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Full Length Research Paper

Spacelike B_2-slant helix in Minkowski 4-space E_1^4

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In this paper, we give the characterizations of spacelike B_2-slant helix by means of curvatures of the spacelike curve in Minkowski 4 - space. Furthermore, we give the integral characterization of the spacelike B_2 - slant helix.

Key words: Minkowski 4 - space, spacelike B_2 -slant helix, Frenet frame.

INTRODUCTION

Helix is one of the most fascinating curves in Science and Nature. Helices can be seen in many subjects of Science such as nanosprings, carbon nanotubes, α-helices, DNA double and collagen triple helix (a DNA molecule has a form of well - known right-handed double helix, wherein two heteropolymer chains are wound around each other. The double helical structure is believed to be the structure of minimum free energy under the normal physiological conditions.), lipid bilayers, bacterial flagella in Escherichia coli and Salmonella, aerial hyphae in actinomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells (helico-spiral structures) (Chouaieb et al., 2006; Lucas Amand and Lambin, 2005; Watson and Crick, 1953). Furthermore, in the fields of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc. (Yang, 2003).

A curve of constant slope or general helix in Euclidean 3-space E^3 is defined by the property that the tangent makes a constant angle with a fixed straight line which is called the axis of the general helix (Barros, 1997). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 is: A necessary and sufficient condition that a curve be a general helix is that the ratio of the first curvature to the second curvature be constant that is k_1/k_2 is constant along the curve, where k_1 and k_2 denote the first and second curvatures of the curve, respectively (Scofield, 1995). Analogue to that Magden (1993) has given a characterization for a curve $x(s)$ to be a helix in Euclidean 4-space E^4. He characterizes a helix if the function

$$\frac{k_1^2}{k_2^2}+\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\right]^2$$

is constant where k_1, k_2 and k_3 are first, second and third curvatures of Euclidean curve $x(s)$, respectively, and they are nowhere zero. Corresponding characterizations of time like helices in Minkowski 4-space E_1^4 were given by Kocayigit and Onder (2007). Latterly Camci et al., (2009) have given some characterizations for a non - degenerate curve to be a generalized helix by using its harmonic curvatures.

Recently, Izumiya and Takeuchi (2004) have introduced the concept of slant helix by saying that the normal lines of the curve make a constant angle with a fixed direction and they have given a characterization of slant helix in Euclidean 3-space E^3 by the fact that the function

$$\frac{k_1^2}{(k_1^2+k_2^2)^{3/2}}\left(\frac{k_2}{k_1}\right)'$$

is constant. After them, Kula and Yayli (2005)

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investigated spherical images, the tangent indicatrix and the binormal indicatrix of a slant helix and they obtained that the spherical images are spherical helices. Analogue to the definition of slant helix, Onder et al. (2008) have defined B_2-slant helix in Euclidean 4-space E^4 by saying that the second binormal vector of the curve make a constant angle with a fixed direction and they have given some characterizations of B_2-slant helix in Euclidean 4-space E^4.

In this paper, we consider spacelike B_2- slant helix in Minkowski 4-space E_1^4 and we give some characterizations and also the integral characterization of spacelike B_2- slant helix.

MATERIALS AND METHODS

Minkowski space-time E_1^4 is a Euclidean space E^4 provided with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4.

Since $\langle , \rangle$ is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of three causal characters; it can be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, time like if $\langle v, v\rangle < 0$ and null (light like) if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $x(s)$ in E_1^4 can locally be spacelike, time like or null (light like), if all of its velocity vectors $x'(s)$ are respectively spacelike, time like or null (light like). Also recall that the pseudo-norm of an arbitrary vector $v \in E_1^4$ is given by $\|v\| = \sqrt{|\langle v, v\rangle|}$. Therefore v is a unit vector if $\langle v, v\rangle = \pm 1$. The velocity of the curve $x(s)$ is given by $\|x'(s)\|$. Next, vectors v, w in E_1^4 are said to be orthogonal if $\langle v, w\rangle = 0$. We say that a time like vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively.

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $x(s)$ in the space E_1^4. Then T, N, B_1, B_2 are the tangent, the principal normal, the first binormal and the second binormal fields, respectively. A timelike (resp. spacelike) curve $x(s)$ is said to be parameterized by a pseudo - arc length parameter s, that is $\langle x'(s), x'(s)\rangle = -1$ (resp. $\langle x'(s), x'(s)\rangle = 1$). Let $x(s)$ be a spacelike curve in Minkowski space-time E_1^4, parameterized by arc length function of s. Then for the curve $x(s)$ the following Frenet equation is given as follows:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -\varepsilon_1 k_1 & 0 & k_2 & 0 \\ 0 & \varepsilon_2 k_2 & 0 & k_3 \\ 0 & 0 & \varepsilon_1 k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \quad (1)$$

where $\langle T, T\rangle = 1,\ \langle N, N\rangle = \varepsilon_1,\ \langle B_2, B_2\rangle = \varepsilon_2,$ $\langle B_1, B_1\rangle = -\varepsilon_1\varepsilon_2\ \varepsilon_1 = \pm 1,\ \varepsilon_2 = \pm 1$

and recall that the functions $k_1 = k_1(s),\ k_2 = k_2(s)$ and $k_3 = k_3(s)$ are called the first, the second and the third curvature of the spacelike curve $x(s)$, respectively and we will assume throughout this work that all the three curvatures satisfy $k_i(s) \neq 0,\ 1 \leq i \leq 3$. Here the signs of ε_1 and ε_2 are changed by a rule. The signature rule between ε_1 and ε_2 can be given as follows

if	ε_1 +1	then	ε_2 +1 or -1
	-1		+1

or

if	ε_2 +1	then	ε_1 +1 or -1
	-1		+1

For the obvious forms of the Frenet equations in (1) we refer to the reader to see Walrave (1995).

RESULTS AND DISCUSSION

In this section, we give the definition and the characterizations of spacelike B_2-slant helix.

Let $x : I \subset IR \to E_1^4$ be a unit speed spacelike curve with nonzero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$ and let $\{T, N, B_1, B_2\}$ denotes the Frenet frame of the curve $x(s)$. We call $x(s)$ as spacelike B_2-slant helix if its second binormal unit vector B_2 makes a constant angle with a fixed direction in a unit vector U; that is

$$\langle B_2, U\rangle = constant \quad (2)$$

along the curve. By differentiation (2) with respect to s

and using the Frenet formulae (1) we have

$$\langle \varepsilon_1 k_3 B_1, U \rangle = 0 .$$

Therefore U is in the subspace $Sp\{T, N, B_2\}$ and can be written as follows

$$U = a_1(s)T(s) + a_2(s)N(s) + a_3(s)B_2(s) \tag{3}$$

where

$$a_1 = \langle U, T \rangle, \ \varepsilon_1 a_2 = \langle U, N \rangle, \\ \varepsilon_2 a_3 = \langle U, B_2 \rangle = constant .$$

Since U is unit, we have

$$a_1^2 + \varepsilon_1 a_2^2 + \varepsilon_2 a_3^2 = M . \tag{4}$$

Here M is +1, -1 or 0 depending if U is spacelike, timelike or lightlike, respectively. The differentiation of (3) gives

$$\left(\frac{da_1}{ds} - \varepsilon_1 a_2 k_1\right) T + \left(\frac{da_2}{ds} - a_1 k_1\right) N \\ + \left(a_2 k_2 + \varepsilon_1 a_3 k_3\right) B_1 + a_3' B_2 = 0$$

and from this equation we get

$$\left.\begin{array}{l} a_2 = -\varepsilon_1 \dfrac{k_3}{k_2} a_3 = \varepsilon_1 \dfrac{1}{k_1} \dfrac{da_1}{ds}, \\ \dfrac{da_2}{ds} = -a_1 k_1, \quad a_3' = 0 \end{array}\right\} \tag{5}$$

Since $\dfrac{da_2}{ds} = -a_1 k_1$ and

$$\frac{da_2}{ds} = -\varepsilon_1 \frac{k_1'}{k_1^2} \frac{da_1}{ds} + \frac{\varepsilon_1}{k_1} \frac{d^2 a_1}{ds^2}$$

we find the second order linear differential equation in a_1 given by

$$\varepsilon_1 \frac{d^2 a_1}{ds^2} - \varepsilon_1 \frac{k_1'}{k_1} \frac{da_1}{ds} + a_1 k_1^2 = 0 . \tag{6}$$

If we change variables in the above equation as $t = \int_0^s k_1(s) ds$ then we get

$$\frac{d^2 a_1}{dt^2} + \varepsilon_1 a_1 = 0 .$$

This equation has two solutions: If $\varepsilon_1 = +1$ then the solution is

$$a_1 = A \cos \int_0^s k_1(s) ds + B \sin \int_0^s k_1(s) ds \tag{7}$$

and if $\varepsilon_1 = -1$ then the solution is

$$a_1 = A \cosh \int_0^s k_1(s) ds + B \sinh \int_0^s k_1(s) ds \tag{8}$$

where A and B are constant.
Assume that $\varepsilon_1 = -1$ and consider the solution (8). From (5) and (8) we have

$$a_2 = \frac{k_3}{k_2} a_3 = -A \sinh \int_0^s k_1(s) ds - B \cosh \int_0^s k_1(s) ds$$

$$a_1 = -\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)' a_3 = A \cosh \int_0^s k_1(s) ds \\ + B \sinh \int_0^s k_1(s) ds$$

From these equations it follows that

$$A = a_3 \left(\frac{k_3}{k_2} \sinh \int_0^s k_1(s) ds - \frac{1}{k_1} \left(\frac{k_3}{k_2}\right)' \cosh \int_0^s k_1(s) ds \right) \tag{9}$$

$$B = a_3 \left(-\frac{k_3}{k_2} \cosh \int_0^s k_1(s) ds + \frac{1}{k_1} \left(\frac{k_3}{k_2}\right)' \sinh \int_0^s k_1(s) ds \right) \tag{10}$$

Hence, using (9) and (10) we get

$$B^2 - A^2 = \left[\left(\frac{k_3}{k_2}\right)^2 - \frac{1}{k_1^2} \left(\left(\frac{k_3}{k_2}\right)' \right)^2 \right] a_3^2 = constant$$

So that

$$\left(\frac{k_3}{k_2}\right)^2 - \frac{1}{k_1^2}\left(\left(\frac{k_3}{k_2}\right)'\right)^2 = constant := m\,. \quad (11)$$

From (4), (9), (10) and (11) we have

$$B^2 - A^2 = a_3^2 m = M - 1\,.$$

Thus, the sign of the constant m agrees with the sign of $B^2 - A^2$. So, if U is timelike or light like then m is negative. If U is spacelike then $m = 0$. Then we can give the following corollary.

Result 1: Let $x(s)$ be a spacelike B_2-slant helix with timelike principal normal N in Minkowski 4-space E_1^4 and U be a unit constant vector which makes a constant angle with the second binormal B_2. Then the vector U is spacelike if and only if there exist a constant K such that

$$\frac{k_3}{k_2}(s) = K\exp\left(\int_0^s k_1(t)dt\right).$$

When $\varepsilon_1 = +1$, by using (7) with similar calculations as above, we get that the spacelike curve $x(s)$ is a spacelike B_2-slant helix if and only if

$$\left(\frac{k_3}{k_2}\right)^2 + \frac{1}{k_1^2}\left(\left(\frac{k_3}{k_2}\right)'\right)^2 = constant\,. \quad (12)$$

Thus, using (11) and (12), we can characterize the spacelike B_2-slant helix $x(s)$ by the fact that

$$\left(\frac{k_3}{k_2}\right)^2 + \varepsilon_1\frac{1}{k_1^2}\left(\left(\frac{k_3}{k_2}\right)'\right)^2 = constant\,. \quad (13)$$

Conversely, if the condition (13) is satisfied for a regular spacelike curve we can always find a constant vector U which makes a constant angle with the second binormal B_2 of the curve.

Consider the unit vector U defined by

$$U = \left[\varepsilon_1\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)' T - \varepsilon_1\frac{k_3}{k_2}N + B_2\right].$$

By taking account of the differentiation of (13), differentiation of U gives that $\frac{dU}{ds} = 0$, this means that U is a constant vector. So that, we can give the following theorem:

Theorem 1: A unit speed spacelike curve $x : I \subset IR \to E_1^4$ with nonzero curvatures $k_1(s),\, k_2(s)$ and $k_3(s)$ is a spacelike B_2-slant helix if and only if the following condition is satisfied,

$$\left(\frac{k_3}{k_2}\right)^2 + \varepsilon_1\frac{1}{k_1^2}\left(\left(\frac{k_3}{k_2}\right)'\right)^2 = constant\,.$$

From Theorem 1 one can easily see that the constant function in Theorem 1 is independent of ε_2. So, we can give the following corollary.

Result 2: The characterizations of the spacelike B_2-slant helix are independent of the Lorentzian causal character of the second binormal vector B_2. It is only related to the Lorentzian causal character of the unit principal normal vector N.

Now, we give another characterization of spacelike B_2-slant helix in Minkowski 4-space.

Let assume that $x : I \subset IR \to E_1^4$ is a spacelike B_2-slant helix. Then, Theorem 1 is satisfied. By differentiating (13) with respect to s we get

$$\left(\frac{k_3}{k_2}\right)\frac{d}{ds}\left(\frac{k_3}{k_2}\right) + \frac{\varepsilon_1}{k_1}\frac{d}{ds}\left(\frac{k_3}{k_2}\right)\frac{d}{ds}\left[\frac{1}{k_1}\frac{d}{ds}\left(\frac{k_3}{k_2}\right)\right] = 0 \quad (14)$$

and hence

$$\frac{\varepsilon_1}{k_1}\left(\frac{k_3}{k_2}\right)' = -\frac{\left(\frac{k_3}{k_2}\right)\left(\frac{k_3}{k_2}\right)'}{\left[\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right]'}\,. \quad (15)$$

If we write

$$f(s) = -\frac{\left(\frac{k_3}{k_2}\right)\left(\frac{k_3}{k_2}\right)'}{\left[\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right]'}. \tag{16}$$

Then

$$f(s)k_1 = \varepsilon_1\left(\frac{k_3}{k_2}\right)'. \tag{17}$$

From (14) it can be written

$$\left[\frac{\varepsilon_1}{k_1}\left(\frac{k_3}{k_2}\right)'\right]' = -k_1\frac{k_3}{k_2}. \tag{18}$$

By using (17) and (18) we have

$$\frac{d}{ds}f(s) = -k_1\frac{k_3}{k_2}. \tag{19}$$

Conversely, let $f(s)k_1 = \varepsilon_1\left(\frac{k_3}{k_2}\right)'$ and $\frac{d}{ds}f(s) = -k_1\frac{k_3}{k_2}$. If we define a unit vector U by

$$U = -f(s)T + \varepsilon_1\frac{k_3}{k_2}N - B_2 \tag{20}$$

We have that U and $\langle B_2, U\rangle$ are constants. So, we have the following theorem:

Theorem 2: A unit speed spacelike curve $x: I \subset IR \to E_1^4$ with nonzero curvatures $k_1(s), k_2(s)$ and $k_3(s)$ is a B_2-slant helix if and only if there exists a C^2-function f such that

$$fk_1 = \varepsilon_1\frac{d}{ds}\left(\frac{k_3}{k_2}\right), \; \frac{d}{ds}f(s) = -k_1\frac{k_3}{k_2} \tag{21}$$

Now, we give the integral characterization of the spacelike B_2-slant helix.

Suppose that, the unit speed spacelike curve $x: I \subset IR \to E_1^4$ with nonzero curvatures $k_1(s), k_2(s)$ and $k_3(s)$ is a spacelike B_2-slant helix. Then Theorem 2 is satisfied. Let us define C^2-function φ and C^1-functions $m(s)$ and $n(s)$ by

$$\varphi = \varphi(s) = \int_0^s k_1(s)ds \tag{22}$$

$$\left.\begin{aligned} m(s) &= \frac{k_3}{k_2}\eta(\varphi) + f(s)\mu(\varphi) \\ n(s) &= \frac{k_3}{k_2}\mu(\varphi) - \varepsilon_1 f(s)\eta(\varphi) \end{aligned}\right\} \tag{23}$$

Where $\eta(\varphi) = \cosh\varphi$, $\mu(\varphi) = \sinh\varphi$ if $\varepsilon_1 = -1$; and $\eta(\varphi) = \cos\varphi$, $\mu(\varphi) = \sin\varphi$ if $\varepsilon_1 = +1$. If we differentiate equations (23) with respect to s and take account of (22) and (21) we find that $m' = 0$ and $n' = 0$. Therefore, $m(s) = C$, $n(s) = D$ are constants. Now substituting these in (23) and solving the resulting equations for $\frac{k_3}{k_2}$, we get

$$\frac{k_3}{k_2} = C\eta(\varphi) + D\mu(\varphi). \tag{24}$$

Conversely if (24) holds then from the equations in (23) we get

$$f = \varepsilon_1\left(C\mu(\varphi) - D\eta(\varphi)\right)$$

which satisfies the conditions of Theorem 2? So, we have the following theorem:

Theorem 3: A unit speed spacelike curve $x: I \subset IR \to E_1^4$ with nonzero curvatures $k_1(s), k_2(s)$ and $k_3(s)$ is a spacelike B_2-slant helix if and only if the following condition is satisfied

$$\frac{k_3}{k_2} = C\eta(\varphi) + D\mu(\varphi)$$

where C and D are constants, $\eta(\varphi) = \cosh\varphi$,

$\mu(\varphi)=\sinh\varphi$ if $\varepsilon_1=-1$; and $\eta(\varphi)=\cos\varphi$, $\mu(\varphi)=\sin\varphi$ if $\varepsilon_1=+1$.

Conclusions

In this paper, the spacelike B_2-slant helix is defined and the characterizations of the spacelike B_2-slant helix are given in Minkowski 4- space E_1^4. It is shown that a spacelike curve $x: I\subset IR\rightarrow E_1^4$ is a B_2-slant helix if an equation holds between the first, second and third curvatures of the curve. Furthermore, the integral characterization of the spacelike B_2-slant helix is given.

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