149470x^{9} - 1063.703653x^{8} \\ & + 2122.157100x^{7} - 02339.491704x^{6} \\ & + 1558.306153x^{5} - 645.2812734x^{4} \\ & + 165.6773181x^{3} - 25.68774288x^{2} \\ & + 3.275655345x + 0.03104332440, \end{aligned}$$

$$\begin{aligned} v(x) = {} & -42845.65844x^{9} + 198904.9387x^{8} \\ & - 389906.1701x^{7} + 419870.7757x^{6} \\ & - 270634.2637x^{5} + 106685.4261x^{4} \\ & - 25219.19223x^{3} + 3360.406582x^{2} \\ & - 223.3096447x + 6.535131213. \end{aligned}$$

Plots of the exact and approximate solutions and error functions are shown in Figures 5 and 10.

Conclusion

In this paper, the Legendre wavelets method is used to find approximate solutions of systems of Abel Volterra integral equations. It is observed that the solution obtained by this method converges rapidly to an exact solution and plots confirm it. Research for finding more applications of this method and other orthogonal basis functions is one of the research fields in our research

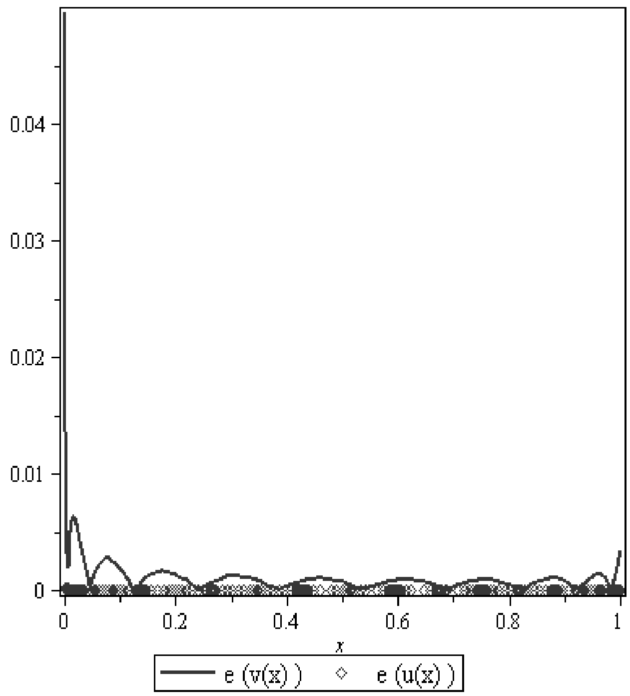


Figure 7. Plots of error functions of Example 2.

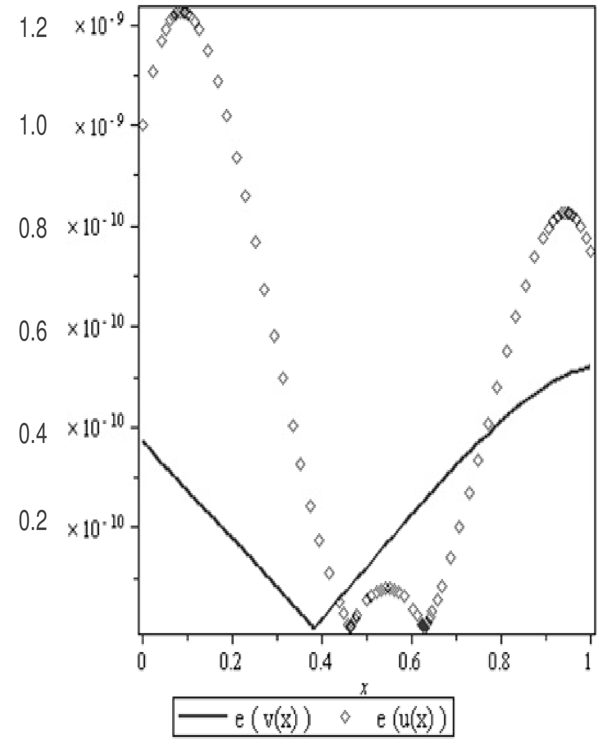


Figure 9. Plots of error functions of Example 4.

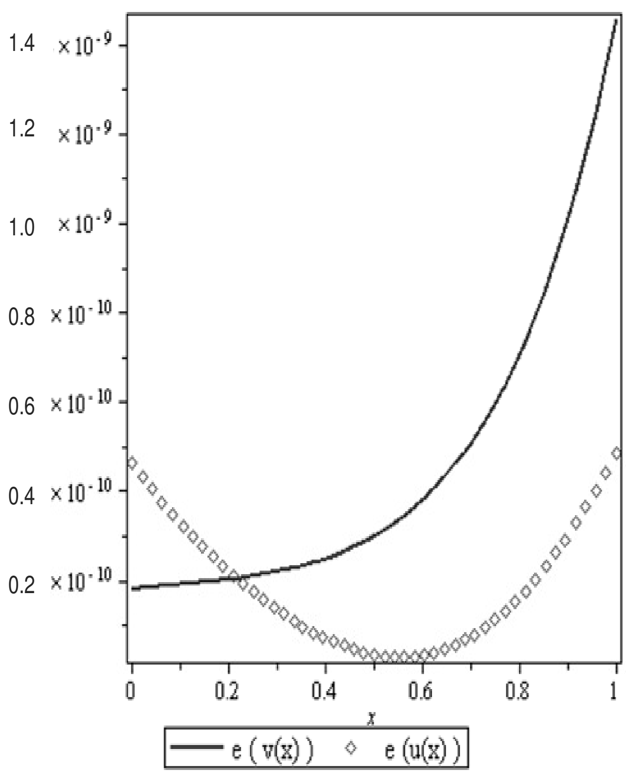


Figure 8. Plots of error functions of Example 3.

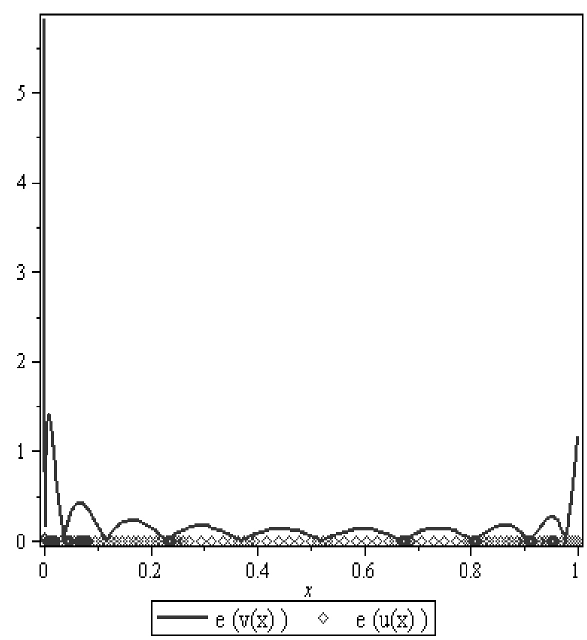


Figure 10. Plots of error functions of Example 5.

group. The computations associated with examples are performed using the package Maple 13.

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