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Full Length Research Paper

Approximate analytic solutions for non-linear oscillators with fractional-order restoring force by means of the optimal variational iteration method

Vasile Marinca[1,2] and Nicolae Herişanu[1,2]*

[1]Politehnica University of Timişoara, Bd. Mihai Viteazu, nr.1, 300222 Timişoara, Romania.
[2]Center for Advanced and Fundamental Technical Research, Romanian Academy, Timişoara Branch, Romania.

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In this paper, a new approach, namely the optimal variational iteration method (OVIM) was proposed to investigate oscillators with fractional-power nonlinearities. It was illustrated that this approach is very effective and convenient and does not require linearization or small parameter. Unlike other classical iteration methods, only one iteration leads to highly accurate results, due to a rigorous procedure for convergence control. Approximate analytical results obtained through the proposed method were compared with numerical results and an excellent agreement was found, which prove the validity of the proposed procedure. This method can be easily applied to other strongly nonlinear problems.

Key words: Optimal variational iteration method, nonlinear differential equation, analytical approximations, fractional-power restoring force.

INTRODUCTION

It is well-known that analytic approximations of nonlinear problems often break down as nonlinearity becomes strong. Classical perturbation approximations are valid only for weakly nonlinear problems (Nayfeh and Mook, 1979; Hagedorn, 1988). The use of perturbation techniques in many important practical problems is invalid, or it simply breaks down for parameters beyond a certain specified range. Therefore, new analytical techniques should be developed to overcome these shortcomings. Such new techniques should work over a larger range of parameters and yield accurate analytical approximate solutions beyond the coverage and ability of the classical perturbation methods.

There are known in the literature some attempts in developing new analytical techniques valid for strongly nonlinear problems. Some extensions of the Lindstedt-Poincare perturbation method to strongly nonlinear systems have been proposed (Lim et al., 2009; Ramos, 2008; Pakdemirli et al., 2009). Other recently proposed analytical approaches intended to solve strongly nonlinear oscillators, are the variational iteration method (Chen et al., 2010; He et al., 2010; Marinca and Herişanu, 2008), the parameter-expanding method (Zengin et al., 2009), the optimal homotopy asymptotic method (Herisanu et al., 2008), the optimal homotopy perturbation method (Marinca and Herişanu, 2010) and so on.

Concerning the specific category of nonlinear oscillators with fractional-power nonlinearities, a mixture of methodologies was employed in an attempt to achieve accurate results. Some piecewise-linearized methods were used by Ramos (2007), the homotopy perturbation method was applied by Beléndez et al. (2007), an analytical approximate technique which incorporates salient features of both Newton's method and harmonic balance method is applied by Lim and Wu (2005), a method based on the combination of the Krylov–Bogoliubov method with Hamilton's variational principle with the uncommutative rule for the variation of velocity is

*Corresponding author. E-mail: herisanu@mec.upt.ro. Tel/Fax: +40-256-403629.

used (Kovacic, 2009). Some of these approaches need higher-order approximations to achieve accurate results, others need piecewise-linearization.

The purpose of this paper is to construct accurate approximations to periodic solutions and frequencies of non-linear oscillators with fractional-power restoring force by applying the optimal variational iteration method (OVIM). The most significant feature of this new approach is the optimal control of the convergence of approximations, which lead to accurate results after only one iteration and therefore higher-order approximations are not needed. Different from other methods, the validity of the OVIM is independent on whether or not there exist small parameters in the considered nonlinear equations. This application will demonstrate the general validity of the OVIM, its effectiveness, accuracy and its potential for solving nonlinear oscillators.

In order to develop an application of the proposed method, we consider the nonlinear oscillator with fractional-power restoring force:

$$\frac{d^2x}{dt^2}+\omega_0^2x+\alpha^2x|x|^{n-1}-f\cos\Omega t=0 \tag{1}$$

where ω_0, α, f, Ω are constants and $n>0$ rational. Initial conditions are given by

$$x(0)=A\ ,\ \dot{x}(0)=0 \tag{2}$$

where $\cdot=\frac{d}{dt}$.

The exact period of vibrations for the oscillator (1) with $\omega_0=f=0$ is calculated by Cveticanin (2009). There exists no small parameter in the equation and therefore the traditional perturbation methods cannot be applied directly. Attention here is restricted primarily to rational powers.

For investigating the nonlinear oscillator described in Equation 1 we propose an approximate analytical solution using OVIM. In case of the classical VIM (He et al., 2009), initial approximations contain unknown parameters which can be identified by initial or boundary conditions after few iterations, while in case of OVIM, initial approximation contains a number of unknown parameters larger than the number of initial/boundary conditions. These parameters can be identified partially from the initial/boundary conditions and the rest of them can be optimally identified so that the residual functional be minimized. Following this procedure, often we need only one iteration for accurately solving the problem.

In general, we consider the following nonlinear equation:

$$\ddot{x}+\omega^2x+h(t,x,\dot{x},\ddot{x})=0 \tag{3}$$

where h is assumed to be a nonlinear function, which may be expanded in a Fourier series. We construct the following iteration formula (He et al., 2010; Marinca and Herişanu, 2008):

$$x_{n+1}(t)=x_n(t)+\int_0^t\lambda(\tau,t)[x_n''(\tau)+\omega^2x_n(\tau)+h(\tau,\tilde{x}_n,\tilde{x}_n',\tilde{x}_n'']d\tau \tag{4}$$

where $\lambda(\tau,t)$ is the Lagrange multiplier which can be identified via variational theory and $\tilde{x}_n$ is a restricted variation $\delta\tilde{x}_n=0$. Calculating variation with respect to x_n, the following stationary conditions are obtained:

$$\begin{aligned}&\lambda''(\tau,t)+\omega^2\lambda(\tau,t)=0\\&\lambda(\tau,t)\big|_{\tau=t}=0\\&1-\lambda'(\tau,t)\big|_{\tau=t}=0\end{aligned} \tag{5}$$

Therefore, the Lagrange multipliers can be readily identified:

$$\lambda(\tau,t)=\frac{1}{\omega}\sin\omega(\tau-t) \tag{6}$$

and as a result, we obtain the following iteration formula

$$x_{n+1}(t)=x_n(t)+\frac{1}{\omega}\int_0^t\sin\omega(\tau-t)\left[x_n''(\tau)+\omega^2x_n(\tau)+h(\tau,x_n,x_n',x_n'')\right]d\tau \tag{7}$$

Equation 7 is equivalent to

$$x_{n+1}(t)=x_0(t)+\frac{1}{\omega}\int_0^t\sin\omega(\tau-t)h(\tau,x_n,x',x'')d\tau \tag{8}$$

Initial conditions for Equation 3 are:

$$x(0)=a\ ,\qquad \dot{x}(0)=0 \tag{9}$$

In our procedure, the initial iteration $x_0(t)$ contains $p>2$ unknown parameters. Two of them can be determined from Equation 9 and the rest of p-2 parameters can be optimally determined from the stationary conditions of the residual functional or by other methods such as Galerkin method, collocation method, least square method (Herişanu and Marinca, 2010).

Analytic solutions for oscillators with fractional-power nonlinearities

It can be shown that

$$x|x|^{n-1} = |x|^n \, signx \tag{10}$$

so that Equation 1 can be rewritten as

$$\ddot{x} + \omega^2 x + (\omega_0^2 - \omega^2)x + \alpha^2 |x|^n signx = f \cos \Omega t \tag{11}$$

Equation 4 describes a system oscillating with the unknown frequency ω and the period T. We switch to a scalar time $\tau = \frac{2\pi t}{T} = \omega t$ and therefore under the transformation

$$\tau = \omega t \tag{12}$$

we can rewrite Equation 3 in the form

$$x''(\tau) + x(\tau) + h(\tau, x(\tau)) = 0 \tag{13}$$

where a prime denotes differentiation with respect to τ and

$$h(\tau, x(\tau)) = \left(\frac{\omega_0^2}{\omega^2} - 1\right)x + \frac{\alpha^2}{\omega^2}|x|^n signx - \frac{f}{\omega^2}\cos\frac{\Omega}{\omega}\tau \tag{14}$$

As an initial approximation for $x_0(\tau)$ we choose:

$$x_0(\tau) = C_1 \cos\tau + C_2 \cos 3\tau + C_3 \cos 5\tau + C_4 \cos 7\tau \tag{15}$$

where C_1, C_2, C_3 and C_4 are unknown constants which partially can be determined from Equation 9:

$$C_1 + C_2 + C_3 + C_4 = a \tag{16}$$

It is known that if *g* is an analytic function, then

$$g(y+p) = g(y) + \frac{p}{1!}g'(y) + \frac{p^2}{2!}g''(y) + \ldots ,$$
$$k! = 1\cdot 2\cdot 3\cdot \ldots \cdot k \tag{17}$$

where a prime denotes differentiation with respect to *y* for any real *y* and *p*. In our case:

$$g(x) = |x|^n signx \quad g'(x) = n|x|^{n-1} \quad , \quad y = C_1\cos\tau \quad ,$$
$$p = C_2\cos 3\tau + C_3 \cos 5\tau + C_4\cos 7\tau \tag{18}$$

From Equation 8 for n=0 it is obtained:

$$x_1(\tau) = x_0(\tau) + \int_0^\tau sin(u-\tau)h(u, x_0(u))du \tag{19}$$

where $h(u, x_0(u))$ is obtained substituting Equation 15 into Equation 14. On the other hand, using only the first two terms into Equation 17, we can approximate *g* in the form:

$$g(x_0(u)) = g(C_1\cos u) + (C_2\cos 3u + C_3\cos 5u + C_4\cos 7u)g'(C_1\cos u) \tag{20}$$

For $g(C_1 \cos u)$, we obtain the following Fourier series expansions:

$$g(C_1\cos u) = C_1^n(a_{1n}\cos u + a_{3n}\cos 3u + a_{5n}\cos 5u + a_{7n}\cos 7u + \ldots) \tag{21}$$

where

$$a_{2j+1,n} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}(\cos u)^n \cos(2j+1)u du \, , \quad j = 0,1,2,..n \in \Re \tag{22}$$

The last term in Equation 20 can be written in the form From Equations 23 we obtain

$$\begin{aligned}(C_2\cos 3u + C_3\cos 5u + C_4\cos 7u)g'(C_1\cos u) = nC_1^{-1}(C_1\cos u)^n[C_2(2\cos 2u - 1) + \\ + C_3(2\cos 4u - 2\cos 2u + 1) + C_4(2\cos 6u - 2\cos 4u + 2\cos 2u - 1)]\end{aligned} \tag{23}$$

$$\begin{aligned}&(C_2\cos 3u + C_3\cos 5u + C_4\cos 7u)g'(C_1\cos u) = \\ &nC_1^{n-1}\{(a_{3n}C_2 + a_{5n}C_3 + a_{7n}C_4)\cos u + \\ &+[a_{1n}C_2 + a_{3n}(C_3 - C_2) + a_{5n}(C_2 - C_3 + C_4) + \\ &a_{7n}(C_3 - C_4)]\cos 3u + [a_{1n}C_3 + \\ &a_{3n}(C_2 - C_3 + C_4) + a_{5n}(C_3 - C_2 - C_4) + \\ &a_{7n}(C_2 - C_3 + C_4)]\cos 5u + [a_{1n}C_4 + \\ &a_{3n}(C_3 - C_4) + a_{5n}(C_2 - C_3 + C_4) \\ &+ a_{7n}(C_2 - C_3 + C_4)]\cos 9u + [a_{5n}C_4 + \\ &+ a_{7n}(C_3 - C_4)]\cos 11u + a_{7n}C_4\cos 13u\}\end{aligned} \tag{24}$$

Substituting Equations 24 and 21 into Equation 20 and then substituting Equation 20 into Equation 1), we obtain

$$\begin{aligned}h(u, x_0(u)) = \Bigg[\left(\frac{\omega_0^2}{\omega^2} - 1\right)C_1 + \frac{\alpha^2}{\omega^2}(a_{1n}C_1^n + na_{3n}C_1^{n-1}C_2 \\ + na_{5n}C_1^{n-1}C_3 + na_{7n}C_1^{n-1}C_4)\Bigg]\cos\tau + N.R.T.\end{aligned} \tag{25}$$

where N.R.T. stands for nonresonant terms.

Avoiding the presence of secular terms in the right-hand side of Equation 19, we obtain from Equation 25 the frequency ω as

$$\omega^2=\omega_0^2+\alpha^2C_1^{n-2}(a_{1n}C_1+na_{3n}C_2+na_{5n}C_3+na_{7n}C_4) \tag{26}$$

With this requirement, Equation 25 becomes

$$h(u,x_0(u))=A\cos 3u+B\cos 5u+C\cos 7u+ D\cos 9u+E\cos 11u+F\cos 13u-\frac{f}{\omega^2}\cos\frac{\Omega}{\omega}u \tag{27}$$

where:

$$A=\left(\frac{\omega_0^2}{\omega^2}-1\right)C_2+\frac{\alpha^2C_1^{n-1}}{\omega^2}[a_{3n}C_1+n(a_{1n}-a_{3n}+a_{5n})C_2+n(a_{3n}-a_{5n}+a_{7n})C_3+n(a_{5n}-a_{7n})C_4];$$

$$B=\left(\frac{\omega_0^2}{\omega^2}-1\right)C_3+\frac{\alpha^2C_1^{n-1}}{\omega^2}[a_{5n}C_1+n(a_{3n}-a_{5n}+a_{7n})C_2+ -n(a_{1n}-a_{3n}+a_{5n}-a_{7n})C_3+n(a_{3n}-a_{5n}+a_{7n})C_4];$$

$$C=\left(\frac{\omega_0^2}{\omega^2}-1\right)C_4+\frac{\alpha^2C_1^{n-1}}{\omega^2}[a_{7n}C_1+n(a_{5n}-a_{7n})C_2+ n(a_{3n}-a_{5n}+a_{7n})C_3+n(a_{1n}-a_{3n}+a_{5n}-a_{7n})C_4];$$

$$D=\frac{\alpha^2}{\omega^2}nC_1^{n-1}[a_{7n}C_2+(a_{5n}-a_{7n})C_3+(a_{3n}-a_{5n}+a_{7n})C_4]; \tag{28}$$

$$E=\frac{\alpha^2}{\omega^2}nC_1^{n-1}[a_{7n}C_3+(a_{5n}-a_{7n})C_4]\ ;\ F=a_{7n}nC_1^{n-1}C_4$$

Substituting Equations 27 and 12 into Equation 19 yields:

$$\begin{aligned}x_1(t)=&C_1\cos\omega t+C_2\cos 3\omega t+C_3\cos 5\omega t+C_4\cos 7\omega t+\\&\frac{A}{8}(\cos 3\omega t-\cos\omega t)+\frac{B}{24}(\cos 5\omega t-\cos\omega t)+\\&+\frac{C}{48}(\cos 7\omega t-\cos\omega t)+\frac{D}{80}(\cos 9\omega t-\cos\omega t)+\\&\frac{E}{120}(\cos 11\omega t-\cos\omega t)+\frac{F}{168}(\cos 13\omega t-\cos\omega t)+G(t)\end{aligned} \tag{29}$$

where ω is given by Equation 26 and

$$G(t)=\begin{cases}\dfrac{f\Omega}{\omega(\omega^2-\Omega^2)}[1-\cos(\omega-\Omega)t] & ,\ \omega\neq\Omega\\[2mm] \dfrac{2f}{\omega^2\varepsilon(2+\varepsilon)}\sin\dfrac{1}{2}\varepsilon\omega t\sin(\omega+\dfrac{1}{2}\varepsilon\omega)t & ,\ \Omega=\omega(1+\varepsilon)\ ,\ \varepsilon\ll 1\end{cases} \tag{30}$$

At this moment the first approximation given by Equation 29 depends on the parameters C_1, C_2, C_3 and C_4. These constants can be identified via various methods such as the collocation method, the Galerkin method, the least square method etc. If $R(t,C_1,C_2,C_3,C_4)$ is the residual obtained substituting the first approximation (29) into Equation 1:

$$R(t,C_1,C_2,C_3,C_4)=\ddot{x}_1+\omega_0x_1+\alpha^2x_1|x_1|^{n-1}-f\cos\Omega t \tag{31}$$

and if the functional J is given by the integral

$$J(C_1,C_2,C_3,C_4)=\int_0^{\frac{2\pi}{\omega}}R^2(t,C_1,C_2,C_3,C_4)dt \tag{32}$$

Then the constants C_1, C_2, C_3 and C_4 can be determined from Equations 16 and from the conditions that J to be minimized, or from the condition of conditioned minimum:

$$\frac{\partial J}{\partial C_1}-\frac{\partial J}{\partial C_2}=0\ ;\ \frac{\partial J}{\partial C_1}-\frac{\partial J}{\partial C_3}=0\ ;\ \frac{\partial J}{\partial C_1}-\frac{\partial J}{\partial C_4}=0 \tag{33a}$$

or, alternatively, the same constants can be achieved imposing the conditions

$$R(t_i,C_1,C_2,C_3,C_4)=0,\quad i=1,2,3,4 \tag{33b}$$

Therefore, it is to be note that OVIM provide us with a simple way to adjust and optimally control the convergence of solutions.

NUMERICAL EXAMPLES

We illustrate the accuracy of the OVIM by comparing the previously obtained approximate solutions (29) with the numerical integration results obtained by a fourth-order Runge-Kutta method.

1. In the first case we consider $n=\frac{1}{3}$ and from Equation 22 it is obtained

$$a_{1,\frac{1}{3}} = 1.159595266; a_{3,\frac{1}{3}} = -0.231919053; a_{5,\frac{1}{3}} = 0.115959526; a_{7,\frac{1}{3}} = -0.073792426$$

a) For a=5, α=1, ω_0=0, f=0, from Equations 16 and 33 we obtain

$$C_1 = 5.083874107; C_2 = 0.005746695; C_3 = -0.057474928; C_4 = -0.032145873$$

The first-order approximate solution (29) becomes:

$$x_1(t) = 5.194375003\cos\omega t - 0.12115924\cos 3\omega t - 0.034843933\cos 5\omega t - \\ - 0.038371881\cos 7\omega t + 0.000007998\cos 9\omega t \quad (34)$$

where $\omega = 0.6269145$.

b) For a=5, α=1, ω_0=1, f=0, the following results are obtained:

$$C_1 = 5.018214118; C_2 = 0.01968471; C_3 = -0.015158401; C_4 = -0.022740428$$

$$x_1(t) = 5.05176786\cos\omega t - 0.016254795\cos 3\omega t - 0.011945177\cos 5\omega t - \\ - 0.023567886\cos 7\omega t + 0.000006078\cos 9\omega t \quad (35)$$

where $\omega = 1.1817241$.

c) For a=5, α=1, ω_0=1, f=0.1, Ω=1, it is obtained

$$C_1 = 5.020224969; C_2 = 0.018260618; C_3 = 0.007441779; C_4 = -0.045927367$$

$$x_1(t) = 5.053982615\cos\omega t - 0.01779699\cos 3\omega t + 0.010444517\cos 5\omega t - 0.046645334\cos 7\omega t + 0.000020455\cos 9\omega t + 0.223011649(1 - \cos 0.17537309t) \quad (36)$$

where $\omega = 1.17537309$.

2. In the second case we consider n=2. From Equation 22 it is obtained:

$$a_{1,2} = 0.848826363; a_{3,2} = 0.169765272; a_{5,2} = -0.02425218; a_{7,2} = 0.00808406$$

a) For a=5, α=1, ω_0=0, f=0, we have

$$C_1 = 4.929081167; C_2 = 0.000890717; C_3 = 0.030884506; C_4 = 0.039143609$$

$$x_1(t) = 4.79659747\cos\omega t + 0.12566055\cos 3\omega t + 0.032727712\cos 5\omega t + \\ + 0.044810445\cos 7\omega t + 0.000203823\cos 9\omega t \quad (37)$$

where $\omega = 2.045312$.

b) For a=5, α=1, ω_0=1, f=0, we obtain

$$C_1 = 4.952586959; C_2 = -0.015000551; C_3 = 0.034838155; C_4 = 0.027575436$$

$$x_1(t) = 4.8543132899\cos\omega t + 0.085782596\cos 3\omega t + 0.030887963\cos 5\omega t + \\ + 0.028913121\cos 7\omega t + 0.000103034\cos 9\omega t \quad (38)$$

where $\omega = 2.2807431$.

c) For a=5, α=1, ω_0=1, f=0.1, Ω=1, we have

$$C_1 = 4.936194326; C_2 = -0.025541995; C_3 = 0.059682125; C_4 = 0.029665543$$

$$x_1(t) = 4.837168737\cos\omega t + 0.075535083\cos 3\omega t + 0.056053186\cos 5\omega t + \\ + 0.031232993\cos 7\omega t + 0.000091955\cos 9\omega t + 0.010503799(1 - \cos 1.2764321t) \quad (39)$$

where $\omega = 2.2764321$.

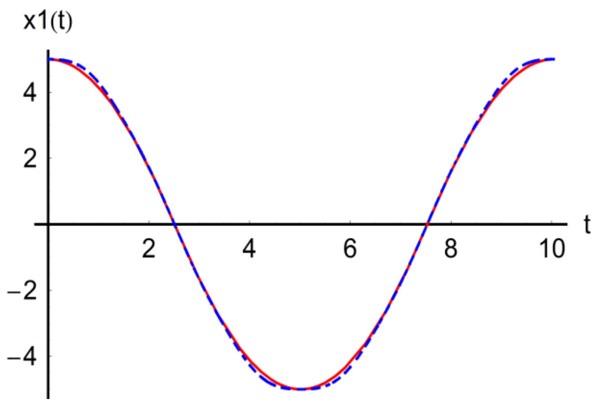

Figure 1. Comparison between the approximate solution (34) and numerical results in case $n = \frac{1}{3}$, *a*=5, *ω₀*=0, *f*=0, ______ numerical solution _ _ _ _ approximate solution (34).

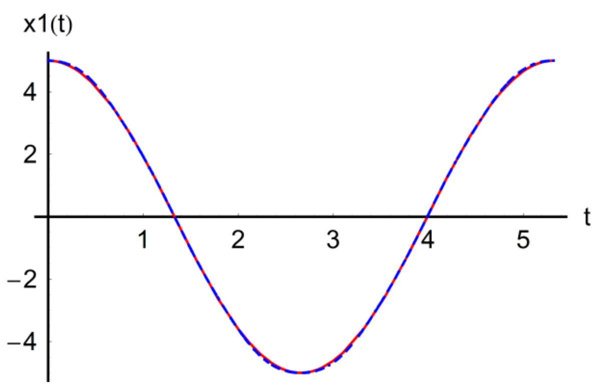

Figure 2. Comparison between the approximate solution (35) and numerical results in case $n = \frac{1}{3}$, *a*=5, ω_0=1, *f*=0 ______ numerical solution_ _ _ _ approximate solution (35).

4.3. In the third case, we consider $n = \frac{5}{3}$. From Equation 22 it is obtained:

$$a_{1,\frac{5}{3}} = 0.891467659; a_{3,\frac{5}{3}} = 0.127352522; a_{5,\frac{5}{3}} = -0.025470504; a_{7,\frac{5}{3}} = 0.009796347$$

a) For *a*=2, *α*=1, ω_0=0, *f*=0, we have:

$$C_1 = 1.971343187; C_2 = 0.020411906; C_3 = 0.001490562; C_4 = 0.006754343$$
$$x_1(t) = 1.958991 \cos \omega t + 0.0334889 \cos 3\omega t + 0.000654408 \cos 5\omega t + 0.0068656 \cos 7\omega t + 0.000001134 \cos 9\omega t \tag{40}$$

where $\omega = 1.1854322$.

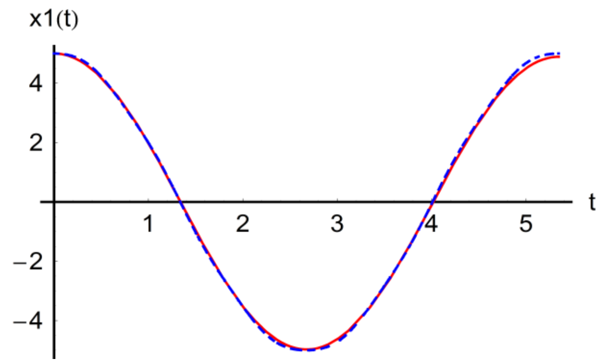

Figure 3. Comparison between the approximate solution (36) and numerical results in case $n = \frac{1}{3}$, *a*=5, ω_0=1, *f*=0.1 ______ numerical solution_ _ _ _ approximate solution (36).

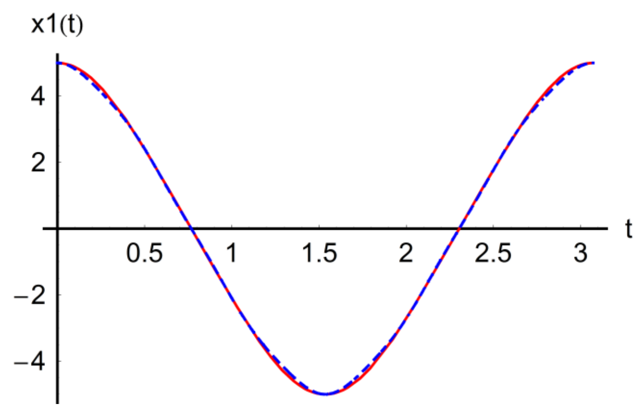

Figure 4. Comparison between the approximate solution (37) and numerical results in case *n*=2, *a*=5, ω_0=0, *f*=0 ______ numerical solution_ _ _ _ approximate solution (37).

b) For *a*=2, *α*=1, ω_0=1, *f*=0, it is obtained

$$C_1 = 1.985630001; C_2 = 0.00505791; C_3 = 0.0000205581; C_4 = 0.00929309$$

$$x_1(t) = 1.977958874 \cos \omega t + 0.013179562 \cos 3\omega t - 0.000490736 \cos 5\omega t + 0.009346505 \cos 7\omega t + 0.000008539 \cos 9\omega t \tag{41}$$

where $\omega = 1.553241$.

Figures 1 to 8 show the comparison between the present solutions and the numerical integration results obtained by a fourth-order Runge-Kutta method in the aforementioned cases.

As it can be observed, Figures 1 to 8 have been obtained for different working conditions of the considered oscillator, for small and also for large initial

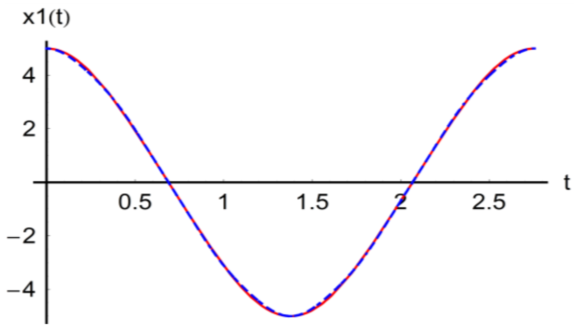

Figure 5. Comparison between the approximate solution (38) and numerical results in case *n*=2, *a*=5, ω_0=1, *f*=0 ______ numerical solution_ _ _ _ approximate solution (38).

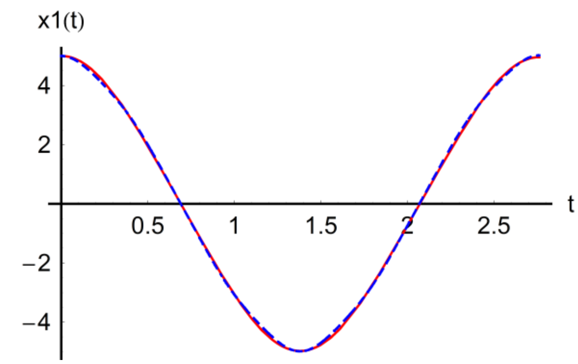

Figure 6. Comparison between the approximate solution (39) and numerical results in case *n*=2, *a*=5, ω_0=1, *f*=0.1 ______ numerical solution_ _ _ _ approximate solution (39).

amplitudes. Since the motion is periodic, only the first period is plotted for each considered case. Two fractional powers were considered for the restoring force: n=1/3 illustrated in Figures 1 to 3 and n=5/3 illustrated in Figures 7 to 8, respectively, for different parameters *a*, ω_0, and *f*. Additionally, an integer power (*n*=2) is considered in a supplementary case illustrated in Figures 4 to 6. Two distinct situations were analysed: functioning with (Figures 3 and 6) and without (Figures 1, 2, 4, 5, 7 and 8) harmonic excitation. These figures show that the first-order analytical approximations obtained by OVIM provide excellent solutions for non-linear oscillators with fractional-order restoring force. In all considered cases, the error between the analytical and numerical results is remarkably good since the solutions obtained by OVIM are nearly identical with the solutions given by the numerical method.

An error analysis is presented in Table 1, where we computed the maximum absolute error between the numerical and analytical results in the considered cases:

$$Err = \left| x_{1app} - x_{R-K} \right|.$$

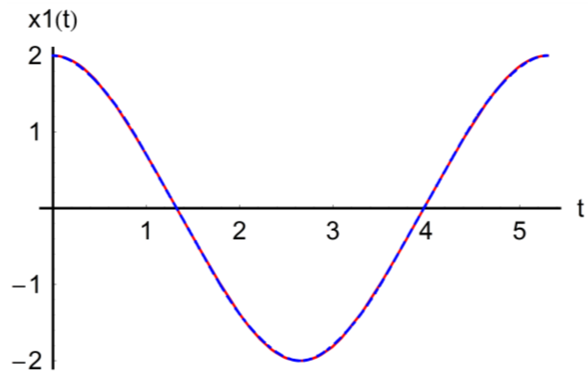

Figure 7. Comparison between the approximate solution (40) and numerical results in case $n=\frac{5}{3}$, *a*=2, ω_0=0, *f*=0 ______ numerical solution, _ _ _ _ approximate solution (40).

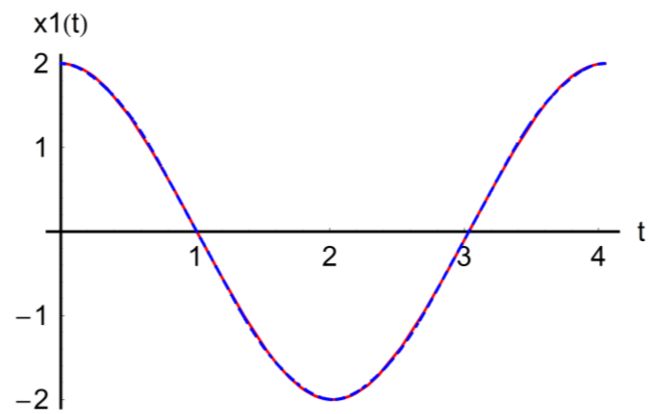

Figure 8. Comparison between the approximate solution (41) and numerical results in case $n=\frac{5}{3}$, *a*=2, ω_0=1, *f*=0 ______ numerical solution_ _ _ _ approximate solution (41).

One can observe that in the cases corresponding to small initial amplitudes (4.3.a and 4.3.b with correspondent Figures 7 and 8, respectively), the obtained error is much better than the error obtained for large amplitudes, which is acceptable. The larger absolute error is obtained in the case 4.1.c (Figure 3) corresponding to the oscillator with non-zero perturbing force in the conditions $n=\frac{1}{3}$, *a*=5, ω_0=1, *f*=0.1, α=1.

Analysing the Figures 1 to 8 we can conclude that an increased value of the power *n* leads to decreasing the period of motion, as it can be observed comparing Figures 1 and 4 or Figures 2 and 5, respectively, which are couples of figures obtained for the same parameters a, ω_0 and f. The same remark is applicable also in the presence of the perturbing force, when one can be seen

Table 1. Maximum absolute error between the numerical and analytical results in the considered cases.

Case	4.1.a (Figure1)	4.1.b (Figure 2)	4.1.c (Figure 3)	4.2.a (Figure 4)	4.2.b (Figure 5)	4.2.c (Figure 6)	4.3.a (Figure 7)	4.3.b (Figure 8)
Maximum error	0.1561	0.0567	0.3158	0.1387	0.0948	0.1321	0.0142	0.0142

that increasing the value of the power n leads to a significant decreasing of the period of motion, as it is illustrated in Figures 3 and 6. For the same frequency of the harmonical excitation (Ω=1), the pulsation ω of the oscillating system increases once the value of the power n is increasing.

Similarly, the presence of a non-zero parameter ω_0 leads to decreasing the period of the motion, which is illustrated comparing Figures 1 and 2 or Figures 4 and 5 or Figures 7 and 8. It is interesting to observe that the period of motion is slowly decreasing for larger values of the powers n (n=2), as it can be seen comparing Figures 4 and 5, while there is abrupt decrease for smaller values of n (n=1/3), as it can be seen comparing Figures 1 and 2.

We note that the Lagrange multipliers $\lambda(\tau,t)$ given by Equation 6 and the initial approximation $x_0(\tau)$ together with the procedure intended to determine the optimal values of the convergence-control parameters C_i involved in the initial approximation guarantee a fast convergence of the solutions in every aforementioned case. This is the reason why the OVIM is valid for different types of highly nonlinear problems. We underline that OVIM provides a great freedom in choosing the initial approximations and implicitly offers a convenient way to control the convergence of the solutions.

Conclusions

In this work, a new analytical technique, called the optimal variational iteration method is employed to propose an analytic approximate solution for some nonlinear oscillations. The validity of the procedure is illustrated on the oscillators with fractional-power nonlinearities. The proposed procedure (OVIM) is valid even if the considered nonlinear equation does not contain any small or large parameter. The OVIM provides us with a simple way to optimally control and adjust the convergence of a solution and can give very good approximations in a few terms. Unlike other classical iteration methods, in this case, only one iteration leads to highly accurate results, due to the rigorous procedure for convergence control. This version of the method proves to be very rapid, effective and accurate. We proved this by comparing the solution obtained through the proposed method with the solution obtained via numerical integration using a fourth-order Runge-Kutta method. This work shows one step in the attempt to develop a new nonlinear analytical technique in the absence of small or large parameters.

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