academicJournals

Vol. Vol. 10(16), pp. 498-502, 30 August, 2015
DOI: 10.5897/IJPS2015.4379
Article Number: F0AB61B54874
ISSN 1992 - 1950

http://www.academicjournals.org/IJPS

International Journal of Physical Sciences

Full Length Research Paper

Existence of solution for a third order three point boundary value problem at resonance

S. A. Iyase and M. C. Agarana*

Department of Mathematics, Covenant University, P. M. B. 1023, Ota, Ogun State, Nigeria.

Received 17 June 2015; Accepted 4 August, 2015

Using coincidence degree arguments we prove some existence results, for a third order three point boundary value problem at resonance.

Key words: Resonance, coincidence degree, third order, three point boundary value problem.

INTRODUCTION

In this paper, we shall discuss the solvability of the three point boundary value problem of the form

$$x'''(t) = f(t, x, x', x'') \tag{1}$$

$$x'(1) = x''(0) = 0,\ x(1) = ax(\eta) \tag{2}$$

Where $a = 1$ and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and $\eta \in (0,1)$.

Multipoint boundary value problems arise in a variety of different areas of Applied Mathematics, Physics and Engineering. For example, in solving partial differential equations by the method of separation of variables, one comes across differential equations whose solutions must satisfy boundary conditions at several points. Similarly bridges of various sizes are sometimes contrived with multipoint supports which correspond to a multipoint boundary condition. Boundary value Problem (1) to (2) is called a problem at resonance if $Lx = x'''(t) = 0$ has non-trivial solutions under the boundary Conditions (2) that is, when $\dim \ker L \geq 1$. On the interval $[0,1]$ second order and third order multipoint boundary value problems have been studied by Aftabizadeh et al. (1989); Constantin (1996); Feng and Webb (2001); Gregus et al. (1971); Gupta and Lakshmikantham (1991); Gupta et al. (1995); Liu and Yu (1995) and Ma (1997, 1998). Our method of proof consists of imposing a decomposition condition on f of the form

$$f(t, x, y) = g(t, x, y) + h(t, x, y) \tag{3}$$

We shall then employ coincidence degree arguments to obtain our existence results. In what follows, we shall use the norm $|x|_\infty = \max_{t\in[0,1]} |x(t)|$. We denote the norm in $L^1[0,1]$ by $|\ |_1$ and on $L^2[0,1]$ by $|\ |_2$.

Examples are the vibrations of a guy wire of a uniform cross composed of N parts of different densities (Moshiasky, 1981) and some problems in the theory of elastic stability (Timoshenko, 1961).

*Corresponding author. E-mail: michael.agarana@covenantuniversity.edu.ng

2010 Mathematics Subject Classification: 34B15, 34B10

PRELIMINARIES

Consider the linear equation

$$Lx = x'''(t) = 0 \tag{4}$$

$$x'(1) = x''(0) = 0,\ x(1) = ax(\eta) \tag{5}$$

If we assume a solution of the form

$$x(t) = \sum_{i=0}^{3} a_i t^i$$

Then this solution exists if and only if

$$a_3(1-a) = 0 \tag{6}$$

The case where $a \neq 1$ which corresponds to the non-resonance case was discussed in Iyase (2005)

When $a = 1$, Equations (4) to (5) has non-trivial solutions. Therefore, problem (1) to (2) is at resonance. We shall prove existence results for the boundary value problem (1) to (2) under the Condition (6) when $a = 1$.

In our proof we shall need the following Continuation Theorem based on Mawhin's coincidence degree.

Theorem 1

Mawhin (1979) Let L be a Fred Holm operator of index zero and let N be $L-\text{compact}$ on $\overline{\Omega}$. Assume that the following conditions are satisfied:

(1) $Lx \neq Nx$ for any $(x,\lambda) \in (\operatorname{dom} L \cap \partial\Omega) \times (0,1)$
(2) $QNx \neq 0$ for $x \in \ker L \cap \partial\Omega$
(3) The Brower degree $\deg_B((JQN)_{\ker}, \Omega \cap \ker L, 0) \neq 0$

Where $J : \operatorname{Im} Q \to \ker L$ is some isomorphism.

Then there exists $x \in \overline{\Omega} \cap \operatorname{dom} L$ such that

$$Lx = Nx \tag{7}$$

Let $X = C^2[0,1],\ Z = L^1[0,1]$
Let $L : \operatorname{dom} L \subset X \to Z$ be defined by

$$Lx = x'''(t) \tag{8}$$

Where
$\operatorname{dom} L = \left\{x \in W^{3,1}(0,1),\ x'(1) = x''(0) = 0,\ x(1) = x(\eta)\right\}$.
We define $N : X \to Z$ by setting

$$Nx = f(t, x(t), x'(t), x''(t)) \tag{9}$$

Then the boundary value Problem (1) to (2) can be put in the form

$$Lx = Nx \tag{10}$$

Lemma 1

If L and N are defined as in Equations (8) and (9), then

$$\operatorname{Im} L = \left\{ y \in Z : \int_\eta^1 \int_t^1 \int_0^r y(s)\, ds dr dt = 0 \right\}$$

$L : \operatorname{dom} L \subset X \to Z$ is a Fred Holm operator of index zero.

Proof

We will show that the problem

$$x'''(t) = y \text{ for } y \in Z \tag{11}$$

has a solution $x(t)$ satisfying

$$x'(1) = x''(0) = 0,\ x(1) = x(\eta) \tag{12}$$

If and only if

$$\int_\eta^1 \int_t^1 \int_0^r y(s)\, ds dr dt = 0 \tag{13}$$

Suppose Equation (11) has a solution $x(t)$ satisfying Equation (12) then from Equation (11) we have

$$0 = \int_\eta^1 \int_t^1 \int^r x'''(s)\, ds dr dt = \int_\eta^1 \int_t^1 \int_0^r y(s)\, ds dr dt$$

Now suppose $\int_\eta^1 \int_t^1 \int_0^r y(v)\, dv dr dt = 0$.
Let

$$x(t) = c - \frac{t}{1-\eta} \int_\eta^1 \int_0^r \int_0^w y(v)\, dv dw dr + \int_0^t \int_0^r \int_0^w y(v)\, dv dw dr \tag{14}$$

Then $x(t)$ is a solution of Equation (11) with $\int_\eta^1 \int_t^1 \int_0^r y(s)\, ds dr dt = 0$.
For $y \in Z$ we define the projection $Q : Z \to Z$ by

$$Qy = \frac{6}{(\eta+2)(\eta-1)^2} \int_\eta^1 \int_t^1 \int_0^r y(s)\, ds dr dt,\ y \in Z \tag{15}$$

Let $y_1 = y - Qy$, that is $y_1 \in \ker Q$. Then by direct calculations we have

$$\int_\eta^1\int_t^1\int_0^r y_1(v)\,dv\,dr\,dt = \int_\eta^1\int_t^1\int_0^r y(v)\,dv\,dr\,dt\left[1 - \frac{6}{(\eta+2)(\eta-1)^2}\int_\eta^1\int_t^1\int_0^r dv\,dr\,dt\right] = 0$$

So $y_1 \in \operatorname{Im} L$. Hence $Z = \operatorname{Im} L + \operatorname{Im} Q$.

Since $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$ we obtain

$Z = \operatorname{Im} L \oplus \operatorname{Im} Q$

Now $\ker L = \{x \in \operatorname{dom} L : x = c,\ c \in \mathbb{R}\}$.

Hence $\dim\ker L = \dim \operatorname{Im} Q = 1$.

Therefore, L is a Fred Holm operator of index zero.

Let $\mathrm{P}: X \to X$ be defined by

$Px(t) = x(0),\ t \in [0,1]$

Let

$$L_p = L|_{\operatorname{dom} L \cap \ker p} \tag{16}$$

The operator $K = L_p^{-1} : \operatorname{Im} L \to \operatorname{dom} L \cap \ker p$ is the linear operator defined by

$$(\mathrm{K}\,y)(t) = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds$$

By the Azela-Ascoli Theorem it can be shown that K is compact. Hence N is $L-\text{compact}$. We recall that a linear mapping

$L : DomLCX \to Z$

With

$KerL = L^{-1}(0)$

And

$\operatorname{Im} L = L(\mathrm{DomL})$

Will be called a Fred Holm mapping if the following two conditions hold:

(i) KerL has a finite dimension
(ii) ImL is closed and has a finite codimension

We also recall that the codimension of ImL is the dimension of Z/ImL that is, the dimension of the cokernol of L. When L is a Fred Holm mapping, its Fred Holm index is the integer.

IndL = dimKorh – CodimImL

We say that a mapping N is L-compact on Ω if the mapping QN: $\bar{\Omega} \to Z$ is continuous, QN($\bar{\Omega}$) is bounded, and Kp(I-Q)N: $\bar{\Omega} \to X$ is compact, that is, it is continuous and Kp(I-Q)N($\bar{\Omega}$) is relatively compact, where Kp: ImL $\to$ DomL$\cap$ KorP is the inverse of restriction Lp to DomL$\cap$ KerP, so that LKp = I and KpL = I – P.

MAIN RESULTS

Theorem 2

Assume that $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and has the decomposition

$$f(t,x,y,z) = g(t,x,y,z) + h(t,x,y,z) \tag{17}$$

such that

(i) $xf(t,x,y,z) > 0$ for $a.e\ t \in [0,1]$ and $(x,y,z) \in \mathbb{R}^3$
(ii) $yg(t,x,y,z) \ge 0$
(iii) $|h(t,x,y,z)| \le M\left\{|x| + |y| + |z|^\beta\right\}$ for $0 \le \beta < 1$
(iv) $zf(t,x,y,z) \le (|z|^2 + 1)(D(t,x,y,z) + \alpha(t))$

Where $D(t,x,y,z)$ is bounded on bounded sets and $\alpha \in L^1[0,1]$

Then the boundary value Problem (1) to (2) has at least one solution in $C^2[0,1]$ provided

$$M < \frac{\pi^3}{16\sqrt{4+\pi^2}} \tag{18}$$

Proof

Let L be defined as in Equation (8). Then $\ker L = \{x \in X : x \text{ is a constant mapping}\} \simeq \mathbb{R}$.

We shall prove that the conditions of Theorem 1 are satisfied. To do this, we shall show that for $\lambda \in (0,1)$ the set of solutions of the family of equations

$$x'''(t) = \lambda f(t,x,x',x'') \tag{19}$$

$$x'(1) = x''(0) = 0,\ x(1) = x(\eta) \tag{20}$$

is a priori bounded and then construct Ω accordingly.

Let $x \in C^2[0,1]$ satisfy Equation (19) to (20). Since $x(1) = x(\eta)$ there exists $\zeta \in (\eta,1)$ such that $x'(\zeta) = 0$ and from $x'(1) = x'(\zeta) = 0$ there exists $t_1 \in (\zeta,1)$ such that $x''(t_1) = 0$. Hence from condition (i) of Theorem 2 we derive that if $x(t) > 0$ then

$$0 = \int_0^{t_1} x'''(s)\,ds = \lambda \int_0^{t_1} f(s, x(s), x'(s), x''(s))\; ds > 0$$

Which is a contradiction. If $x(t) < 0$ we derive a similar contradiction. Hence there exists $t_0 \in (0,t_1)$ such that $x(t_0) = 0$. Therefore for each $t \in [0,1]$ we have

$$|x|_2^2 \le \frac{4}{\pi^2}|x'|_2^2 \tag{21}$$

Multiplying Equation (19) by $x'(t)$ and using the relation

$x'(1) = x''(0) = 0$

yields

$$\int_0^1 x'(t)x'''(t)\,dt = -\int_0^1 |x''|^2 dt\,.$$

Hence

$$\int_0^1 |x''|^2 dt = -\lambda\int_0^1 x'(t)g(t,x,x',x'')\,dt - \lambda\int_0^1 x'(t)h(t,x,x',x'')\,dt$$

Using the Cauchy inequality

$$|ab| \le \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon} \quad \text{for } \varepsilon > 0$$

we have

$$\int_0^1 |x'(t)||h(t,x,x',x'')|\,dt \le \frac{\varepsilon}{2}\int_0^1 |x'|^2 dt + \frac{1}{2\varepsilon}\int_0^1 \;|h(t,x,x',x'')|^2 dt$$

From condition (iii) we obtain the estimate

$$|h(t,x,y,z)|^2 \le 4M^2\left\{|x|^2 + |y|^2 + |z|^{2\beta}\right\}$$

Therefore from Holder's inequality we get

$$|x''|_2^2 - \frac{\varepsilon}{2}|x'|_2^2 \le \frac{2M^2}{\varepsilon}\left\{|x|_2^2 + |x'|_2^2 + |x''|_2^{2\beta}\right\}$$

since $x'(1) = 0$ we obtain

$$\frac{1}{2}|x''|_2^2 + \left(\frac{\pi^2}{8} - \frac{\varepsilon}{2}\right)|x'|_2^2 \le \frac{2M^2}{\varepsilon}\left\{|x|_2^2 + |x'|_2^2 + |x''|_2^{2\beta}\right\} \tag{22}$$

Using Equation (21) in (22) we get

$$\frac{1}{2}|x''|_2^2 + \left(\frac{\pi^2}{8} - \frac{\varepsilon}{2} - \frac{8M^2}{\varepsilon\pi^2} - \frac{2M^2}{\varepsilon}\right)|x'|_2^2 \le \frac{2M}{\varepsilon}|x''|_2^{2\beta} \tag{23}$$

Since $0 \le \beta < 1$ we infer the existence of a constant M_1 such that

$$|x'|_2 < |x''|_2 < M_1 \tag{24}$$

Provided

$$\frac{\pi^4}{8} > \frac{8M^2}{\varepsilon} + \frac{2\pi^2 M^2}{\varepsilon} + \frac{\varepsilon\pi^2}{2} \tag{25}$$

The choice $\varepsilon = 2M\sqrt{4+\pi^2}$ minimizes the right hand side of Equation (25) and the minimum value is $2M\pi\sqrt{4+\pi^2}$. Therefore Equation (25) holds provided

$$M < \frac{\pi^3}{16\sqrt{4+\pi^2}} \tag{26}$$

Furthermore, since $x(t_0) = x'(1) = 0$ for $t_0 \in (\eta,1)$, we get from Equation (24) that

$$|x|_\infty < |x'|_\infty < M_2,\; M_2 > 0 \tag{27}$$

From condition (iv) of Theorem 2 we obtain

$$\frac{x''x'''}{|x''|^2 + 1} \le D(t,x,x') + \alpha(t) \tag{28}$$

Integrating Equation (28) from 0 to t we get

$$log_e|x''| \le \int_0^t \frac{x''(s)x'''(s)}{|x''(s)|^2+1}ds = \left[\frac{1}{2}log_e\left(|x''(s)|^2+1\right)\right]_0^t < D + |\alpha|_1 = N_0 \tag{29}$$

Where the constant D depends only on M_2. Furthermore since $x''(0) = 0$ we get from Equation (29) that

$$|x''|_\infty < e^{N_0} = M_3 \tag{30}$$

Let

$$\|x\| = \max(|x|_\infty, |x'|_\infty, |x''|_\infty) < \max(M_2, M_3) = A$$

It follows that $\|x\| < A$.

We take $\Omega = \{x \in X : \|x\| < A\}$ then if $x \in \operatorname{dom} L \cap \partial\Omega$ then $Lx \neq \lambda Nx$, $0 < \lambda < 1$.

If $x \in \ker L \cap \partial\Omega$ then $x = \pm A$.

Now if $x = A$ we derive from condition (i) of Theorem 2 that $QNx = \frac{6}{(\eta+2)(\eta-1)^2}\int_\eta^1\int_t^1\int_0^r f(s, A, 0, 0)dsdrdt > 0$

and if $x = -A$ we get $QNx < 0$

Thus $QNx \neq 0$ for $x \in \ker L \cap \partial\Omega$.

Verifying Condition (2) of Theorem 1

It is easily verified that

$$H(\mu, x) = \mu x + (1-\mu)QNx,\ 0 \le \mu < 1$$

is a homotopy from the identity I to QN on $\overline{\Omega}$ and is such that $H(\mu, x) \neq 0$ on $[0,1] \times (\partial\Omega \cap \ker L)$

Hence taking J in Condition (3) of Theorem 1 to be the identity we get

$$\deg_\beta\left[QN|_{\ker L}, \Omega \cap \ker L, 0\right] = \deg_\beta\left[\mathrm{I}, \Omega \cap \ker L, 0\right] = 1$$

This completes the proof of Theorem 2.

Remark 1

The results of Theorem 2 still hold if condition (i) is replaced by

$$xf(t, x, y, z) < 0.$$

Remark 2

The results of Theorem 2 remain valid if assumption (i) is replaced by any of the following assumptions:

(1) $|h(t, x, y, z)| \le M(|x| + |y|^\beta + |z|)$ for $0 \le \beta < 1$

provided $M < \frac{\pi}{8\sqrt{16+\pi^4}}$

(2) $|h(t, x, y, z)| \le M(|x|^\beta + |y| + |z|)$ for $0 \le \beta < 1$

provided $M < \frac{1}{\pi}$

(3) $|h(t, x, y, z)| \le M(|x|^r + |y|^\beta + |z|^p)$ for $0 \le \beta, r, p < 1$

for some constant M.

CONCLUSION

In this paper, by using a continuation theorem based on coincidence degree theory, we obtained the existence of solution for a third order boundary value problem at resonance. For the research on the existence of solution of linear and non-linear boundary value problem. Coincidence degree theory plays an important role.

Conflict of Interest

The authors have not declared any conflict of interest.

REFERENCES

Aftabizadeh AR, Gupta CP, Jian MX (1989). Existence and Uniqueness Theorems for three point boundary value problems. SIAM. J. Math. Anal. 20:716-726.

Constantin A (1996). A note on a boundary value problem. Nonlinear Anal. 27(1):13-16.

Feng W, Webb JRL (2001). Solvability of three point boundary value problems at resonance. Nonlinear Anal. Theory methods. Appl. 47:2407-2418.

Gregus RE, Neumann F, Arscott M (1971). Three point boundary value problem for differential equations. J. London Math. Soc. 3:429-436.

Gupta CP, Lakshmikantham V (1991). Existence and uniqueness theorems for a third order three point boundary value problem. Nonlinear Anal. 16:949-957.

Gupta CP, Ntouyas K, Tsamatos P (1995). Solvability of an m-point boundary value problem for second order ordinary differential equations. J. Math. Anal Appl. 189:575-584.

Iyase SA (2005). Existence Theorems for a third order three point boundary value problem. J. Nig. Math. Soc. 24:17-23.

Liu B, Yu S (1995). Solvability of multipoint boundary value problem set resonance (III). Appl. Math. Comput. 129:1483–1489.

Ma RY (1997). Existence Theorem for a second order three point boundary value problem. J. Math. Anal. Appl. 212:430-442.

Ma RY (1998). Multiplicity results for a third order boundary value problem at resonance. Non Linear Anal. Theory Methods. Appl. 32:493-499.

Mawhin J (1979). Topological degree Methods in Non-Linear boundary value problems. NSF. BMS Reg.Conf. Ser. Math 40. Americ. Math. Soc. Providence R.I.

Moshiasky M (1981). Sobre Los prolemas de condiciones a La frontiers en una dimension de caracteristicas discontinues. BoL soc. Mat Mericana 7.

Timoshenko S (1961). Theory of Elastic Stability. McGraw-Hill. NewYork 1961.