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Full Length Research Paper

Analysis of inventory-production systems with Weibull distributed deterioration

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We studied the inventory-production system with two-parameter Weibull distributed deterioration items. The solution of optimal inventory production control problem with Weibull distribution deteriorating item was revisited based on Baten and Kamil (2009) and carried out using Pontryagin maximum principle. It was also illustrated with an example. The overall inventory and production control were tested for different demand patterns. The inventory production controlled systems of non-linear differential equations were solved numerically.

Key words: Inventory-production system, Weibull distributed deterioration, optimal control, Pontryagin maximum principle, performance measure.

INTRODUCTION

Inventory-production system consists of a manufacturing plant and a finished goods warehouse to store those products which are manufactured but not immediately sold. The control of dynamic inventory production systems that evolve over time called continuous time systems or discrete-time systems depending on whether time varies continuously or discretely which is a rich research area (Sethi and Thompson, 2000). In this paper we are interested to find optimal ways to control a dynamic system using optimal control theory. The optimal control theory has been applied to different inventory-production control problems to analyze the effect of deterioration and the variations in the demand rate with time. A number of studies have been done with the assumption that the deterioration rate follows the Weibull distribution (Chakrabarty et al., 1998; Chen and Lin, 2003; Ghosh and Chaudhuri, 2004; Goel and Aggarwal, 1980; Wu and Lee, 2003).

The assumption of the constant deterioration rate was relaxed by Covert and Philip (1973) who used a two-parameter Weibull distribution to represent the distribution of time to deterioration. This model was further generalized by Philip (1974) by taking a three-parameter Weibull distribution. Mishra (1975) also adopted a two-parameter, Weibull distribution deterioration to develop an inventory model with a finite rate of replenishment. Al-khedhairi and Tadj (2007) studied the optimal control model of a production inventory system with Weibull distributed deterioration. Shah and Acharya (2008) minimized the total cost per unit of an inventory system with two parameter Weibull distributed deterioration under assumption of exponentially decreasing demand. These types of problems have been studied to determine the optimum order quantity for different demand patterns (Happing and Wang, 1990; Bahari-Kashani, 1989; Andijani and AL-Dajani, 1998; AL-Majed, 2002; Bounkhel et al., 2005; Benhadid et al., 2008; Baten and Kamil, 2009).

We setup an optimal control model and utilize the optimal control theory to obtain optimal production policy for inventory production systems where the novelty we take into consideration is that the time of deterioration is a random variable followed by the two-parameter Weibull distribution. This distribution can be used to model either increasing or decreasing rate of deterioration, according to the choice of the parameters. The probability density function of a two-parameter Weibull distribution is given by

$$f(t) = \eta\gamma t^{\gamma-1}\exp\left\{-\eta t^{\gamma}\right\}, \qquad t > 0,$$

where $0 \le \eta \le 1$ is the scale parameter, $\gamma \ge 1$ is the

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shape parameter, and $t > 0$ is the time of deterioration (Mishra and Shah, 2008; Shah and Acharya, 2008). The probability distribution function is

$$F(t) = 1 - \exp\left\{-\eta t^{\gamma}\right\}, \quad t > 0.$$

The instantaneous rate of deterioration of the on-hand inventory is given by

$$\delta(t) = \frac{f(t)}{1 - F(t)} = \eta\gamma t^{\gamma-1}, \quad 0 \le t \le T.$$

In this study we revisited the article of Baten and Kamil (2009) and give a numerical solution of the inventory production controlled system with two parameter Weibull distributed deterioration. Pontryagin maximum principle is used to obtain an optimal control policy of the inventory production model. We focus on the analysis of a production inventory system in which the nonlinear holding and production cost are treated as the function of the inventory level and production rate respectively.

This paper is organized as follows. In section-2 we explain an inventory-production model and setup an optimal control problem. In section-3 we briefly present the optimal solution of such problem and give an example. Section-3 discusses the numerical solution of the inventory production controlled system using different types of demand functions. Finally conclusions of the results are presented in the last section.

OPTIMAL CONTROL OF INVENTORY PRODUCTION MODEL

In this section, we are concerned with mathematical formulation and optimal control of inventory production model for Weibull deteriorating items. The problem is represented as an optimal control problem with state and control variables which are the inventory level and production rate respectively. We develop the analytical solution of the inventory production controlled system and then we give the numerical solution as well as display it graphically. The solution of the inventory production controlled systems include different cases of the demand which are: sinusoidal function of time t, twice-sinusoidal function of time t, co-sinusoidal function of time t, exponential decreasing function of time t and exponential increasing function of time t.

The model assumptions

This subsection is devoted to discuss the model assumptions.We consider that a firm can manufactures a certain product, selling some and stocking the rest in a warehouse. We assume that the demand rate varies with time. The production rates are itself the rates of continuous supply to inventory levels. We assume that the firm has no shortage, the instantaneous rate of deterioration of the on-hand inventory follows the two-parameter Weibull distribution and the production is continuous. We also assume that the firm has set an inventory goal level $\hat{x}(t)$ and a production goal rate $\hat{u}(t)$ and is looking for a pair $(u(t); x(t))$ which converges to $(\hat{u}(t); \hat{x}(t))$ and minimizes the cost function.

Let us define the following variables and parameters:

$x(t)$ is the inventory level in the warehouse at any instant of time $t \in [0, T]$,

$\hat{x}(t)$ is an inventory goal level,

$h > 0$ is the inventory holding cost coefficient incurred for the inventory level,

$u(t) > 0$ is the firm manufactured units of the production rate at any instant of time $t \in [0, T]$,

$\hat{u}(t)$ is the production goal rate.

$C > 0$ is the production cost coefficient.

$y(t)$ is the demand rates.

$T > 0$ represents the fixed length of the planning horizon.

We will setup the optimal inventory production control problem using the above assumptions and we will give the analytical and a numerical solution of this problem with Weibull deteriorating items. The inventory production controlled system will be solved numerically for different types of the demand functions.

Mathematical modeling and previous work

Setup of optimal control problem

This subsection studies to arrive at a mathematical description and to predict the response of the inventory production model with Weibull deterioration rate. Since our objective is to give a numerical solution of the inventory production controlled system with revisiting the article of Baten and Kamil (2009) which minimizes the setup and the inventory costs, the objective function can be expressed as the quadratic form:

$$\text{minimize } J = \frac{1}{2}\int_0^T \left\{h\left[x(t) - \hat{x}(t)\right]^2 + C\left[u(t) - \hat{u}(t)\right]^2\right\} dt \qquad (2.1)$$

The economic interpretation of this objective function (2.1) is that we want to keep the inventory $x(t)$ as close as possible to its goal $\hat{x}(t)$ and also keep the production rate $u(t)$ as close to its goal level $\hat{u}(t)$. The quadratic terms $h\left[x(t) - \hat{x}(t)\right]^2$ and $C\left[u(t) - \hat{u}(t)\right]^2$ impose 'penalties' for having either $x(t)$ or $u(t)$ not being close to its corresponding goal level (Sethi and Thomson, 2000).

The dynamics of the state equation of this objective function (2.1) which says that the inventory at time t is increased by the production rate and decreased by the demand rate and the instantaneous rate of deterioration $\eta\gamma t^{\gamma-1}$ of Weibull distribution can be written as according to:

$$dx(t)=\left[u(t)-y(t)-\eta\gamma t^{\gamma-1}x(t)\right]dt \quad x(0)=x, \quad x>0 \tag{2.2}$$

Note that the goal pair $\left(\hat{u}(t);\hat{x}(t)\right)$ must satisfy the differential equation (2.2), to be feasible. The solution of (2.2) is given by

$$x(t)=x(0)\exp\left\{-\eta t^{\gamma}\right\}-\int_0^t\left[y(t)-u(t)\right]dt, \quad \text{for all } t\in[o,T]. \tag{2.3}$$

Assuming that $x(0)=x$ is known and note that the production goal rate $\hat{u}(t)$ can be computed using the state equation (2.2) as:

$$\hat{u}(t) = y(t)+\eta\gamma t^{\gamma-1}\hat{x}(t) \tag{2.4}$$

The inventory dynamics represented by equation (2.2) can be written in terms of the new variables as:

$$dz(t)=\left[-\eta\gamma t^{\gamma-1}z(t)+k(t)+\mu(t)\right]dt \tag{2.5}$$

Where; $z(t)=x(t)-\hat{x}(t)$ is the deviation of the inventory level $x(t)$, from the desired inventory goal rate; $k(t)=u(t)-\hat{u}(t)$ is the deviation of the production rate from the desired production goal rate and a function of the actual demand expressed as

$$\mu(t)=\hat{u}(t)-y(t)-\eta\gamma t^{\gamma-1}\hat{x}(t). \tag{2.6}$$

The optimal control model becomes

$$\text{minimize } J=\frac{1}{2}\int_0^{\mathrm{T}}\left\{h\left[z(t)^2\right]+C\left[k(t)^2\right]\right\}dt, \tag{2.7}$$

together with (2.5) form a standard linear quadratic regulator (LQR) problem with known disturbance $\mu(t)$ defined in (2.6). The general form of this LQR optimal control problem for a finite time horizon $[0,T]$ is as follows;

$$\text{minimize } J=\frac{1}{2}\int_0^{\mathrm{T}}\left\{z^T(t)Q(t)z(t)+k^T(t)R(t)k(t)\right\}dt, \tag{2.8}$$

subject to an ordinary differential equation:

$$dz(t)=\left[B(t)z(t)+D(t)k(t)+\mu(t)\right]dt, \tag{2.9}$$

Where; $Q(t)$ and $R(t)$ are real symmetric positive semi-definite matrices of appropriate dimension; and $B(t)$ and $D(t)$ are the system of dynamics matrices.

SOLUTION TO THE OPTIMAL CONTROL PROBLEM

The optimal control policy is given by Baten and Kamil (2009)

$$k^*(t)=-R^{-1}(t)D^T(t)P(t)z(t), \tag{3.1}$$

Where;

$$P(t)=-B^T(t)^{-1}Q(t). \tag{3.2}$$

By comparing equations (2.7) and (2.5) to (2.8) and (2.9), we have $B(t)=-\eta\gamma t^{\gamma-1}$, $D=1$, $Q=h$ and $R=C$. Then from (3.2) we obtain:

$$P(t) = h(\eta\gamma t^{\gamma-1})^{-1}, \tag{3.3}$$

and the optimal control policy (3.1) becomes:

$$k^*(t)=-hC^{-1}(\eta\gamma t^{\gamma-1})^{-1}z(t). \tag{3.4}$$

Example 3.1 If we choose $B=-1(by\,\eta=1,\gamma=1), \quad D=1,\ Q=h=1 \ \ and \ \ R=C=1$ then the optimal control model (2.8) and (2.9) becomes over a finite time horizon $[0,T]$

$$\text{minimize} \quad J(k(t))=\frac{1}{2}\int_0^T\{z^2(t)+k^2(t)\}dt, \tag{3.5}$$

subject to the control system

$$dz(t)=[-z(t)+k(t)+\mu(t)]dt \quad z(0)=z, \quad z>0.$$

Here the optimal (state) feedback control (3.4) becomes

$$k^*(t)=-P(t)z(t). \tag{3.6}$$

NUMERICAL SOLUTION AND SENSITIVITY ANALYSIS

To illustrate, we consider a numerical solution of an inventory-production system where the planning horizon has length $T = 12$ months; the inventory holding cost coefficient $h = 1$; the production cost coefficient $C = 1$. Following the assumption that the demand function varies with time, we can consider the different demand rates with changing the shape of the demand function by taking $y_1(t) = 1 + \sin(t)$; $y_2(t) = 2\sin(t) + 3$; $y_3(t) = 1 + \cos(t)$; $y_4(t) = \exp(-t)$; $y_5(t) = \exp(t)$ and keeping all other parameters unchanged yielded the figures represented by the figures 3 to 12. The goal inventory level is considered as $\hat{x}(t) = 1 + t + \sin(t)$. The shape and scale parameters of the Weibull distribution rate are considered as $\eta = 1$, and $\gamma = 1$ respectively. The production level with time t given $\hat{u}(t)$ from the equation (2.3) and the inventory level $x(t)$ in-terms of the first-order differential equation from (2.2) are solved numerically using the version 6.5 of the mathematical package MATLAB. Whenever the goal inventory level is considered as the sinusoidal function of time t that is, $\hat{x}(t) = 1 + t + \sin(t)$, then figure 1 does not show the convergence of the optimal level. Whenever if we take the inventory goal level is as $\hat{x}(t) = 10$, then figure 2 shows the convergence of the optimal inventory towards inventory goal level. In addition, in order to solve the objective function (3.5), the optimal control policy (3.4) and the solution of the inventory level (2.3) are used. So, the value of the objective function in this case is 7008.5264 cost units.

However, in the subsections we present the model to measure the performance using different demand patterns. figures 3, 5, 7 and 9 show the slight variations of the optimal inventory level with time but figure 11 shows the large variations of the optimal inventory level with time with changing the shape of the demand functions. On the other hand, the optimal production level with time represented by figures 4, 6, 8, 10 and 12 that show variations with changing the shape of the demand functions. It is observed that the optimal production rates are very sensitive to changes in the demand functions. The different shape of the demand functions in the inventory controlled system increase the production systems significantly.

Sinusoidal demand function

In this subsection, we present the model with sinusoidal demand function. Substituting $y_1(t) = 1 + \sin(t)$ in the controlled system (2.2) instead of $y(t)$ we have;

$$dx_1(t) = \left[u_1(t) - (1 + \sin(t)) - \eta\gamma t^{\gamma-1} x_1(t)\right] dt \quad x(0) = x,$$

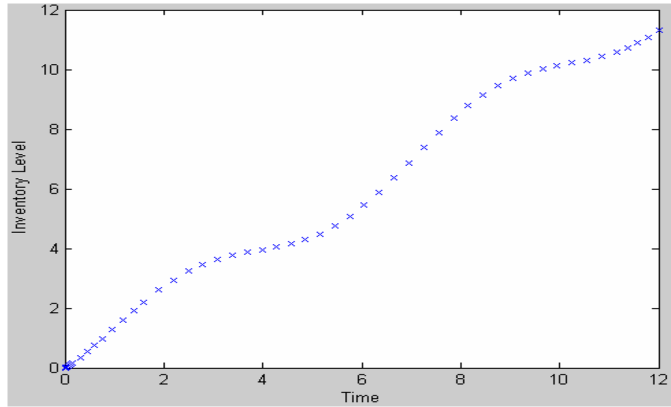

Figure 1. Optimal inventory level with time.

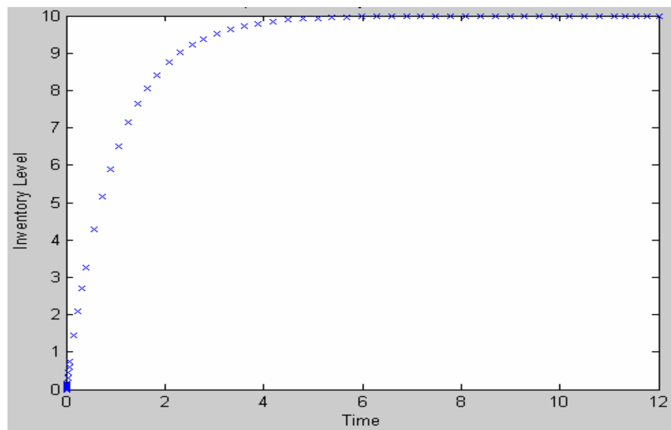

Figure 2. Optimal inventory level with time.

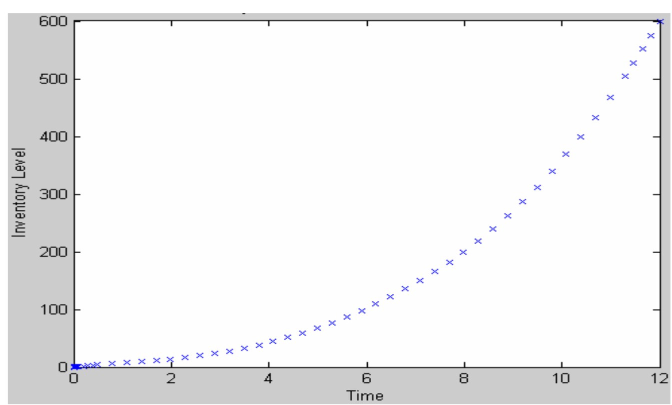

Figure 3. Inventory level with sinusoidal demand function.

x>0, displayed by figure 3 from which the production goal rate $\hat{u}(t)$ can be computed (assuming $x(0) = x$) as displayed in Figure 4.

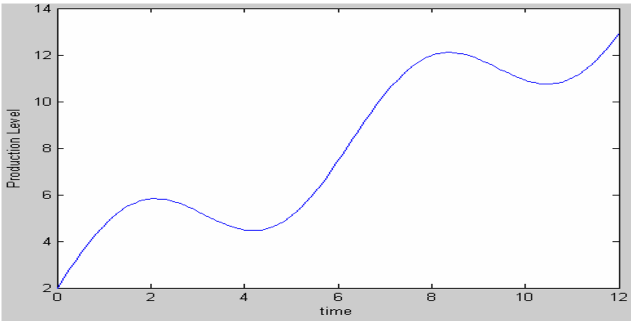

Figure 4. Production level with time.

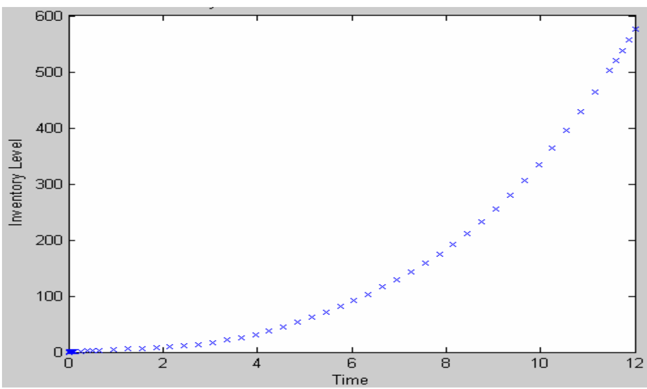

Figure 5. Inventory level with twice-sinusodial demand function.

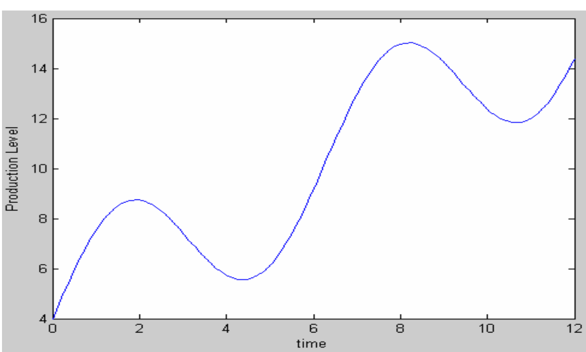

Figure 6. Production level with time.

Twice-sinusoidal demand function

In this subsection, we present the model with twice-sinusoidal demand function. Substituting $y_2(t) = 2\sin(t) + 3$ in the controlled system (2.2) instead of $y(t)$ we have

$$dx_2(t) = \left[u_2(t) - (2\sin(t)+3) - \eta\gamma t^{\gamma-1} x_2(t)\right]dt \quad x(0) = x, \quad x > 0, \quad \text{displayed by figure-5}$$

from which the production goal rate $\hat{u}(t)$ can be

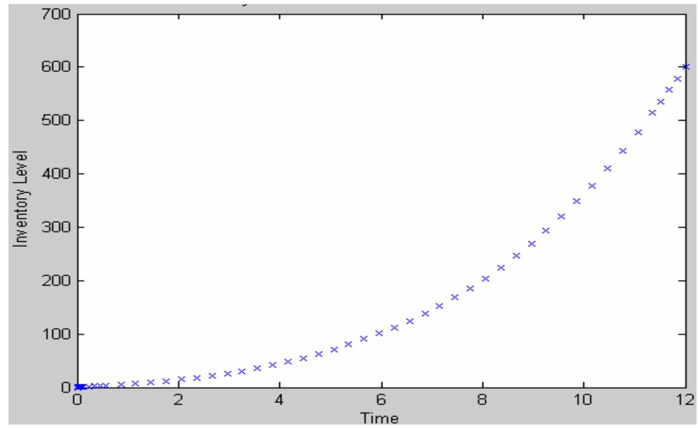

Figure 7. Inventory level with co-sinusodial demand function.

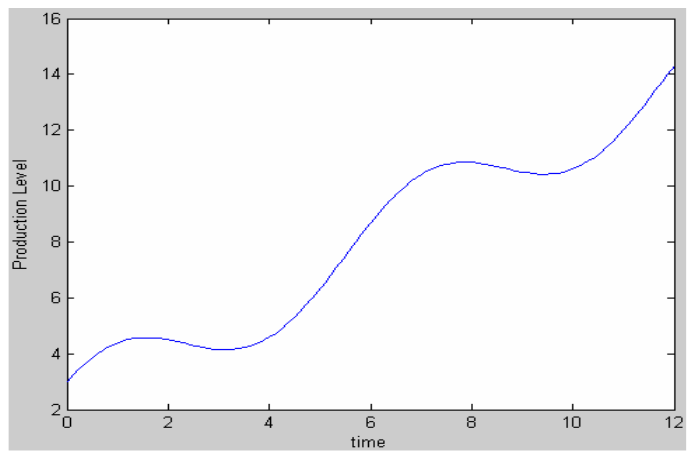

Figure 8. Production level with time.

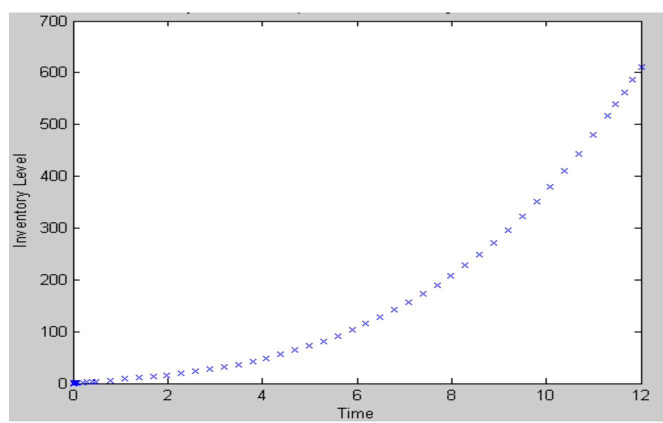

Figure 9. Inventory level with exponential decreasing demand function.

computed (assuming $x(0) = x$)

$$\hat{u}_2(t) = (2\sin(t)+3) + \eta\gamma t^{\gamma-1}\hat{x}(t), \quad \text{displayed by figure-6}$$

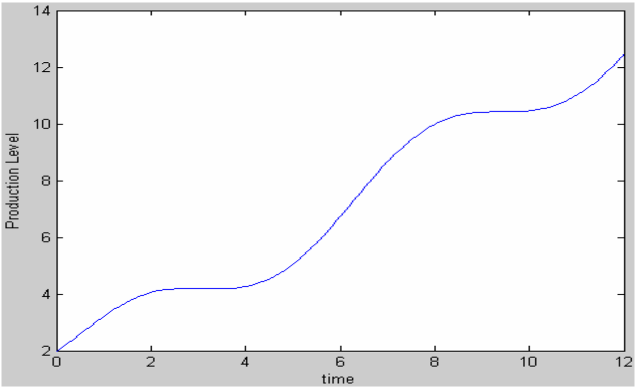

Figure 10. Production level with time.

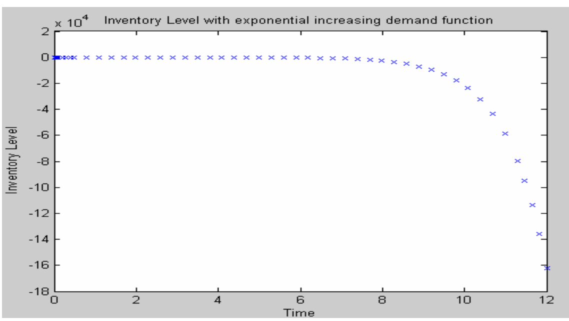

Figure 11. Inventory level with exponential increasing demand function.

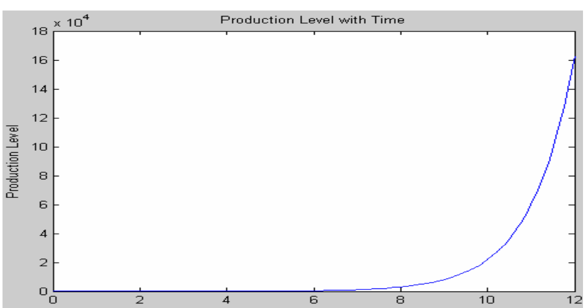

Figure 12. Production level with time.

Co-sinusoidal demand function

In this subsection, we present the model with twice-sinusoidal demand function. Substituting $y_3(t) = 1 + \cos(t)$ in the controlled system (2.2) instead of $y(t)$ we have;

$$dx_3(t) = \left[u_3(t) - (1+\cos(t)) - \eta\gamma t^{\gamma-1} x_3(t) \right] dt \quad x(0) = x,$$

x>0, displayed by figure-7 from which the production goal rate $\hat{u}(t)$ can be computed (assuming $x(0) = x$)

$$\hat{u}_3(t) = (1+\cos(t)) + \eta\gamma t^{\gamma-1}\hat{x}(t), \quad \text{displayed by figure-8}$$

Exponential decreasing demand function

In this subsection, we present the model with sinusoidal demand function. Substituting $y_4(t) = \exp(-t)$ in the controlled system (2.2) instead of $y(t)$ we have;

$$dx_4(t) = \left[u_4(t) - \exp(-t) - \eta\gamma t^{\gamma-1} x_4(t) \right] dt \quad x(0) = x,$$

x>0, displayed by figure-9 from which the production goal rate $\hat{u}(t)$ can be computed; (assuming $x(0) = x$)

$$\hat{u}_4(t) = \exp(-t) + \eta\gamma t^{\gamma-1}\hat{x}(t), \quad \text{displayed by figure-10}$$

Exponential increasing demand function

In this subsection, we present the model with twice-sinusoidal demand function. Substituting $y_5(t) = \exp(t)$ in the controlled system (2.2) instead of $y(t)$ we have;

$$dx_5(t) = \left[u_5(t) - \exp(t) - \eta\gamma t^{\gamma-1} x_5(t) \right] dt \quad x(0) = x, \quad x > 0,$$

As displayed in figure 11, from which the production goal rate $\hat{u}(t)$ can be computed (assuming $x(0) = x$)

$$\hat{u}_5(t) = \exp(-t) + \eta\gamma t^{\gamma-1}\hat{x}(t), \quad \text{displayed by figure-12}$$

CONCLUSION

The optimal inventory production control problem with problem Weibull distributed deterioration is studied and the solution of this problem is described by Pontryagin maximum principle. The inventory level and the optimal production control policy are derived. However, we give numerical solution of optimal inventory production controlled system with Weibull distribution deteriorating items using different types of the demand functions.

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